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SUBALGEBRAS AND SUBLOGICS OF σ-LOGICS

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$(L, \leqq, ')$ is supposed to be an orthocomplemented partially ordered set. It will be explicitely said when it is considered as σ-complemented. The notion of a sublogic or a subalgebra is used in various papers more or less intuitively. The differences and even misunderstandings are caused by the fact that sometimes the logic L is supposed to be a lattice while in some papers the assumption that L be a lattice is omitted.

This paper contains some results concerning generated systems having their analogies in Boolean algebras or in σ-logics of subsets of a given set X. One of them is a proof of a necessary and sufficient condition for the existence of a sub-σ-algebra A containing $E \subset L$ and such that $E \subset A \subset L$. This result was proved for the logic of subsets in [1] or in a different way in [2] but as far as the author knows it has not been proved for the general case. We shall also show that whenever the answer to the last mentioned problem is positive, the generated logic (σ-logic) coincides with the generated algebra (σ-algebra).

Notations

A partially ordered set L with the first element o and the last element i will be considered. The symbols $x \vee y$, $x \wedge y$ stand for $\sup\{x, y\}$, $\inf\{x, y\}$, respectively, if the mentioned elements exist. In what follows x' denotes the uniquely determined complement which is supposed to exist for any $x \in L$. We say that a and b are *disjoint* or *orthogonal* write $a \perp b$ iff $a \leqq b'$. If $a \perp b$, we write $a + b$ instead of $a \vee b$. L is said to be a *logic* iff the following axioms are satisfied:

(i) $(a')' = a$ for all $a \in L$,
(ii) $a \leqq b$ implies $b' \leqq a'$,
(iii) $a \vee a'$ exists and $a \vee a' = i$ for each $a \in L$,
(iv) if $a \leqq b$, then there is $c \in L$ such that $a + c = b$,
(v) if $a, b \in L$ and $a \perp b$, then $a + b$ exists as an element of L.

It is shown in [4] that the element c from (iv) is uniquely determined and $c = b \wedge a'$. In this case we use the notation $c = b - a$.

L is said to be a *σ-logic* iff (v) is substitued by

(v′) if $a_1, a_2, \ldots$ are mutually disjoint elements from L, then $\vee_i a_i$ exists in L.

If $a_1, a_2, \ldots$ are mutually disjoint, we write $\Sigma_i a_i$ instead of $\vee_i a_i$. The elements of a logic are called events.

An *isomorphism* from a logic L_1 into a logic L_2 is an injection $h\colon L_1 \to L_2$ which satisfies:

(i) $a, b \in L_1$ and $a \perp b$ implies $h(a) \perp h(b)$,

(ii) if $a, b \in L_1$ and $a \vee b$ exists in L_1, then there exists $h(a) \vee h(b)$ in L_2 and moreover

$$h(a \vee b) = h(a) \vee h(b),$$

(iii) $h(o_1) = o_2$, $h(i_1) = i_2$, where o_j, i_j $(j = 1, 2)$ are the first and the last elements of L_j, respectively.

h is said to be a *σ-isomorphism* if L_1 and L_2 are logics and (ii) is substituted by

(ii′) if $a_1, a_2, \ldots$ is a sequence of elements of L_1 and $\vee_i a_i$ exists in L_1, then there exists $\vee_i h(a_i)$ in L_2 and

$$h(\vee_i a_i) = \vee_i h(a_i).$$

Remark. It is easy to see that any isomorphism $h\colon L_1 \to L_2$ has the following property: $h(a') = (h(a))'$.

$$(h(i_1) = h(a + a') = h(a) + h(a') = i_2 \quad \text{and} \quad a + b = i, \quad a + b_1 = i$$

in any logic implies $b = b_1 = a'$.)

Lemma 1. *Let $h\colon L_1 \to L_2$ be an isomorphism from a logic L_1 into a logic L_2. Then $a \wedge b = o$ in L_1 implies $h(a) \wedge h(b) = o$ in L_2.*

We leave the simple proof to the reader.

