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THE SOLUTION OF THE TWO-POINT BOUNDARY VALUE PROBLEM FOR A NONLINEAR DIFFERENTIAL EQUATION OF THE THIRD ORDER

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The main aim of this papers is to prove the existence theorem for the problem (1), (2), The Green function for this problem, demanded in the proof of this theorem, has also been constructed.

The method of constructing this function is sufficiently described, for instance, in [3]. V. Šeda in [4] deals extensively with an application of Green's function in differential equations.

In many papers Green's functions for various boundary value problems were published, however, for the problem (1), (2) not yet.

Some estimates of this function have also been formed and existence theorems of the problem (1), (2) have been proved.

Take the nonlinear differential equation

$$x'''(t)+F(t, x(t), x'(t), x''(t))+x(t)=e(t) \tag{1}$$

and the boundary conditions

$$\begin{aligned} x(0)-x(2\pi/\sqrt{3})&=0\\ x'(0)-x'(2\pi/\sqrt{3})&=0\\ x''(0)-x''(2\pi/\sqrt{3})&=0. \end{aligned} \tag{2}$$

In what follows we shall assume the functions $F(t, x, y, z)$ and $e(t)$ to be continuous in their domains.

Let us consider the linearized problem

$$x'''(t)+x(t)=0 \tag{1'}$$

with boundary conditions (2).

By solving the characteristic equation

$$r^3+1=0$$

we get the general solution in the form of

$$x(t)=c_1\,e^{-t}+c_2\,e^{t/2}\cos(\sqrt{3}/2)\,t+c_3\,e^{t/2}\sin(\sqrt{3}/2)\,t. \tag{3}$$

Lemma 1. *The problem* (1′), (2) *has only a trivial solution.*
Proof. Let us introduce the notation

$$e^{(-2\pi/\sqrt{3})}=\alpha$$

$$e^{\pi/\sqrt{3}}=\beta.$$

After putting $x'(t)$ and $x''(t)$ into (2) we get the linear homogeneous system for the coefficients c_1, c_2, c_3. The determinant of this system equals $(\sqrt{3}/4)(1+\beta)^2(1+\alpha)$ and therefore differs from zero. Consequently the system (1′), (2) has only trivial solution.

Lemma 1 ensures the existence of the Green function $G(t, s)$ which enables the solution of the problem (1), (2) in the form

$$x(t)=\int_0^{2\pi/\sqrt{3}} G(t, s)[e(s)-F(s, x(s), x'(s), x''(s))]\,ds \tag{4}$$

The Green function can be found in the form

$$G(t, s)=\begin{cases} c_1\,e^{-t}+c_2\,e^{t/2}\cos(\sqrt{3}/2)\,t+c_3\,e^{t/2}\sin(\sqrt{3}/2)\,t \\ \quad\text{for}\quad 0\leqslant t<s<2\pi/\sqrt{3} \\ c_4\,e^{-t}+c_5\,e^{t/2}\cos(\sqrt{3}/2)\,t+c_6\,e^{t/2}\sin(\sqrt{3}/2)\,t \\ \quad\text{for}\quad 0<s<t\leqslant 2\pi/\sqrt{3} \end{cases} \tag{5}$$

where the coefficients $c_1, \ldots, c_6$ are uniquely determined under the following conditions

$$\begin{aligned} &G(0, s)-G(2\pi/\sqrt{3}, s)=0 \\ &G_t(0, s)-G_t(2\pi/\sqrt{3}, s)=0 \\ &G_{tt}(0, s)-G_{tt}(2\pi/\sqrt{3}, s)=0 \end{aligned} \tag{6a}$$

$$\begin{aligned} &\lim_{t\to s-} G(t, s)-\lim_{t\to s+} G(t, s)=0 \\ &\lim_{t\to s-} G_t(t, s)-\lim_{t\to s+} G_t(t, s)=0 \end{aligned} \tag{6b}$$

$$\lim_{t\to s-} G_{tt}(t, s) - \lim_{t\to s+} G_{tt}(t, s) = -1$$

where $G_t(t, s)$ denotes the partial derivative of $G(t, s)$ with respect to t.

