Svatoslav Staněk

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Mathematica Slovaca, Vol. 27 (1977), No. 4, 423--430

Persistent URL: http://dml.cz/dmlcz/131953

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ON AN ALGEBRAIC STRUCTURE OF A GROUP OF PHASES

SVATOSLAV STANĚK

New subgroups of the group of the phases of all differential equations (q): $y''=q(t)y$ oscillatory on $\mathbf{R}=(-\infty, \infty)$ are found and investigated.

1. Throughout this paper the differential equation:

$$\text{(q)} \qquad y''=q(t)y,\ q\in C^0_{\mathbf{R}},$$

will be inderstood to be oscillatory on $\mathbf{R}$. Hence any nontrivial solution of this equation has infinitely many zeros on the right and on the left of each point t_0, $t_0\in\mathbf{R}$.

Following O. Borůvka [1] we introduce the basic definitions and properties. We say that the function α is a first phase of (q) if it is continuous on $\mathbf{R}$ and there exist independent solutions u, v of (q) such that for all t, $t\in\mathbf{R}$ where $v(t)\neq 0$: $\operatorname{tg}\alpha(t)=\dfrac{u(t)}{v(t)}$. Any first phase α of (q) has three properties as follows:

$$\begin{aligned}
&1)\ \alpha\in C^3_{\mathbf{R}};\\
&2)\ \alpha'(t)\neq 0 \quad \text{for} \quad t\in\mathbf{R};\\
&3)\ \lim_{t\to\nu\cdot\infty}\alpha(t)=(\nu\cdot\operatorname{sgn}\alpha')\cdot\infty \quad (\nu=\pm 1).
\end{aligned} \tag{1}$$

On the other hand, every function α with all the foregoing three properties is the first phase of (q) — oscillatory on $\mathbf{R}$ — where $q(t):=-\{\alpha, t\}-\alpha'^2(t)$, $t\in\mathbf{R}$; $\{\alpha, t\}=\dfrac{\alpha'''(t)}{2\alpha'(t)}-\dfrac{3}{4}\left(\dfrac{\alpha''(t)}{\alpha'(t)}\right)^2$.

If we define an algebraic operation of composition of functions on the set of all functions α possessing the properties stated in (1), then the set is a group called the group of phases which we denote by $\mathfrak{G}$. The elements of $\mathfrak{G}$ are exactly all the first phases of all equations (q) that are oscillatory on $\mathbf{R}$.

Let $t\in\mathbf{R}$ and y be a nontrivial solution of (q) vanishing at t: $y(t)=0$. Let us denote by $\varphi(t)$ the first zero point of y lying to the right of t. By assumption the equation (q) is oscillatory on $\mathbf{R}$ and thus the function φ is defined on $\mathbf{R}$. The function φ is called the basic central dispersion (hereafter more briefly dispersion) of (q).

If α, φ are a first phase and the dispersion of (q), respectively, then there applies the Abelian relation: $\alpha\circ\varphi(t)=\alpha(t)+\pi\cdot\operatorname{sgn}\alpha'$, $t\in\mathbf{R}$.

In the group $\mathfrak{G}$ there exist two important subgroups, namely, the fundamental subgroup $\mathfrak{E}$ and the elementary subgroup $\mathfrak{H}$. The elements of the fundamental subgroup $\mathfrak{E}$ are exactly all the first phases of the equation $y''=-y$. It is well know that any element in the decomposition $\mathfrak{G}/_r\mathfrak{E}$ of the group $\mathfrak{G}$ into the right classes generated by the fundamental subgroup $\mathfrak{E}$ contains exactly all the first phases of one and only one equation (q). The subgroup $\mathfrak{H}$ is formed by the elementary phases, that is, by those phases α for which $\alpha(t+\pi)=\alpha(t)+\pi\cdot\operatorname{sgn}\alpha'$. It follows from [1], page 147, that any element of the decomposition $\mathfrak{G}/_r\mathfrak{H}$ contains exactly all the first phases of those equations (q) having the same dispersion. There holds the inclusion $\mathfrak{G}\supset\mathfrak{H}\supset\mathfrak{E}$ between $\mathfrak{G}$, $\mathfrak{H}$ and $\mathfrak{E}$.

A further subgroup of $\mathfrak{G}$ is the subgroup $\mathfrak{G}_\alpha$ conjugate to $\mathfrak{E}$ with respect to the element α, $\alpha\in\mathrm{G}$: $\mathrm{G}_\alpha=\alpha^{-1}\mathfrak{E}\alpha$. Thereby if α is the first phase of (q), the elements $\mathfrak{G}_\alpha$ are exactly all the solutions on $\mathbf{R}$ of

$$\text{(qq)}\qquad -\{X,t\}+X'^2(t)\cdot q\circ X(t)=q(t).$$

In what follows we present some further subgroups of $\mathfrak{G}$ in particular such whose elements belong to "the union of some elements of the decomposition $\mathfrak{G}/_r\mathfrak{E}$". In other words there are involved subgroups of $\mathfrak{G}$ having the following property: If α is an element of such a subgroup and α is the first phase of (q), then all the first phases of (q) belong to this subgroup. Such subgroups must always contain the fundamental subgroup $\mathfrak{E}$.