We say that $L_1 \subset L$ is a *sublogic (sub-σ-logic)* of L if L_1 is a logic (σ-logic) and the identity map from L_1 into L is an isomorphism (σ-isomorphism). If nothing else is said, then L may be interpreted as a logic or a σ-logic.

$A \subset L$ is said to be a *subalgebra (sub-σ-algebra)* whenever A is a Boolean algebra (Boolean-σ-algebra) and A is a sublogic (sub-σ-logic) of L.

B will be called a *monotone system* iff B is a partially ordered set and for every increasing (decreasing) sequence of elements of B there exists $\vee_i a_i$ $(\wedge_i a_i)$ as an element of B. It is easy to see that a σ-logic is a monotone system.

An *m-isomorphism* h from the monotone system B_1 into the monotone system B_2 is the injection $h\colon B_1 \to B_2$ such that if $a_1, a_2, \ldots$ is a sequence of elements of B_1 and $\vee_i a_i$ $(\wedge_i a_i)$ exists in B_1, then there exists $\vee_i h(a_i)$ $(\wedge_i h(a_i))$ in B_2 and moreover

$$h(\vee_i a_i) = \vee_i h(a_i) \quad (h(\wedge_i a_i) = \wedge_i h(a_i)).$$

If $B_1 \subset B$ is a monotone system and the identity map from B_1 into B is an m-isomorphism, then we say that B_1 is a *submonotone system* of B.

For any non-void set of events $E \subset L$ let $L(E)$, $L_\sigma(E)$, $A(E)$, $A_\sigma(E)$ and $M(E)$ denote respectively the smallest sublogic, sub-σ-logic, subalgebra, sub-σ-algebra and submonotone system in L containing E (if they exist).

Two events a, $b \in L$ will be said *compatible* (notation $a \leftrightarrow b$) if there are mutually disjoint events a_1, b_1, $c \in L$ such that $a = a_1 + c$ and $b = b_1 + c$. If $E \subset L$ and $a \leftrightarrow b$ for every $a, b \in E$, then E is said to be *compatible*. The logic (or σ-logic) L will be said *conditionally three compatible* if for any three mutually compatible events a, b, c belonging to $L, b \vee c$ exists and $a \leftrightarrow b \vee c$.

A special type of a logic and a σ-logic are an s-class and a σ-class, respectively (see [1], [2]). Recall that an s-class S is a collection of subsets of a given set Ω which is closed under the forming of the union of any two disjoint sets and under the complementations, while the σ-class is an s-class which is closed under the forming of countable unions of pairwise disjoint sets. The examples of s-classes (σ-classes) which are not Boolean algebras (σ-algebras) are well known.

Example 1. Let Ω be the set of positive integers. Let S be the class of all subsets of Ω which have a density. Then S is an s-class. Recall that the set $E \subset \Omega$ has a density if there exists

$$\lim_n \operatorname{card}(A \cap \{1, 2, \ldots, n\})/n .$$

Example 2. Let Ω be as in Example 1. Let S be the class of all such subsets E of Ω that either E or its complement has an even number of elements. Then S is an s-class.

Generated algebras and σ-algebras

The following notes about the terminology seem to be useful. Given an s-class (σ-class) S of subsets of Ω and an s-class (σ-class) $T \subset S$, then T is sometimes called a sub-s-class (sub-σ-class) of S. The sub-s-class (sub-σ-class) T considered as a logic with respect to the partial ordering given by the inclusion and to the set theoretical complementation need not be a sublogic (sub-σ-logic) according to our definition.

Example 3. Let S be the σ-class consisting of all subsets of $\Omega = \{1, 2, \ldots, 8\}$, and let T be the σ-class consisting of all subsets of Ω with an even number of elements. Clearly $T \subset S$. Let h denote the identity map $h: T \to S$. Then h is not an isomorphism (see Lemma 1).