Lemma 2. *The coefficients of Green's function of the problem* (1′), (2) *are*

$$c_1 = \frac{\alpha}{3(1-\alpha)} e^s$$

$$c_2 = \frac{\beta}{3(1+\beta)} e^{-s/2} (\cos(\sqrt{3}/2)s + \sqrt{3}\sin(\sqrt{3}/2)s)$$

$$c_3 = \frac{\beta}{3(1+\beta)} e^{-s/2} (\sin(\sqrt{3}/2)s - \sqrt{3}\cos(\sqrt{3}/2)s)$$

$$c_4 = \frac{1}{3(1-\alpha)} e^s$$

$$c_5 = \frac{-1}{3(1+\beta)} e^{-s/2} (\cos(\sqrt{3}/2)s + \sqrt{3}\sin(\sqrt{3}/2)s)$$

$$c_6 = \frac{-1}{3(1+\beta)} e^{-s/2} (\sin(\sqrt{3}/2)s - \sqrt{3}\cos(\sqrt{3}/2)s).$$

Proof. By direct substitution into (5) we get

$$\begin{aligned}
G(0, s) &= c_1 + c_2 \\
G(2\pi/\sqrt{3}, s) &= \alpha c_4 - \beta c_5 \\
G_t(0, s) &= -c_1 + (1/2)c_2 + (\sqrt{3}/2)c_3 \\
G_t(2\pi/\sqrt{3}, s) &= -\alpha c_4 - (1/2)\beta c_5 - (\sqrt{3}/2)\beta c_6 \\
G_{tt}(0, s) &= c_1 - (1/2)c_2 + (\sqrt{3}/2)c_3 \\
G_{tt}(2\pi/\sqrt{3}, s) &= \alpha c_4 + (1/2)\beta c_5 - (\sqrt{3}/2)\beta c_6.
\end{aligned}$$

The boundary conditions (6a) for Green's function attain the form

$$\begin{aligned}
c_1 + c_2 - \alpha c_4 + \beta c_5 &= 0 \\
-c_1 + (1/2)c_2 + (\sqrt{3}/2)c_3 + \alpha c_4 + (1/2)\beta c_5 + (\sqrt{3}/2)\beta c_6 &= 0 \\
c_1 - (1/2)c_2 + (\sqrt{3}/2)c_3 - \alpha c_4 - (1/2)\beta c_5 + (\sqrt{3}/2)\beta c_6 &= 0
\end{aligned}$$

By proper linear transformations we get

$$\begin{aligned}
c_1 \qquad\qquad -\alpha c_4 \qquad\qquad\qquad &= 0 \\
c_2 \qquad\qquad\qquad +\beta c_5 \qquad &= 0 \qquad\qquad (7a) \\
c_3 \qquad\qquad\qquad\qquad +\beta c_6 &= 0.
\end{aligned}$$

Introducing the notations

$$\begin{aligned}
\sin(\sqrt{3}/2)s &= \gamma \\
\cos(\sqrt{3}/2)s &= \delta \qquad\qquad (8) \\
e^{s/2} &= \tau
\end{aligned}$$

conditions (6b) for Green's function can be adjusted as follows

$$\begin{aligned}
&c_1 e^{-s} + c_2\tau\delta + c_3\tau\gamma - c_4 e^{-s} - c_5\tau\delta - c_6\tau\delta = 0 \\
&-c_1 e^{-s} + c_2(\tau/2)(\delta - \sqrt{3}\gamma) + c_3(\tau/2)(\gamma + \sqrt{3}\delta) + c_4 e^{-s} - c_5(\tau/2)(\delta - \sqrt{3}\gamma) - c_6(\tau/2)(\gamma + \sqrt{3}\delta) = 0 \\
&c_1 e^{-s} - c_2(\tau/2)(\delta + \sqrt{3}\gamma) - c_3(\tau/2)(\gamma - \sqrt{3}\delta) - c_4 e^{-s} + c_5(\tau/2)(\delta + \sqrt{3}\gamma) + c_6(\tau/2)(\gamma - \sqrt{3}\delta) = -1.
\end{aligned}$$

This can be rewritten in the form

$$\begin{aligned}
c_1 \qquad - c_4 \qquad\qquad &= -e^{s}/3 \\
c_2 \qquad\qquad - c_5 \qquad &= (1/3\tau)(\delta + \sqrt{3}\gamma) \qquad\qquad (7b) \\
c_3 \qquad\qquad\qquad - c_6 &= (1/3\tau)(\gamma - \sqrt{3}\delta).
\end{aligned}$$

Comparing (7a) with (7b) and by using (8) again the proof of the Lemma is completed.