Before proceeding with the main part of this paper we will explain the use of our notation. f^{-1} denotes the inverse to f (if there exists one). For a positive integer n the n-th iteration of the function f, $\underbrace{f\circ f\circ\ldots\circ f}_{n}$, will be marked by $f^{[n]}$. For a negativ integer n, $f^{[n]}$ means $\underbrace{f^{-1}\circ f^{-1}\circ\ldots\circ f^{-1}}_{-n}$. This notation will not be used only in case of dispersions, where the established notation φ_n is used in place of $\varphi^{[n]}$. $\mathbf{Z}$ and $\mathbf{N}$ denote the sets of all non-zero integers and of positive integers, respectively.

2. We shall be concerned with the subsets $\mathfrak{S}$, $\mathfrak{S}_r$ of $\mathfrak{G}$, where $r\in\mathbf{R}$, $r>0$: $\alpha\in\mathfrak{S}$ iff there exist $m\in\mathbf{N}$, $n\in\mathbf{N}$ such that $\alpha(t+m\pi)=\alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$ and $\alpha\in\mathfrak{S}_r$ iff there exist $m\in\mathbf{N}$, $n\in\mathbf{N}$ such that $\alpha(t+mr)=\alpha(t)+nr\cdot\operatorname{sgn}\alpha'$ $(t\in\mathbf{R})$.

Lemma 1. *$\mathfrak{S}$ is a subgroup of the group $\mathfrak{G}$, $\mathfrak{H}\subset\mathfrak{S}\subset\mathfrak{G}$, and $\mathfrak{E}\alpha\subset\mathfrak{S}$ for every $\alpha\in\mathfrak{S}$.*

Proof: Let $\alpha\in\mathfrak{S}$, $\alpha_1\in\mathfrak{S}$. Then $\alpha(t+m\pi)=\alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$, $\alpha_1(t+m_1\pi)=$

$= \alpha_1(t) + n_1\pi\cdot\operatorname{sgn}\alpha_1'$, where m, m_1, n, n_1 are from $\mathbf{N}$. The following relations $\alpha_{1\circ}\alpha(t+mm_1\pi) = \alpha_1(\alpha(t) + nm_1\pi\cdot\operatorname{sgn}\alpha') = \alpha_{1\circ}\alpha(t) + nn_1\pi\cdot\operatorname{sgn}\alpha'\cdot\operatorname{sgn}\alpha_1' = \alpha_{1\circ}\alpha(t) + nn_1\pi\cdot\operatorname{sgn}(\alpha_{1\circ}\alpha)'$ and $\alpha^{-1}(t+n\pi) = \alpha^{-1}(t) + m\pi\cdot\operatorname{sgn}\alpha^{-1\prime}$ yield $\alpha_{1\circ}\alpha \in \mathfrak{S}$, $\alpha^{-1} \in \mathfrak{S}$. The statement $\mathfrak{H} \subset \mathfrak{S}$ is immediate.

Let $\varepsilon \in \mathfrak{C}$. Then $\varepsilon(t+\pi) = \varepsilon(t) + \pi\cdot\operatorname{sgn}\varepsilon'$, $\varepsilon_\circ\alpha(t+m\pi) = \varepsilon(\alpha(t) + n\pi\cdot\operatorname{sgn}\alpha') = \varepsilon_\circ\alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'\cdot\operatorname{sgn}\varepsilon' = \varepsilon_\circ\alpha(t) + n\pi\cdot\operatorname{sgn}(\varepsilon_\circ\alpha)'$. Therefore $\varepsilon_\circ\alpha \in \mathfrak{S}$ for every $\alpha \in \mathfrak{S}$ and we have $\mathfrak{C}\alpha \subset \mathfrak{S}$.

Remark 1. The elements of $\mathfrak{S}$ play an important role in searching for equations of the type $(\mathrm{q}+\lambda)$: $y'' = (q(t)+\lambda)y$, where $\lambda \in \mathbf{R}$ and q is a periodic function with the period π, $q \in C_{\mathbf{R}}^0$, with the property that there exists such a sequence (finite or infinite) $\{\lambda_n\}$, $\lambda_n \in \mathbf{R}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, where the equations $(\mathrm{q}+\lambda_n)$ have all solutions periodic or halfperiodic with period π (see [2]).