If T is a sub-s-class of an s-class S, then evidently there exists the smallest algebra A of subsets of Ω containing T. The same is true for a σ-algebra of subsets of Ω. But it is well known that the existence of A satisfying $T \subset A \subset S$ cannot be guaranteed in general. Hence two kinds of problems arise. The first is to give conditions under which such an algebra A exists, the other is to give conditions under which this algebra is a subalgebra of S in the sense of our definition. The first problem was discussed, e.g., in [1]. As to the second it is useful to note that, in general, not only the inclusion but also the existence is problematic.

Remark. There exists an s-class S of subsets of a set Ω such that S considered as a logic cannot be immersed in any σ-logic H.

Proof. Let Ω be a set of positive integers. Let S be the s-class of all such subsets E of Ω that either E or its complement has an even number of elements. Let H be a σ-logic. We shall show that there exists no σ-isomorphism $h: S \to H$. Suppose that such a σ-isomorphism exists. Denote $b_1 = h(\{1, 2\})$ $b_2 = h(\{1, 3\})$ and $a_n = h(\{2n+2,\ 2n+3\})$ for $n = 1, 2, \ldots$. Clearly $a_n \perp a_m$ if $n \neq m$, and $b_k \neq o$ for $k = 1, 2$. Since Ω is the unique upper bound of the set $\{\{1, 2\}, \{4, 5\}, \ldots, \{2n+2, 2n+3\}, \ldots\}$, we get $\sup\{\{1, 2\}, \{4, 5\}, \ldots, \{2n+2, 2n+3\}, \ldots\} = \Omega$. Since h is a σ-isomorphism, we obtain $\Sigma_n a_n + b_1 = i$. Similarly $\Sigma_n a_n + b_2 = i$ is valid. And so $b_1 = b_2 = i - \Sigma_n a_n = (\Sigma_n a_n)'$. But this is impossible because $\{1, 2\} \wedge \{1, 3\} = \emptyset$ implies $h(\{1, 2\}) \wedge h(\{1, 3\}) = o$. Hence $b_1 = b_2 = o$, which is a contradiction. This contradiction shows that the existence of a σ-isomorphism $h: S \to H$ is impossible. Thus S cannot be immersed in any σ-logic H.

First we need some known results for our considerations. In Propositions 1 and 2 we assumed that L is a logic and a, b, c are events from L.

Proposition 1. (see [4]) *If $a \leftrightarrow b$ and $a = a_1 + c$, $b = b_1 + c$, where a_1, b_1, c are mutually disjoint, then*

(i) $a \vee b = a_1 + b_1 + c$, $a \wedge b = c$,

(ii) $a \leftrightarrow b'$, $a' \leftrightarrow b$ and $a' \leftrightarrow b'$.

Proposition 2. (Proposition 3.8 [4]) *If $a_1, a_2, \ldots$ is a sequence of elements of L, if $a \leftrightarrow a_j$ for each j and if $\vee_j a_j$ and $\vee_j (a \wedge a_j)$ both exist, then $a \leftrightarrow \vee_j a_j$. Moreover we have $a \wedge (\vee_j a_j) = \vee_j (a \wedge a_j)$. Particularly if $a_1, a_2, \ldots$ are pairwise disjoint and $a \leftrightarrow a_j$ for all j, then $a \leftrightarrow (a_1 + a_2 + \ldots)$ and $a \wedge (a_1 + a_2 + \ldots) = (a \wedge a_1) + (a \wedge a_2) + \ldots$.*

In the following propositions until Proposition 10 L is always assumed conditionally three compatible and $E \subset L$ is always assumed non-void and compatible. The following is a consequence of Proposition 1 and the definition of the conditional three compatibility.