Lemma 3. *Green's function for the problem* (1), (2) *is*

$$G(t, s) = \frac{\alpha}{3(1-\alpha)} e^{s-t} + \frac{\beta}{3(1+\beta)} e^{(t-s)/2} [\cos(\sqrt{3}/2)(s-t) + \sqrt{3}\sin(\sqrt{3}/2)(s-t)]$$

$$\text{for} \quad 0 \leqslant t < s < 2\pi/\sqrt{3}$$

and

$$G(t, s) = \frac{1}{3(1-\alpha)} e^{s-t} - \frac{1}{3(1+\beta)} e^{(t-s)/2} [\cos(\sqrt{3}/2)(s-t) + \sqrt{3}\sin(\sqrt{3}/2)(s-t)]$$

$$\text{for} \quad 0 < s < t \leqslant 2\pi/\sqrt{3}.$$

Proof. The proof follows from setting the coefficients of lemma 2 into (5) and by using proper trigonometric formulas.

Lemma 4. *The Green function* $G(t, s)$ *of the problem* (1), (2) *is positive on the interval* $[0,2\pi/\sqrt{3}]\times[0,2\pi/\sqrt{3}]$.

Proof. Putting $u=s-t$ into the function $G(t, s)$ for $0\leqslant t<s<2\pi/\sqrt{3}$ we get

$$G(t, s)=\frac{\alpha}{3(1-\alpha)}\,e^{u}+\frac{\beta}{3(1+\beta)}\,e^{-u/2}[\cos(\sqrt{3}/2)\,u+\sqrt{3}\sin(\sqrt{3}/2)\,u]$$

$$\text{for}\quad 0<u<2\pi/\sqrt{3}.$$

The function $f(u)=\cos(\sqrt{3}/2)\,u+\sqrt{3}\sin(\sqrt{3}/2)\,u$ has such a course that there exists the argument u_0 with

$$f(u)>0\quad\text{for}\quad 0\leqslant u<u_0$$

$$f(u_0)=0$$

$$f(u)<0\quad\text{for}\quad u_0<u\leqslant 2\pi/\sqrt{3}.$$

Here $u_0=(2/\sqrt{3})(5\pi/6)$.
The coefficients α, β are real numbers, for which the following estimations hold good

$$4.52<\beta<4.58,\quad 0.04<\alpha<0.05.$$

This implies that

$$0.01<\frac{\alpha}{3(1-\alpha)}<0.02,\quad 0.22<\frac{\beta}{3(1+\beta)}<0.34.$$

The inequalities
$e^{u}>0$, $\dfrac{\alpha}{3(1-\alpha)}>0$ hold good for the case $0<u\leqslant u_0$
from which it follows that $G(t, s)>0$ for $0<u=s-t\leqslant u_0$.
For $u>3$ we have $\dfrac{\alpha}{3(1-\alpha)}\,e^{u}>0.2$.
The following estimations are true as well

$$e^{-u/2}<0.224$$

$$f(u)>-1$$

$$\frac{\beta}{3(1+\beta)}\,e^{-u/2}f(u)>-0.077\quad\text{for}\quad u_0<u<2\pi/\sqrt{3}.$$

Hence $G(t, s)>0.2+(-0.077)>0$ for $u_0<u=s-t<2\pi/\sqrt{3}$.