Lemma 2. *$\alpha \in \mathfrak{S}$ iff $\alpha(t) = n\cdot\gamma\left(\frac{t}{m}\right)$ for $t \in \mathbf{R}$, where $m \in \mathbf{N}$, $n \in \mathbf{N}$ and $\gamma \in \mathfrak{H}$.*

Proof: Let $\alpha \in \mathfrak{S}$. Then there exist $m \in \mathbf{N}$, $n \in \mathbf{N}$: $\alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$. Put $\gamma(t) := \frac{1}{n}\alpha(mt)$, $t \in \mathbf{R}$. Then $\operatorname{sgn}\gamma' = \operatorname{sgn}\alpha'$, $\gamma(t+\pi) = \frac{1}{n}\alpha(mt+m\pi) = \frac{1}{n}[\alpha(mt) + n\pi\cdot\operatorname{sgn}\alpha'] = \frac{1}{n}\alpha(mt) + \pi\cdot\operatorname{sgn}\gamma' = \gamma(t) + \pi\cdot\operatorname{sgn}\gamma'$; thus $\gamma \in \mathfrak{H}$. Let now $\gamma \in \mathfrak{H}$, $m \in \mathbf{N}$, $n \in \mathbf{N}$ and put $\alpha(t) := n\cdot\gamma\left(\frac{t}{m}\right)$, $t \in \mathbf{R}$. Then $\operatorname{sgn}\alpha' = \operatorname{sgn}\gamma'$, $\alpha(t+m\pi) = n\cdot\gamma\left(\frac{t}{m}+\pi\right) = n\cdot\gamma\left(\frac{t}{m}\right) + n\pi\cdot\operatorname{sgn}\alpha' = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$; thus $\alpha \in \mathfrak{S}$.

Remark 2. For $\alpha \in \mathfrak{S}$ there exist $m \in \mathbf{N}$, $n \in \mathbf{N}$: $\alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$. The integers m, n are not uniquely associated with α, which becomes readily apparent from the fact that for every k, $k \in \mathbf{N}$, there is $\alpha(t+km\pi) = \alpha(t) + kn\pi\cdot\operatorname{sgn}\alpha'$. However, among all the positive integers m with $\alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$ we can always find the smallest positive integer that we again denote by m. To this number m there belongs exactly one positive integer n.

Lemma 3. *Let α and φ be a first phase and the dispersion of (q), respectively. Then $\alpha \in \mathfrak{S}$, $\alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$, $m \in \mathbf{N}$, $n \in \mathbf{N}$ if and only if $\varphi_n(t) = t + m\pi$.*

Proof: Let α and φ be a first phase and the dispersion of (q), respectively. Let $\alpha \in \mathfrak{S}$, $\alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$, $m \in \mathbf{N}$, $n \in \mathbf{N}$. Then from the Abelian relation $\alpha_\circ\varphi_n(t) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$ it follows that $\varphi_n(t) = t + m\pi$. Let there now exist positive integers m, n: $\varphi_n(t) = t + m\pi$. Then $\alpha_\circ\varphi_n(t) = \alpha(t+m\pi) = \alpha(t) + n\pi\cdot\operatorname{sgn}\alpha'$ and therefore $\alpha \in \mathfrak{S}$.

Lemma 4. *Let α, $\varphi(\beta, \tilde{\varphi})$ be a first phase and the dispersion of (q) ((p)),* *respectively. Then* $\alpha=\varrho\circ\beta$, *where* $\varrho\in\mathfrak{S}$, $\varrho(t+m\pi) = \varrho(t)+n\pi\cdot\operatorname{sgn}\varrho'$, $m\in\mathbf{N}$, $n\in\mathbf{N}$, *if and only if* $\varphi_n=\tilde{\varphi}_m$.

Proof. Let therefore α, $\varphi(\beta, \tilde{\varphi})$ be a first phase and the dispersion of (q) ((p)), respectively, and let $\alpha=\varrho\circ\beta$, where $\varrho\in\mathfrak{S}$, $\varrho(t+m\pi) = \varrho(t)+n\pi\cdot\operatorname{sgn}\varrho'$, $m\in\mathbf{N}$, $n\in\mathbf{N}$. Then $\alpha\circ\tilde{\varphi}_m(t) = \varrho\circ\beta\circ\tilde{\varphi}_m(t) = \varrho(\beta(t) + m\pi\cdot\operatorname{sgn}\beta') = \varrho\circ\beta(t) + n\pi\cdot\operatorname{sgn}\beta'\cdot\operatorname{sgn}\varrho'=\alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'=\alpha\circ\varphi_n(t)$ and from this $\varphi_n=\tilde{\varphi}_m$. Let us now suppose that $\varphi_n=\tilde{\varphi}_m$. Then $\alpha\circ\varphi_n(t) = \alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$, $\beta\circ\tilde{\varphi}_m(t)=\beta(t)+ m\pi\cdot\operatorname{sgn}\beta'$, hence $\beta\circ\varphi_n\circ\beta^{-1}(t) = t+m\pi\cdot\operatorname{sgn}\beta'$. Let us put $\varrho:=\alpha\circ\beta^{-1}$. Then $\varrho(t+m\pi\cdot\operatorname{sgn}\beta') = \alpha\circ\beta^{-1}\circ\beta\circ\varphi_n\circ\beta^{-1}(t)=\alpha\circ\varphi_n\circ\beta^{-1}(t) = \alpha\circ\beta^{-1}(t)+n\pi\cdot\operatorname{sgn}\alpha' = \varrho(t)+n\pi\cdot\operatorname{sgn}\alpha'$. Thus we proved: $\varrho(t+m\pi) = \varrho(t)+n\pi\cdot\operatorname{sgn}\alpha'\cdot\operatorname{sgn}\beta' = \varrho(t)+n\pi\cdot\operatorname{sgn}\varrho'$. Consequently $\varrho\in\mathfrak{S}$ and $\varrho(t+m\pi) = \varrho(t)+n\pi\cdot\operatorname{sgn}\varrho'$.