Proposition 3. *If $a \in E$, then for any finite sequence $a_1, a_2, \ldots, a_n \in E$ one has $a \leftrightarrow a_1 \vee a_2 \vee \ldots \vee a_n$.*

Proposition 4. *If $a \in E$, then, for any finite sequence $a_1, a_2, \ldots, a_n \in E$, $a \leftrightarrow a_1 \wedge a_2 \wedge \ldots \wedge a_n$ is true.*

Proof. We shall prove the proposition only for the case $n = 2$. For $n > 2$ it can be easily proved by induction. Since E is compatible, $a \leftrightarrow a_1'$ and $a \leftrightarrow a_2'$. Since L is conditionally three compatible $a \leftrightarrow a_1' \vee a_2'$ and hence

$$a \leftrightarrow (a_1' \vee a_2') = a_1 \wedge a_2.$$

Proposition 5. *Let $E^{\vee} = \{x \in L\,;\, x = a \vee b$ and $a, b \in E\}$, $E^{\wedge} = \{x \in L\,;\, x = a \wedge b$ and $a, b \in E\}$ and $E' = \{x \in L\,;\, x' \in E\}$. Then E compatible infers $(E \cup E')^{\vee}$ and $(E \cup E')^{\wedge}$ compatible.*

Proof. The proof is a simple conclusion of Propositions 1, 3 and 4.

Proposition 6. *There exists a subalgebra $A \subset L$ such that $E \subset A$.*

Proof. Define $E_1 = (E \cup E' \cup \{o, i\})^{\vee}$ and for $n \geqq 2$ put $E_n = (E_{n-1} \cup E'_{n-1})^{\vee}$ for odd n and $E_n = (E_{n-1} \cup E'_{n-1})^{\wedge}$ for even n. Then $E_1 \subset E_2 \subset E_3 \ldots$. Denote $A = \cup_n E_n$. Clearly A is compatible and therefore A is a sublattice of L. By Proposition 2 the distributive laws hold in A. A is closed under the union and complementation, hence A is an algebra. Clearly $A \subset L$ and A is a subalgebra.

Corollary 7. *The algebra A constructed in the last proof is a subalgebra generated by E, i.e. $A = A(E)$.*

Proposition 8. *The algebra and the logic generated by E coincide, i.e. $A(E) = L(E)$.*

For the proof we need the following:

Lemma 2. *If $a, b \in L$ and $a \leftrightarrow b$, then $A(\{a, b\}) = L(\{a, b\})$.*

Proof. Let $a = a_1 + c$ and $b = b_1 + c$, where a_1, b_1 and c are mutually disjoint. Denote $d = a_1 + b_1 + c$. $A(\{a, b\})$ is a sublogic of L, hence $A(\{a, b\}) \supset L(\{a, b\})$. To show the opposite inclusion we prove first that $c \in L(\{a, b\})$. Suppose $c \notin L(\{a, b\})$, evidently $c \neq o$. Let $z \in L(\{a, b\})$ and $z \leqq a, b$. Then, by Proposition 1, $z < c$. Since $L(\{a, b\})$ is a sublogic of $A(\{a, b\})$ and $A(\{a, b\})$ is a Boolean algebra generated by the elements a, b, we obtain that $c = a \wedge b$ is an atom in $A(\{a, b\})$. ($A(\{a, b\})$ is a homomorphic image of the free Boolean algebra on two generators and $c \neq o$.) We know that $z < c$ and $z \in A(\{a, b\})$. Therefore $o = \inf\{a, b\}$ in $L(\{a, b\})$. Hence $o = a \wedge b = c$ by Lemma 1 because $L(\{a, b\})$ is a sublogic of L, a contradiction. Thus we have shown that $c \in L(\{a, b\})$. From this we obtain that $a_1 = a - c$ and $b_1 = b - c$ are in $L(\{a, b\})$, too. Since all elements of $A(\{a, b\})$ are unions of suitable elements from $\{a_1, b_1, c, d'\}$ (and these are pairwise orthogonal) we obtain that $L(\{a, b\}) \supset A(\{a, b\})$. Hence $L(\{a, b\}) = A(\{a, b\})$.

Proof of Proposition 8. It follows from Proposition 6 that $A(E)$ exists and $L(E) \subset A(E)$ is valid. Let $a, b \in L(E)$, then $a \leftrightarrow b$. By Lemma 2 we get $a \vee b \in A(\{a, b\}) = L(\{a, b\}) \subset L(E)$. Thus $L(E)$ is closed under the formation of unions and since $L(E)$ is closed under the complementation, we get that $L(E)$ is an algebra and $L(E) = A(E)$.