Again by substituting $u=t-s$ for $0<s<t\leq 2\pi/\sqrt{3}$ we can write the Green function in the form

$$G(t, s)=\frac{1}{3(1-\alpha)}\,e^{-u}+\frac{1}{3(1+\beta)}\,e^{u/2}[\sqrt{3}\sin(\sqrt{3}/2)\,u-\cos(\sqrt{3}/2)\,u]$$

$$\text{for}\quad 0<u<2\pi/\sqrt{3}.$$

As to the function

$$f(u)=\sqrt{3}\sin(\sqrt{3}/2)\,u-\cos(\sqrt{3}/2)\,u,$$

it has such a course that there exists the argument u_1 for which it holds good that

$$f(u)<0 \quad \text{for} \quad 0<u<u_1$$

$$f(u_1)=0$$

$$f(u)>0 \quad \text{for} \quad u_1<u<2\pi/\sqrt{3}$$

where $u_1=(2/\sqrt{3})(\pi/6)$.
With a similar consideration as above we can show that $G(t, s)>0$ for $u_1<u=t-s<2\pi/\sqrt{3}$.
Now let us consider the interval $(0, u_1)$.
As $f(u)>-1$

$$G(t, s)>\frac{1}{3(1-\alpha)}\,e^{-u}-\frac{1}{3(1+\beta)}\,e^{u/2}=L.$$

When we multiply the inequality

$$L>0$$

by the positive number $e^u3(1+\beta)$ we get the equivalent relation

$$\frac{1+\beta}{1-\alpha}>e^{3u/2}.$$

If $u<u_1$, then $3u/2<1$. By using $1+\beta>5$ and $1-\alpha<0.96$ we get the truth inequality

$$e^{3u/2}<e^1<5/0.96<(1+\beta)/(1-\alpha).$$

Thus the inequality $L>0$ holds good and it implies that

$$G(t, s)>0 \quad \text{for} \quad 0<u=t-s<u_1.$$

Thus the Lemma is completely proved.

Lemma 5.

$$\int_0^{2\pi/\sqrt{3}} |G(t, s)|\, ds = 1 \quad \textit{for} \quad 0 \leqslant t \leqslant 2\pi/\sqrt{3}.$$

Proof. According to Lemma 4 $|G(t, s)| = G(t, s)$. Let us consider the equation

$$x''' + x = 1 \tag{9}$$

with boundary conditions (2).
The solution x of this problem satisfies the relation

$$x(t) = \int_0^{2\pi/\sqrt{3}} G(t, s) \cdot 1\, ds.$$

On the other hand it is easy to verify that the function $x(t) = 1$ is the solution of the problem (9), (2).

Lemma 6. *The estimations*

$$|G_t(t, s)| \leqslant 1.04 \qquad |G_{tt}(t, s)| \leqslant 1.04$$

are valid.

Proof. By differentiating we get

$$G_t(t, s) = \frac{-\alpha}{3(1-\alpha)} e^{s-t} + \frac{\beta}{3(1+\beta)} e^{(t-s)/2}[\sqrt{3} \sin (\sqrt{3}/2)(s-t) - \cos (\sqrt{3}/2)(s-t)]$$

$$G_{tt}(t, s) = \frac{\alpha}{3(1-\alpha)} e^{s-t} + \frac{-\beta}{3(1+\beta)} e^{(t-s)/2} \cdot 2 \cdot \cos (\sqrt{3}/2)(s-t)$$

$$\text{for} \quad 0 \leqslant t < s < 2\pi/\sqrt{3}.$$

And

$$G_t(t, s) = \frac{-1}{3(1-\alpha)} e^{s-t} + \frac{-1}{3(1+\beta)} e^{(t-s)/2}[\sqrt{3} \sin (\sqrt{3}/2)(s-t) - \cos (\sqrt{3}/2)(s-t)]$$

$$G_{tt}(t, s) = \frac{1}{3(1-\alpha)} e^{s-t} + \frac{1}{3(1+\beta)} e^{(t-s)/2} \cdot 2 \cdot \cos (\sqrt{3}/2)(s-t)$$

$$\text{for} \quad 0 < s < t \leqslant 2\pi/\sqrt{3}.$$

In the case $0 \leqslant t < s < 2\pi/\sqrt{3}$ we introduce the substitution of $u = s - t$. Then