Definition 1. *Let* φ *and* $\tilde{\varphi}$ *be the dispersions of* (q) *and* (p), *respectively. We say that* (p), (q) *are in the relation* $\sim$ *and write* (p)$\sim$(q), *iff there exist positive integers* m, n: $\varphi_n=\tilde{\varphi}_m$.

Lemma 5. *The binary relation* $\sim$ *is an equivalence on the set* $\mathfrak{Q}$ *of all equations* (q) *oscillatory on* **R**. *The decomposition of the set* $\mathfrak{Q}$ *defined by the equivalence* $\sim$ *will be denoted by* $\bar{\mathfrak{Q}}$.

Proof: It is evident that the relation $\sim$ is reflexive and symmetric. We will prove its transitivity, too. Let φ, $\tilde{\varphi}$ and $\tilde{\tilde{\varphi}}$ be the dispersions of (q), (q_1) and (q_2), respectively and let (q)$\sim$(q_1), (q_1)$\sim$(q_2). Then there exist positive integers n, n_1, n_2, $\bar{n}_1$ for which $\varphi_n=\tilde{\varphi}_{n_1}$, $\tilde{\varphi}_{\bar{n}_1}=\tilde{\tilde{\varphi}}_{n_2}$. Then, naturally, $\varphi_{n\bar{n}_1}=\tilde{\varphi}_{n_1\bar{n}_1}=\tilde{\tilde{\varphi}}_{n_1n_2}$ and consequently (q)$\sim$(q_2).

Theorem 1. *The sets* $\mathfrak{G}/_r\mathfrak{S}$ *and* $\bar{\mathfrak{Q}}$ *are isomorphic. Each element of the decomposition* $\mathfrak{G}/_r\mathfrak{S}$ *contains all the first phases of exactly those equations* (q) *that belong to the same decomposition* $\bar{\mathfrak{Q}}$.

Proof: Let α and β be first phases of (q) and (p), respectively, which lie in the same element of the decomposition $\mathfrak{G}/_r\mathfrak{S}$. Then by Lemma 4 (q)$\sim$(p) and the equations (p), (q) belong to the same element of the decomposition $\bar{\mathfrak{Q}}$. Let (p), (q) be equations for which (p)$\sim$(q) and let $\alpha(\beta)$ be a first phase of (q) ((p)). Then by Lemma 4 $\alpha=\varrho\circ\beta$, $\varrho\in\mathfrak{S}$, which signifies that α and β belong to the same element of the decomposition $\mathfrak{G}/_r\mathfrak{S}$.

Let us look now at the subsets $\mathfrak{S}_r$, $r\in\mathbf{R}$, $r>0$ and at their relation to $\mathfrak{S}$.

Lemma 6. *Let* $r\in\mathbf{R}$, $r>0$. *Then* $\mathfrak{S}_r$ *is a subgroup of the group of phases* $\mathfrak{G}$. *Besides,* $\gamma\in\mathfrak{S}_r$ *if and only if* $\alpha(t):=\frac{\pi}{r}\cdot\gamma\left(\frac{r}{\pi}t\right)$, $t\in\mathbf{R}$ *is an element of* $\mathfrak{S}$.

Proof: Let $r\in\mathbf{R}$, $r>0$. By analogy with the proof of Lemma 1 we can verify that $\mathfrak{S}_r$ is a subgroup of $\mathfrak{G}$ (this time we write r in place of π). Let now $\gamma\in\mathfrak{S}_r$, $\gamma(t+mr) = \gamma(t)+nr\cdot\operatorname{sgn}\gamma'$, $m\in\mathbf{N}$, $n\in\mathbf{N}$ and $\alpha(t):=\frac{\pi}{r}\cdot\gamma\left(\frac{r}{\pi}t\right)$, $t\in\mathbf{R}$.

Then $\alpha(t+m\pi) = \frac{\pi}{r}\cdot\gamma\left(\frac{r}{\pi}t+mr\right) = \frac{\pi}{r}\left[\gamma\left(\frac{r}{\pi}t\right)+nr\cdot\operatorname{sgn}\gamma'\right] = \frac{\pi}{r}\cdot\gamma\left(\frac{r}{\pi}t\right) + {}$ $+ n\pi\cdot\operatorname{sgn}\alpha' = \alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$, hence $\alpha\in\mathfrak{S}$. Conversely, suppose that $\alpha\in\mathfrak{S}$, $\alpha(t+m\pi) = \alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$, $m\in\mathbf{N}$, $n\in\mathbf{N}$. Put $\gamma(t):= \frac{r}{\pi}\cdot\alpha\left(\frac{\pi}{r}t\right)$, $t\in\mathbf{R}$. Then $\gamma(t+mr) = \frac{r}{\pi}\cdot\alpha\left(\frac{\pi}{r}t+m\pi\right) = \frac{r}{\pi}\left[\alpha\left(\frac{\pi}{r}t\right) + n\pi\cdot\operatorname{sgn}\alpha'\right] = \frac{r}{\pi}\cdot\alpha\left(\frac{\pi}{r}t\right) + {}$ $+ nr\cdot\operatorname{sgn}\alpha' = \gamma(t)+nr\cdot\operatorname{sgn}\gamma'$, therefore $\gamma\in\mathfrak{S}_r$.

