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LATTICE UNIFORMITIES ON ORTHOMODULAR STRUCTURES

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ABSTRACT. We prove that every lattice uniformity on an orthomodular lattice is generated by a family of weakly subadditive functions and that every modular measure on a difference-lattice generates a topological structure as modular functions on orthomodular lattices.

Introduction

Starting from the seventies, many authors, as L. Drewnowski, Z. Lipecki, H. Weber and others (see for example [D_1], [L], [W_1]), introduced, in classical measure theory, topological methods based on the theory of Fréchet-Nikodym topologies, which gave many contributions to the study of measures on Boolean algebras.

In the last years, similar topological methods have been developed for the study of modular functions on orthomodular lattices in non-commutative measure theory (see for example [A_1], [A_2], [A-B-C], [A-D], [A-L], [W_4], [W_5], [W_6], [W_8]) and for the study of measures on fuzzy structures in fuzzy measure theory (see [B-W], [B-L-W], [G]). In this context, the theory of Fréchet-Nikodym topologies is replaced by the theory of lattice uniformities — i.e. uniformities which makes the lattice operations uniformly continuous — developed in [W_2], [W_3], [W_4], [W_7], [A-W], starting from the fact that every modular function on an orthomodular lattice generates a lattice uniformity which makes the orthocomplementation uniformly continuous ([W_4; 1.1]) and every measure on a Δ-ℓ-semigroup or on a Vitali space generates a lattice uniformity which makes uniformly continuous the operations of these structure ([B-W; 3.1.2], and [G; 5.3]).

It is known (see, for example, [D]) that, in a Boolean algebra, every Frechet-Nikodym topology is generated by a family of subadditive functions. In the first

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part of the present paper, we prove that a similar result also holds for lattice uniformities on orthomodular lattices: we introduce a class of weakly subadditive functions — the k-submeasures — and we prove that, for every k-submeasure η, there exists the weakest lattice uniformity which makes η uniformly continuous (see 2.6) and, conversely, every lattice uniformity is generated by a family of k-submeasures (see 2.8). In particular, every lattice uniformity with a countable base coincides with the uniformity generated by a k-submeasure.

In the second part, we prove that it is possible to use topological methods also in the study of modular measures on difference-lattices (D-lattices), since every modular measure on a D-lattice L generates a lattice uniformity which makes the difference operation of L uniformly continuous (see 3.2.2).

As example of consequence of this result, we derive by standard topological methods the equivalence in any D-lattice between Vitali-Hahn-Saks and Brooks-Jewett theorems for modular measures (see 3.6). In particular, as consequence of [D-P] — in which the Brooks-Jewett theorem has been proved for measures on quasi-σ-complete D-posets — we obtain the Vitali-Hahn-Saks theorem for modular measures on quasi-σ-complete D-lattices (see 3.7).

We recall that D-posets and D-lattices have been introduced in [C-K_1] as a generalization of many structures as orthomodular lattices, MV-algebras, orthoalgebras, weakly complemented posets and others. For a study, see for example [B-F], [C-K_1], [C-K_2], [D-D-P], [F-G-P], [P_2], [P_3], [R].

1. Preliminaries

Let L be a lattice. If L has a smallest or a greatest element, we denote these elements by 0 and 1, respectively. We set $\Delta = \{(a,b) \in L \times L : \ a = b\}$. If $\{a_n\}$ is an increasing sequence and $a = \sup\limits_n a_n$ in L (respectively, $\{a_n\}$ is decreasing and $a = \inf\limits_n a_n$ in L), we write $a_n \uparrow a$ (respectively, $a_n \downarrow a$). If $a \le b$, we set $[a,b] = \{c \in L : \ a \le c \le b\}$.

A *lattice uniformity* $\mathcal{U}$ on L is a uniformity on L which makes the lattice operations of L uniformly continuous (for a study, see [W_1]). $\mathcal{U}$ is called *exhaustive* if every monotone sequence in L is Cauchy in $(L,\mathcal{U})$ and σ*-order continuous* (σ-o.c.) if $a_n \uparrow a$ or $a_n \downarrow a$ imply $a_n \to a$ in $(L,\mathcal{U})$.

If $(G,+)$ is an Abelian group, a function $\mu\colon L \to G$ is called *modular* if, for every $a,b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$. If G is a topological group and $\mu\colon L \to G$ is a modular function, by [W_5; (3.1)] there exists the weakest lattice uniformity $\mathcal{U}(\mu)$ which makes μ uniformly continuous and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets

$$\{(a,b) \in L \times L : \ \mu(c) - \mu(d) \in W \ \text{ for all } \ c,d \in [a \wedge b, a \vee b]\},$$

where W is a 0-neighbourhood in G.

L is called *orthomodular lattice* if it has 0 and 1 and there exists a map $'$: $a \in L \to a' \in L$, with the following properties:

(1) $a \vee a' = 1$ and $a \wedge a' = 0$.
(2) $a \leq b \implies a' \geq b'$.
(3) $(a')' = a$.
(4) $a \leq b \implies b = a \vee (b \wedge a')$.

For a study, we refer to [K] or [P-P].