$$G_t(t, s) = \frac{-\alpha}{3(1-\alpha)} e^{u} + \frac{\beta}{3(1+\beta)} e^{-u/2}[\sqrt{3} \sin (\sqrt{3}/2)\, u - \cos (\sqrt{3}/2)\, u]$$

$$G_{tt}(t, s) = \frac{\alpha}{3(1-\alpha)} e^{u} + \frac{-\beta}{3(1+\beta)} e^{-u/2} \cdot 2 \cdot \cos(\sqrt{3}/2)\, u$$

$$\text{for} \quad 0 < u < 2\pi/\sqrt{3}.$$

Similarly in the case $0 < s < t \leqslant 2\pi/\sqrt{3}$ we introduce the substitution of $u = t - s$. Then

$$G_t(t, s) = \frac{-1}{3(1-\alpha)} e^{-u} + \frac{-1}{3(1+\beta)} e^{u/2} [\sqrt{3} \sin(\sqrt{3}/2)\, u + \cos(\sqrt{3}/2)\, u]$$

$$G_{tt}(t, s) = \frac{1}{3(1-\alpha)} e^{-u} + \frac{1}{3(1+\beta)} e^{u/2} \cdot 2 \cdot \cos(\sqrt{3}/2)\, u$$

$$\text{for} \quad 0 < u < 2\pi/\sqrt{3}.$$

The function $f(u) = |\sqrt{3} \sin(\sqrt{3}/2)\, u - \cos(\sqrt{3}/2)\, u|$ attains its maximum at the point $u_0 = (2/\sqrt{3})(2\pi/3)$ and $f(u_0) = 2$. Then for the case $0 \leqslant t < s < 2\pi/\sqrt{3}$ we have

$$G_t(t, s) \leqslant \left| \frac{-\alpha}{3(1-\alpha)} \right| e^{2\pi/\sqrt{3}} + \left| \frac{\beta}{3(1+\beta)} \right| \cdot 2 = \frac{\alpha}{3(1-\alpha)} \frac{1}{\alpha} + \frac{\beta}{3(1+\beta)} \cdot 2 \leqslant 1.04.$$

After similar considerations the proof is completed. In further considerations we shall need the estimations given in the next Lemma.

Lemma 7.

$$\max_{0 \leqslant t \leqslant 2\pi/\sqrt{3}} \int_0^{2\pi/\sqrt{3}} |G(t, s)|\, ds = K_0 = 1$$

$$\max_{0 \leqslant t \leqslant 2\pi/\sqrt{3}} \int_0^{2\pi/\sqrt{3}} |G_t(t, s)|\, ds = K_1 \leqslant 1.04(2\pi/\sqrt{3})$$

$$\max_{0 \leqslant t \leqslant 2\pi/\sqrt{3}} \int_0^{2\pi/\sqrt{3}} |G_{tt}(t, s)|\, ds = K_2 \leqslant 1.04(2\pi/\sqrt{3})$$

Proof. The first part of this assertion results directly from Lemma 5.

The integral $\int_0^{2\pi/\sqrt{3}} |G_t(t, s)|\, ds$ is a continuous function of the variable t in the compact interval $[0, 2\pi/\sqrt{3}]$ and in this interval it attains its maximum K_1. It holds good according to Lemma 6 that $K_1 \leqslant 1.04(2\pi/\sqrt{3})$.

The integral $\int_0^{2\pi/\sqrt{3}} |G_{tt}(t, s)|\, ds$ is a piecewise continuous function of t. Hence its maximum K_2 exists in the interval $[0, 2\pi/\sqrt{3}]$ and the estimation $K_2 \leqslant 1.04(2\pi/\sqrt{3})$ is true.

Into the linear space $C^2[0, 2\pi/\sqrt{3}]$ let us introduce the norm

$$\|x\|_2 = \max(\|x\|, \|x'\|, \|x''\|)$$

where $\|x^{(k)}\| = \max\limits_{0 \leq t \leq 2\pi/\sqrt{3}} |x^{(k)}(t)|$. Here $k = 0, 1, 2$ denotes derivation.