Lemma 7. *Let $r\in\mathbf{R}$, $r>0$. Then $\mathfrak{S}_r=\mathfrak{S}$ iff $\frac{r}{\pi}$ is a rational number.*

Proof: Let $r=\frac{k}{l}\pi$, $k\in\mathbf{N}$, $l\in\mathbf{N}$ and $\gamma\in\mathfrak{S}_r$. Then there exist $m\in\mathbf{N}$, $n\in\mathbf{N}$: $\gamma(t+mr) = \gamma(t)+nr\cdot\operatorname{sgn}\gamma'$. From this we obtain $\gamma(t+mk\pi) = \gamma(t+mlr) = {}$ $= \gamma(t) + nl\pi\cdot\operatorname{sgn}\gamma'$, hence $\gamma\in\mathfrak{S}$. This proves $\mathfrak{S}_r\subset\mathfrak{S}$. Let $\alpha\in\mathfrak{S}$. Then: $\alpha(t+m\pi) = \alpha(t)+n\pi\cdot\operatorname{sgn}\alpha'$, $m\in\mathbf{N}$, $n\in\mathbf{N}$. Therefore $\alpha(t+mlr) = {}$ $= \alpha(t+mk\pi) = \alpha(t)+nk\pi\cdot\operatorname{sgn}\alpha' = \alpha(t)+nlr\cdot\operatorname{sgn}\alpha'$, hence $\alpha\in\mathfrak{S}_r$ and thus also $\mathfrak{S}\subset\mathfrak{S}_r$. Consequently $\mathfrak{S}=\mathfrak{S}_r$.

Let now $\mathfrak{S}_r=\mathfrak{S}$, where $r\in\mathbf{R}$, $r>0$. It follows from Lemma 1 that $\mathfrak{E}\subset\mathfrak{S}$ and thus also $\mathfrak{E}\subset\mathfrak{S}_r$. To each ε, $\varepsilon\in\mathfrak{E}$ there exist $m=m(\varepsilon)\in\mathbf{N}$, $n=n(\varepsilon)\in\mathbf{N}$: $\varepsilon(t+mr)=\varepsilon(t)+nr\cdot\operatorname{sgn}\varepsilon'$. Let $\varepsilon_0\in\mathfrak{E}$ be such that $\varepsilon_0'(t+c)=\varepsilon_0'(t)$, $t\in\mathbf{R}$, $c>0$ if and only if $c=k\pi$, $k\in\mathbf{N}$. Such ε_0 always exists and to it also $m_0\in\mathbf{N}$, $n_0\in\mathbf{N}$: $\varepsilon_0(t+m_0r)=\varepsilon_0(t)+n_0r\cdot\operatorname{sgn}\varepsilon_0'$, so that we get $\varepsilon_0'(t+m_0r)=\varepsilon_0'(t)$. Then, naturally, there exists $s\in\mathbf{N}$ with $m_0r=s\pi$, $r=\frac{s}{m_0}\pi$. This proves our lemma.

Corollary 1. *Let $r\in\mathbf{R}$, $r>0$. Then $\mathfrak{E}\subset\mathfrak{S}_r$ if and only if $r=\frac{k}{l}\pi$, $k\in\mathbf{N}$, $l\in\mathbf{N}$.*

Proof: If $\mathfrak{E}\subset\mathfrak{S}_r$, then we readily get from the second part of the proof of Lemma 7 $r=\frac{k}{l}\pi$, $k\in\mathbf{N}$, $l\in\mathbf{N}$. If, conversely, $r=\frac{k}{l}\pi$, $k\in\mathbf{N}$, $l\in\mathbf{N}$, then we get from Lemma 7 $\mathfrak{S}=\mathfrak{S}_r$ and from this, according to Lemma 1, we come to $\mathfrak{E}\subset\mathfrak{S}_r$.

Remark 3. It follows from Lemma 7 that instead of $\mathfrak{S}$ we can consider $\mathfrak{S}_r$, where r is a rational multiple of π, because just in this case $\mathfrak{S}=\mathfrak{S}_r$. Further, from Lemma 7 and Corollary 1 it follows that $\mathfrak{S}_r$ is formed "by the union of some elements of the decomposition of $\mathfrak{G}/_r\mathfrak{E}$" iff $\mathfrak{S}=\mathfrak{S}_r$.

3. Now we shall be concerned with the subsets $\mathfrak{A}_\vartheta$ of $\mathfrak{G}$, where $\vartheta\in\mathfrak{G}$ defined as follows: $\alpha\in\mathfrak{A}_\vartheta$ iff there exist $m\in\mathbf{Z}$, $n\in\mathbf{Z}$ such that $\vartheta^{[m]}\circ\alpha=\alpha\circ\vartheta^{[n]}$.