Proposition 9. *Let L be a σ-logic and let $E \subset L$, then there exists a sub-σ-logic $B \subset L$ such that $E \subset B$.*

Proof. Denote

$$\mathscr{A} = \{A\,;\ E \subset A \subset L \ \text{ and } \ A \ \text{ is a subalgebra of } \ L\}.$$

By Proposition 6, $\mathscr{A}$ is nonempty. Let $\{A_t\}_{t \in T}$ be a chain from $\mathscr{A}$, then clearly $\cup\{A_t\,;\, t \in T\}$ is from $\mathscr{A}$. By Zorn's Lemma there exists a maximal element B of $\mathscr{A}$. Let $x_1, x_2, \ldots$ be a sequence of mutually disjoint elements from B. L is a σ-logic, hence $x = \Sigma_i x_i$ is in L. Since $x_i \leftrightarrow a$ for any $a \in B$ and for $i = 1, 2, \ldots$, we have by Proposition 2 that $x = \Sigma_i x_i \leftrightarrow a$. Hence $B \cup \{x\}$ is a compatible subset of L. By Proposition 6 there exists a subalgebra $C \subset L$ such that $B \cup \{x\} \subset C$, clearly $E \subset C$, and so $B = C$ by the maximality of B. Since $B \cup \{x\} \subset C = B$, it follows that $x \in B$. We have proved that B is a σ-algebra. Obviously it is also a sub-σ-algebra.

Proposition 10. *If L is a σ-logic and $E \subset L$, then the σ-logic generated by E and the σ-algebra generated by E coincide, i.e. $L_\sigma(E) = A_\sigma(E)$.*

Proof. By Proposition 8 we have $L(E) = A(E)$, and so $M(L(E)) = M(A(E)) = A_\sigma(E)$, but $L_\sigma(E)$ is a monotone system so that $L_\sigma(E) \supset M(L(E)) = A_\sigma(E)$. Hence $L_\sigma(E) \supset A_\sigma(E)$. The opposite inclusion is trivial.

We may formulate the result obtained here as follows:

Theorem 1. *If a σ-logic L is conditionally three compatible and $E \subset L$ is compatible, then there exists a sub-σ-algebra B of L such that $E \subset B \subset L$. Moreover $L_\sigma(E) = A_\sigma(E)$.*

Theorem 2. *If L is a σ-logic $E \subset L$, then a necessary and sufficient condition for the existence of $A_\sigma(E)$ is the simultaneous fulfillement of the following two conditions:*

(i) *E is compatible,*

(ii) *There exists a sublogic L_1 of L which is conditionally three compatible such that $E \subset L_1 \subset L$.*

Proof. The necessity is trivial. The sufficiency follows by Theorem 1.

Theorem 3. *Let L be a logic or a σ-logic and let $E \subset L$. If $A_\sigma(E)$ exists, then $A_\sigma(E) = L_\sigma(E)$.*

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ПОДАЛГЕБРЫ И ПОДЛОГИКИ σ-ЛОГИКИ

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Резюме

Пусть L-логика, т.е. частично упорядоченное множество с наименьшим и наибольшим элементом и ортодополнением $a \to a'$.

Скажем, что элементы a, b логики L ортогональны, еслы $a \leqq b'$. Скажем, что элементы a, б логики L согласованы, если существуют попарно ортогональные элементы a_1, b_1, c, для которых

$$a = a_1 + c, \qquad b = b_1 + c.$$

Скажем, что логика L условно 3 – согласована, если для любых трех попарно согласованных элементов a, b, c из L существует $b \vee c$ и a согласованные с $b \vee c$.

В статье доказывается, кроме других следующая теорема:

Если σ-логика L условно 3 – согласованная и $E \subset L$ согласована – то существует под-σ-алгебра $B \subset L$ такая, что $E \subset B \subset L$, и кроме того σ-алгебра порожденная E совпадает с логикой порожденной E.