Let us consider the integral equation

$$x(t) = \int_0^{2\pi/\sqrt{3}} G(t, s)[e(s) - F(s, x(s), x'(s), x''(s))]\, ds. \tag{4}$$

Its solution belongs to $C^2[0, 2\pi/\sqrt{3}]$.
The function $x(t)$ is a solution of the equation (4) if and only if it is the solution of the boundary value problem (1), (2).

Let us consider the operator T on the set $C^2[0, 2\pi/\sqrt{3}]$ which has the form

$$(Tx)(t) = \int_0^{2\pi/\sqrt{3}} G(t, s)[e(s) - F(s, x(s), x'(s), x''(s))]\, ds. \tag{11}$$

Then the solution of the problem (1), (2) means the solution of the operator equation

$$x = Tx.$$

Let us introduce the notation

$$E(t) = \int_0^{2\pi/\sqrt{3}} G(t, s)\, e(s)\, ds.$$

On the basis of the Schauder fix-point theorem we can now formulate the existence theorem for the problem (1), (2).

Theorem 1. *Let $F = F(t, x, y, z)$ be a continuous function on the set*

$$B = \{(t, x, y, z) \colon\ 0 \leq t \leq 2\pi/\sqrt{3},\ |x| \leq R_0,\ |y| \leq R_1,\ |z| \leq R_2\}.$$

Let $|F(t, x, y, z)| \leq q$ on B. Let the following conditions hold

$$\begin{aligned} \|E\| + qK_0 &\leq R_0 \\ \|E'\| + qK_1 &\leq R_1 \\ \|E''\| + qK_2 &\leq R_2. \end{aligned} \tag{12}$$

Then the boundary value problem (1), (2) has at least one solution $x(t)$ for which it holds that

$$|x(t)| \leq R_0, \quad |x'(t)| \leq R_1, \quad \|x''(t)\| \leq R_2 \tag{12a}$$

for all $t \in [0, 2\pi/\sqrt{3}]$.

Proof. The set

$$M=\{x\in C^2[0, 2\pi/\sqrt{3}]:\ \|x\|\leqslant R_0,\quad \|x'\|\leqslant R_1,\quad \|x''\|\leqslant R_2\}$$

is a closed, convex subset of the space $C^2[0, 2\pi/\sqrt{3}]$. The T mapping of the set M into $C^2[0, 2\pi/\sqrt{3}]$ defined by (4) maps the M set on the basis of (12) into itself. The function $F(t, x, y, z)$ is uniformly continuous in the compact B set. Thus to all $\varepsilon_1>0$ there exists a $\delta>0$ that for all $x_1, x_2\in M$, $\|x_1-x_2\|_2<\delta$ there holds

$$|F(t, x_1(t), x_1'(t), x_1''(t))-F(t, x_2(t), x_2'(t), x_2''(t))|<\varepsilon_1.$$

For $k=0, 1, 2$ (k denoting derivatives) the following inequalities hold good:

$$\|(Tx_1)^{(k)}-(Tx_2)^{(k)}\|\leqslant K_k\varepsilon_1.$$

This implies that

$$\|Tx_1-Tx_2\|_2\leqslant\max(K_0\varepsilon_1, K_1\varepsilon_1, K_2\varepsilon_1)=\varepsilon_1\max_{k=0,1,2}K_k=\varepsilon.$$

Thus, to an $\varepsilon>0$ there exists a $\delta>0$ so that for $x_1, x_2\in M$, $\|x_1-x_2\|_2<\delta$ there holds $\|Tx_1-Tx_2\|_2<\varepsilon$.
This means that the operator T: $M\rightarrow T(M)\subset M$ is continuous. The functions of the $T(M)$ set, their first and second derivatives are uniformly bounded: $|y(t)|\leqslant R_0$, $|y'(t)|\leqslant R_1$, $|y''(t)|\leqslant R_2$ for $t\in[0, 2\pi/\sqrt{3}]$.
According to the mean value theorem the uniform boundedness of the first derivatives $y'(t)$ ensures equicontinuity of the functions $y(t)$.
Uniform boundedness of the second derivatives $y''(t)$ ensures equicontinuity of the first derivatives $y'(t)$. For the functions $y=T(x)$ from the set $T(M)$ the equation

$$y'''(t)+y(t)=e(t)-F(t, x(t), x'(t), x''(t)) \tag{13}$$

holds good.