First we will prove that for each $\vartheta\in\mathfrak{G}$ the set $\mathfrak{A}_\vartheta$ is a subgroup of $\mathfrak{G}$ which is in particular evident for $\vartheta=\mathrm{id}$, when $\mathfrak{A}_\vartheta=\mathfrak{G}$. As the main result we shall prove that if the order ϑ, $\vartheta\in\mathfrak{G}$ is equal to ∞, $\operatorname{ord}\vartheta=\infty$ (that is $\vartheta^{[m]}\neq\vartheta^{[n]}$ for all $n\in\mathbf{N}$, $m\in\mathbf{N}$, $n\neq m$), and $\mathfrak{A}_\vartheta$ has the property: $\mathfrak{E}_\alpha\subset\mathfrak{A}_\vartheta$ for every $\alpha\in\mathfrak{A}_\vartheta$, then $\mathfrak{A}_\vartheta=\mathfrak{S}$.

Lemma 8. *Let $\vartheta \in \mathfrak{G}$. Then $\mathfrak{A}_\vartheta$ is a subgroup of the group $\mathfrak{G}$ of phases.*

Proof: Let $\alpha \in \mathfrak{A}_\vartheta$, $\alpha_1 \in \mathfrak{A}_\vartheta$. Then $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}$, $\vartheta^{[m_1]} \circ \alpha_1 = \alpha_1 \circ \vartheta^{[n_1]}$, where m, n, m_1, n_1 are from $\mathbf{Z}$. From the relations $\vartheta^{[mm_1]} \circ \alpha \circ \alpha_1 = \alpha \circ \vartheta^{[nm_1]} \circ \alpha_1 = \alpha \circ \alpha_1 \circ \vartheta^{[nn_1]}$, $\vartheta^{[-n]} \circ \alpha^{-1} = \alpha^{-1} \circ \vartheta^{[-m]}$ we obtain: $\alpha \circ \alpha_1 \in \mathfrak{A}_\vartheta$, $\alpha^{-1} \in \mathfrak{A}_\vartheta$.

Remark 4. Let $\alpha \in \mathfrak{A}_\vartheta$, $\vartheta \in \mathfrak{G}$. Then $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}$, $m \in \mathbf{Z}$, $n \in \mathbf{Z}$. At the same time $\vartheta^{[-m]} \circ \alpha = \vartheta^{[-m]} \circ \vartheta^{[m]} \circ \alpha \circ \vartheta^{[-n]} = \alpha \circ \vartheta^{[-n]}$. Thus without any loss of generality we can suppose $m \in \mathbf{N}$. Let $k \in \mathbf{N}$. Then $\vartheta^{[km]} \circ \alpha = \alpha \circ \vartheta^{[kn]}$ from which it becomes apparent that m $(\in \mathbf{N})$ and thus also n $(\in \mathbf{Z})$ are not uniquely associated with α. From now on we shall suppose m to be the smallest positive integer with $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}$. In this way the positive integer m is uniquely associated with every $\alpha \in \mathfrak{A}_\vartheta$. Now we will investigate when exactly one and only one n $(\in \mathbf{Z})$ corresponds to the above mentioned m. If $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}$ and $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n_1]}$, then $\vartheta^{[n]} = \vartheta^{[n_1]}$, therefore $\vartheta^{[n-n_1]}(t) = t (t \in \mathbf{R})$ and $n \neq n_1$ exactly in the case of ϑ being of a finite order, $\operatorname{ord} \vartheta < \infty$ (exist $m \in \mathbf{N}$, $n \in \mathbf{N}$, $m \neq n$: $\vartheta^{[m]} = \vartheta^{[n]}$). Herefrom we can conclude: With every $\alpha \in \mathfrak{A}_\vartheta$ we can uniquely associate an ordered pair (m, n), $m \in \mathbf{N}$, $n \in \mathbf{Z}$, $\vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}$ exactly if $\operatorname{ord} \vartheta = \infty$.

Theorem 2. *Let $\vartheta \in \mathfrak{G}$, $\operatorname{ord} \vartheta = \infty$. Then there is $\mathfrak{E}\alpha \subset \mathfrak{A}_\vartheta$ for every $\alpha \in \mathfrak{A}_\vartheta$ if and only if $\mathfrak{A}_\vartheta = \mathfrak{S}$.*

Proof: Let $\vartheta \in \mathfrak{G}$, $\operatorname{ord} \vartheta = \infty$. Let $\mathfrak{A}_\vartheta = \mathfrak{S}$. If $\alpha \in \mathfrak{A}_\vartheta$, then we get from Lemma 1 $\mathfrak{E}\alpha \subset \mathfrak{A}_\vartheta$.

Let now $\mathfrak{A}_\vartheta$ have the property by which $\mathfrak{E}\alpha \subset \mathfrak{A}_\vartheta$ for every $\alpha \in \mathfrak{A}_\vartheta$, thus in particular $\mathfrak{E} \subset \mathfrak{A}_\vartheta$. Therefore there exists to every $\varepsilon \in \mathfrak{E}$ $m = m(\varepsilon)$ $(\in \mathbf{N})$ and $n = n(\varepsilon)$ $(\in \mathbf{Z})$: $\vartheta^{[m]} \circ \varepsilon = \varepsilon \circ \vartheta^{[n]}$. By Remark 4 we can uniquely associate the numbers m, n with every $\varepsilon \in \mathfrak{E}$ which will still be presupposed in the next part of the proof. Since the function $t + a$ is an element of $\mathfrak{E}$ for every a $(\in \mathbf{R})$, there exist numbers $m = m(a)$ $(\in \mathbf{N})$, $n = n(a)$ $(\in \mathbf{Z})$ uniquely associated with every a, $a \in \langle 0, 1 \rangle$:

$$\vartheta^{[m]}(t + a) = \vartheta^{[n]}(t) + a, \; t \in \mathbf{R}.$$

The cardinality of $\mathbf{N}$ and $\mathbf{Z}$ is less than the cardinality of $\langle 0,1 \rangle$, hence there exist i_0 $(\in \mathbf{N})$, k_0 $(\in \mathbf{Z})$ and a sequence $\{a_n\}$, $a_n \in \langle 0,1 \rangle$, $a_i \neq a_j$ for $i \neq j$, $\lim_{n \to \infty} a_n = a_0$:

$$\vartheta^{[i_0]}(t + a_n) = \vartheta^{[k_0]}(t) + a_n, \; t \in \boldsymbol{R}, \; n = 1, 2, 3, \dots. \tag{2}$$

Letting n tend to ∞ in (2), we conclude

$$\vartheta^{[i_0]}(t + a_0) = \vartheta^{[k_0]}(t) + a_0, \; t \in \mathbf{R}.$$

Herefrom and according to (2) we get

$$\frac{\vartheta^{[i_0]}(t + a_n) - \vartheta^{[i_0]}(t + a_0)}{a_n - a_0} = 1, \quad t \in \mathbf{R}, \; n = 1, 2, 3, \dots. \tag{3}$$

In assigning to the limit in (3) ($n \to \infty$) from $\vartheta \in C^3_{\mathbf{R}}$ we get $\vartheta^{[i_0]\prime}(t+a_0)=1$ for $t \in \mathbf{R}$, which gives $\vartheta^{[i_0]\prime}(t)=1$ for $t \in \mathbf{R}$. Thus there exist d ($\in \mathbf{R}$) satisfying

$$\vartheta^{[i_0]}(t)=t+d, \quad t \in \mathbf{R}. \tag{4}$$

Hereby $d \neq 0$, which follows from the assumption $\operatorname{ord}\vartheta=\infty$. From (2) and (4) we come to $\vartheta^{[i_0]}(t+a_n)=t+a_n+d=\vartheta^{[k_0]}(t)+a_n$, hence $\vartheta^{[k_0]}(t)=t+d$ and again, with respect to $\operatorname{ord}\vartheta=\infty$, we obtain $i_0=k_0$, so that (2) may be written as

$$\vartheta^{[i_0]}(t+a_n)=\vartheta^{[i_0]}(t)+a_n, \ t \in \mathbf{R}, \quad n=1, 2, 3, \ldots. \tag{5}$$

By derivation of (4) we get: $\vartheta' \circ \vartheta^{[i_0-1]}(t) \cdot \vartheta' \circ \vartheta^{[i_0-2]}(t) \cdot \ldots \vartheta'(t)=1$. Substituting $\vartheta(t)$ instead of t into the last formula and applying (4) we get: $\vartheta'(t+$ $+d) \cdot \vartheta' \circ \vartheta^{[i_0-1]}(t) \cdot \ldots \vartheta' \circ \vartheta(t)=1$, which gives $\vartheta'(t+d)=\vartheta'(t)$ for $t \in \mathbf{R}$. Thus there exist b ($\in \mathbf{R}$): $\vartheta(t+d)=\vartheta(t)+b$. From $\vartheta(t+d)=\vartheta \circ \vartheta^{[i_0]}(t)=\vartheta^{[i_0+1]}(t)=$ $=\vartheta(t)+b$ it follows $\vartheta^{[i_0]}(t)=t+b$ and by taking account of (4) we have $b=d$ and $\vartheta(t+d)=\vartheta(t)+d$. Consequently $\operatorname{sgn}\vartheta'=1$.