The function $e(t)$ is continuous on the compact interval $[0, 2\pi/\sqrt{3}]$, hence there exists such a $C>0$ that $|e(t)|\leqslant C$ on this interval. From (13) there follows a uniform boundedness of the functions $y'''(t)$

$$|y'''(t)|\leqslant R_0+C+q.$$

This implies the equicontinuity of the functions $y''(t)$. According to the Ascoli lemma the $T(M)$ set is conditionally compact. Thus the conditions of the Schauder fix point theorem guaranteeing the existence of a fix point of the T mapping in the M set are fulfilled.
This fix point is a solution of the problem (1), (2) and has the properties of (12a).

Definition. *We say that the function $F(t, x, y, z)$ satisfies Lipschitz's condition with the constants L_0, L_1, L_2 if*

$$|F(t, x_1, y_1, z_1)-F(t, x_2, y_2, z_2)| \leqslant L_0|x_1-x_2|+L_1|y_1-y_2|+L_2|z_1-z_2|$$

is valid.

In the space $C^2[0, 2\pi/\sqrt{3}]$ we introduce the metric

$$\varrho(x_1, x_2)=L_0\|x_1-x_2\|+L_1\|x_1'-x_2'\|+L_2\|x_1''-x_2''\| \tag{14}$$

where $\|x_1^{(k)}-x_2^{(k)}\|=\max\limits_{0\leqslant t\leqslant 2\pi/\sqrt{3}}|x_1^{(k)}-x_2^{(k)}|, \quad k=0, 1, 2.$

Definition. *The T operator is contractive if the constant $K<1$ exists in such a way that the relation*

$$\varrho(Tx_1, Tx_2) \leqslant K\varrho(x_1, x_2)$$

holds

Lemma 8. *If the function $F(t, x, y, z)$ satisfies the Lipschitz condition with the constants L_0, L_1, L_2 and*

$$L_0K_0+L_1K_1+L_2K_2<1$$

holds, the T operator is contractive.

Proof. Let us denote

$$L_0K_0+L_1K_1+L_2K_2=K.$$

Then for $i=1, 2$, $k=0, 1, 2$ the T operator maps the functions $x_i \in C^2[0, 2\pi/\sqrt{3}]$ into the set $C^2[0, 2\pi/\sqrt{3}]$ in such a way that

$$(Tx_i)^{(k)}(t)=\int_0^{2\pi/\sqrt{3}}\frac{\partial^k G(t, s)}{\partial t^k}[e(s)-F(s, x_i(s), x_i'(s), x_i''(s))]\,ds=$$

$$=E^{(k)}(t)-\int_0^{2\pi/\sqrt{3}}\frac{\partial^k G(t, s)}{\partial t^k}F(s, x_i(s), x_i'(s), x_i''(s))\,ds.$$

Then

$$\|(Tx_1)^{(k)}-(Tx_2)^{(k)}\|=$$

$$=\max_{0\leqslant t\leqslant 2\pi/\sqrt{3}}\left|\int_0^{2\pi/\sqrt{3}}\frac{\partial^k G(t, s)}{\partial t^k}[F(s, x_1(s), x_1'(s), x_1''(s))-\right.$$

$$\left.-F(s, x_2(s), x_2'(s), x_2''(s))]\,ds\right| \leqslant$$

$$\leqslant(L_0\|x_1-x_2\|+L_1\|x_1'-x_2'\|+L_2\|x_1''-x_2''\|)\max_{0\leqslant t\leqslant 2\pi/\sqrt{3}}\int_0^{2\pi/\sqrt{3}}\left|\frac{\partial^k G(t, s)}{\partial t^k}\right|ds=$$