Let $\varepsilon \in \mathfrak{E}$ such that $\varepsilon'(t+h)=\varepsilon'(t)$, $h \neq 0$, only if $h=k\pi$ for some $k \in \mathbf{Z}$. The existence of such an ε follows from the properties of the elements of the group $\mathfrak{E}$. To such an ε there uniquely exists m ($\in \mathbf{N}$), n ($\in \mathbf{Z}$): $\vartheta^{[m]} \circ \varepsilon=\varepsilon \circ \vartheta^{[n]}$ and therefore there is also $\vartheta^{[i_0 m]} \circ \varepsilon=\varepsilon \circ \vartheta^{[i_0 n]}$, with respect to (4) it gives $\varepsilon(t)+md=\varepsilon(t+nd)$. Herefrom $\varepsilon'(t)=\varepsilon'(t+nd)$ and thus $nd=k\pi$, $k \in \mathbf{Z}$. Consequently $d=\frac{k}{n}\pi$, $\vartheta\left(t+\frac{k}{n}\pi\right)=\vartheta(t)+\frac{k}{n}\pi$. From this $\vartheta^{[i_0 n]}(t)=t+nd=t+k\pi$ and from definition $\mathfrak{S}$ it becomes immediately obvious that $\mathfrak{A}_\vartheta=\mathfrak{S}$. This completes the proof of the Theorem.

Remark 5. If $\operatorname{ord}\vartheta=\infty$ and $\mathfrak{E}\alpha \subset \mathfrak{A}_\vartheta$ for every $\alpha \in \mathfrak{A}_\vartheta$, then it follows from the proof of Theorem 2: $\operatorname{sgn}\vartheta'=1$, $\vartheta\left(t+\frac{k}{l}\pi\right)=\vartheta(t)+\frac{k}{l}\pi$, $\vartheta^{[i_0]}(t)=t+\frac{k}{l}\pi$, where $i_0 \in \mathbf{N}$, $k \in \mathbf{Z}$, $l \in \mathbf{Z}$.

Corollary 2. *Let $\vartheta \in \mathfrak{G}$. Then $\mathfrak{A}_\vartheta=\mathfrak{G}$ if and only if $\operatorname{ord}\vartheta<\infty$.*

Proof: Let $\operatorname{ord}\vartheta=i$, $i<\infty$. Then $\vartheta^{[i]}(t)=t$ and for every $\alpha \in \mathfrak{G}$ $\vartheta^{[i]} \circ \alpha=\alpha \circ \vartheta^{[i]}$, hence $\alpha \in \mathfrak{A}_\vartheta$ and $\mathfrak{A}_\vartheta=\mathfrak{G}$.

Let $\mathfrak{A}_\vartheta=\mathfrak{G}$ and suppose $\operatorname{ord}\vartheta=\infty$. From the proof of Theorem 2 we have the existence of i_0 ($\in \mathbf{N}$) and d ($\in \mathbf{R}$): $\vartheta^{[i_0]}(t)=t+d$. We can always find an α in $\mathfrak{G}$ such that α' is not a periodic function; let α_0 be one of them. Then there exist m_0 and n_0: $\vartheta^{[m_0 i_0]} \circ \alpha_0=\alpha_0 \circ \vartheta^{[n_0 i_0]}$, which is equivalent to the equality $\alpha_0(t)+m_0 d=\alpha_0(t+n_0 d)$ from which it follows that $\alpha_0'(t)=\alpha_0'(t+n_0 d)$. According to the assumption, α_0' is not a periodic function and therefore $d=0$. Herefrom $\vartheta^{[i_0]}(t)=t$ and ϑ has a finite order ($\leqq i_0$). Thus, we have proved that $\operatorname{ord}\vartheta<\infty$ when $\mathfrak{A}_\vartheta=\mathfrak{G}$.

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Received April 21, 1976

Katedra matematickej analýzy
a numerickej matematiky PF UP
Leninova 26
772 00 Olomouc

О АЛГЕБРАИЧЕСКОЙ СТРУКТУРЕ ГРУППЫ ФАЗ

Сватослав Станек

Резюме

В работе введены и исследуются новые подгруппы группы $\mathfrak{G}$ фаз дифференциальных уравнений (q): $y'' = q(t)y$, $q \in C^0_{\mathbf{R}}$, $\mathbf{R} = (-\infty, \infty)$, решения которых колебются в $\mathbf{R}$. Если обозначить $\mathbf{N}$ множество натуральных чисел, $\mathbf{Z}$ множество целых чисел и $\vartheta^{[n]}$ n-ю итерацию функции ϑ, то

$$\mathfrak{S} = \{\alpha \in \mathfrak{G} : \alpha(t + m\pi) = \alpha(t) + n\pi \cdot \operatorname{sgn} \alpha', m \in \mathbf{N}, n \in \mathbf{N}\}$$

и

$$\mathfrak{A}_\vartheta = \{\alpha \in \mathfrak{G} : \vartheta^{[m]} \circ \alpha = \alpha \circ \vartheta^{[n]}, m \in \mathbf{Z}, n \in \mathbf{Z}\}, \quad \vartheta \in \mathfrak{G},$$

являются подгруппами группы $\mathfrak{G}$. Исследуются тоже связи между $\mathfrak{S}$ и $\mathfrak{A}_\vartheta$. Показано, что если $\mathfrak{E}$ группа фаз дифференциального уравнения $y'' = -y$ и ϑ, $\vartheta \in \mathfrak{G}$, имеет бесконечный порядок, то $\mathfrak{A}_\vartheta = \mathfrak{S}$ тогда и только тогда, если $\mathfrak{E}\alpha \subset \mathfrak{A}_\vartheta$ для всех $\alpha \in \mathfrak{A}_\vartheta$.