$$=K_k(L_0\|x_1-x_2\|+L_1\|x_1'-x_2'\|+L_2\|x_1''-x_2''\|).$$

If we multiply the above inequality

for $k=0$ by L_0,
for $k=1$ by L_1,
for $k=2$ by L_2

and add all these three inequalities we get

$$L_0\|Tx_1-Tx_2\|+L_1\|(Tx_1)'-(Tx_2)'\|+L_2\|(Tx_1)''-(Tx_2)''\|\leqslant$$
$$\leqslant(L_0K_0+L_1K_1+L_2K_2)(L_0\|x_1-x_2\|+L_1\|x_1'-x_2'\|+L_2\|x_1''-x_2''\|).$$

This, however, means contractivity of the T mapping in the metric (14).

Theorem 2. *Let the conditions of Theorem 1 be fulfilled and moreover the function $F(t, x, y, z)$ fulfills the Lipschitz condition*

$$|F(t, x_1, y_1, z_1)-F(t, x_2, y_2, z_2)|\leqslant L_0|x_1-x_2|+L_1|y_1-y_2|+L_2|z_1-z_2|$$

where the Lipschitz constants satisfy

$$L_0K_0+L_1K_1+L_2K_2<1.$$

Then the boundary value problem (1), (2) *has just one solution $x(t)$. Moreover the estimation*

$$\varrho(x, x_n)\leqslant\frac{K^n}{1-K}\varrho(x_0, x_1)$$

is valid, where the functions x_n are members of the sequence of successive approximations

$$x_n=T(x_{n-1})$$

with x_0 chosen arbitrarily in the set M.

Proof. The M set is closed and according to (12) $T(M)\subset M$ holds good. So the conditions of the Banach fix point theorem are fulfilled.

Theorem 3. *If the function $F(t, x, y, z)$ is continuous on the set $B_1=\{(t, x, y, z): -\infty<t<\infty, |x|\leqslant R_0, |y|\leqslant R_1, |z|\leqslant R_2\}$ with R_0, R_1, $R_2>0$ and the function $e(t)$ is continuous on R, both functions $F(t, x, y, z)$ and $e(t)$ are periodic in the variable t with the period $2\pi/\sqrt{3}$ and $|F(t, x, y, z)|\leqslant q$ on B_1. Further let the conditions* (12) *be satisfied. Then there exists a periodic solution $x_1(t)$ of the differential equation* (1) *with the same period.*

Proof. Since all assumptions of Theorem 1 are satisfied, by this theorem there exists a solution $x(t)$ of the problem (1), (2). Let us denote the periodic extension in $(-\infty, \infty)$ of $x(t)$ by $x_1(t)$. The conditions (2) imply that $x_1(t)\in C^2(-\infty, \infty)$. Let us consider now an arbitrary $t \in (-\infty, \infty)$. There exists a unique integer k that $t \in (k \cdot 2\pi/\sqrt{3}, (k+1) \cdot 2\pi/\sqrt{3}]$ and

$$x_1'''(t) = x'''(t - k \cdot 2\pi/\sqrt{3}) =$$

$$= e(t - k \cdot 2\pi/\sqrt{3}) - x(t - k \cdot 2\pi/\sqrt{3}) - F(t - k \cdot 2\pi/\sqrt{3},$$

$$x(t - k \cdot 2\pi/\sqrt{3}), x'(t - k \cdot 2\pi/\sqrt{3}), x''(t - k \cdot 2\pi/\sqrt{3})) =$$

$$= e(t) - x_1(t) - F(t, x_1(t), x_1'(t), x_1''(t)).$$

Thus the periodic extension $x_1(t)$ of the function $x(t)$ satisfies the equation (1) and the proof of the Theorem is completed

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РЕШЕНИЕ КРАЕВОЙ ЗАДАЧИ ДЛЯ НЕЛИНЕЙНОГО УРАВНЕНИЯ ТРЕТЬЕГО ПОРЯДКА

Ľudmila Zajacová

Резюме

В работе найдена функция Грина для проблемы (1), (2) и рассматриваются некоторые свойства этой функции. Доказана тоже теорема существования и единственности для проблемы (1), (2).