A *difference-poset* (or *D-poset*) is a non-empty partially ordered set $(L, \leq)$ with a greatest element 1 and a binary partial operation $\ominus$, called *difference*, such that $a \ominus b$ is defined if and only if $b \leq a$ and the following properties hold:

(1) $b \ominus a \leq b$.
(2) $b \ominus (b \ominus a) = a$.
(3) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$.
(4) If $a \leq b \leq c$, then $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

If $(L, \leq)$ is a lattice, a D-poset is called *D-lattice*. For every $a, b \in L$, we set $a \triangle b = (a \vee b) \ominus (a \wedge b)$ and $a^{\perp} = 1 \ominus a$. It is easy to see that $(a^{\perp})^{\perp} = a$ for every $a \in L$ and $a \leq b$ implies $a^{\perp} \geq b^{\perp}$. If $a, b \in L$, we say that $a \perp b$ if $a \leq b^{\perp}$. If $a \perp b$, we set $a \oplus b = (a^{\perp} \ominus b)^{\perp}$. It is easy to see that $\oplus$ is commutative and, if $b \oplus c$ and $a \oplus (b \oplus c)$ are defined, then $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, too, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. More in general, for $n \geq 3$, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ if $a_1 \oplus \cdots \oplus a_{n-1}$ and $(a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ are defined, and the definition is independent on any permutation of the elements.

We say that a family $\{a_1, \dots, a_n\}$ in L is *orthogonal* if $\bigoplus_{i \leq n} a_i = a_1 \oplus \cdots \oplus a_n$ is defined. We say that $\{a_n\}$ is *orthogonal* if, for every finite $M \subseteq \mathcal{N}$, $\{a_n : n \in M\}$ is orthogonal.

We use the following properties of D-lattices.

PROPOSITION (1.1). ([R; 1.3, 1.4, 1.7, 2.2, 2.4, 2.6]) *Let L be a D-lattice. Then:*

(1) *If $c \leq a$ and $c \leq b$, then $(a \vee b) \ominus c = (a \ominus c) \vee (b \ominus c)$ and $(a \wedge b) \ominus c = (a \ominus c) \wedge (b \ominus c)$.*
(2) *If $a \leq b$, then $b = a \oplus (b \ominus a)$.*
(3) *If $a \leq b \leq c$, then $b \ominus a \leq c \ominus a$.*
(4) *If $a \perp b$, then $a \leq a \oplus b$ and $(a \oplus b) \ominus a = b$.*
(5) *If $a \leq b \leq c$, then $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(6) *If $c \geq a$ and $c \geq b$, then $c \ominus (a \vee b) = (c \ominus a) \wedge (c \ominus b)$ and $c \ominus (a \wedge b) = (c \ominus a) \vee (c \ominus b)$.*
(7) *If $a \leq b \leq c$, then $(c \ominus b) \oplus a$ exists and $(c \ominus b) \oplus a = c \ominus (b \ominus a)$.*

If G is an Abelian group, a function $\mu\colon L \to G$ is called *a measure* if, for every $a, b \in L$ with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. By (1.1)(2) and (4), it is easy to see that μ is a measure if and only if, for every $a, b \in L$, with $b \leq a$, $\mu(a \ominus b) = \mu(a) - \mu(b)$. Moreover, by induction, we obtain that, if $\{a_1, \dots, a_n\}$ is orthogonal, then $\mu\Bigl(\bigoplus\limits_{i \leq n} a_i\Bigr) = \sum\limits_{i \leq n} \mu(a_i)$.

Many structures are examples of D-lattices (see [P_2; Chapter 12]). In particular, every orthomodular lattice is a D-lattice if we define, for $b \leq a$, $a \ominus b = a \wedge b'$. In this case, $a^{\perp} = a'$ and, if $a \perp b$, then $a \oplus b = a \vee b$.

In the following, we denote by $\mathcal{N}$ the set of the positive integer numbers. Moreover, for $x, y \in [0, \infty]$, we set $d_{\infty}(x, y) = |x - y|$, where $\infty - \infty = 0$ and $\infty - x = x - \infty = \infty$ for every $x \in [0, +\infty[$.

2. Lattice uniformities on orthomodular lattices

In this section, L is an orthomodular lattice, and $\eta\colon L \to [0, +\infty]$.

DEFINITION (2.1). If $k \geq 1$, we say that η is *k-subadditive* if, for every $a, b \in L$, $\eta(a \vee b) \leq k\eta(a) + \eta(b)$.

If η is 1-subadditive, we say that η is *subadditive*.

DEFINITION (2.2). We say that η is a *k-submeasure* if $\eta(0) = 0$, η is monotone and k-subadditive and, for every $a, b \in L$, $\eta\bigl((a \vee b) \wedge b'\bigr) \leq k\eta(a)$.

A 1-submeasure is called *submeasure*.

Every k-submeasure is k-triangular and null-additive in the sense of [P_2].

If L is a Boolean algebra, every monotone and k-subadditive function η, with $\eta(0) = 0$, is a k-submeasure.

EXAMPLES (2.3).

(1) Every positive real-valued modular function μ with $\mu(0) = 0$, is a submeasure, since μ is monotone and subadditive and, for every $a, b \in L$, $a \vee b = b \vee \bigl((a \vee b) \wedge b'\bigr)$, where $b \perp (a \vee b) \wedge b'$, from which

$$\mu\bigl((a \vee b) \wedge b'\bigr) = \mu(a \vee b) - \mu(b)\,. \qquad (*)$$

(2) A positive real-valued measure μ is a k-submeasure if and only if μ is k-subadditive, because $(*)$ of (1) holds.

(3) Let μ be a positive real-valued modular function with $\mu(0) = 0$, $k \geq 1$, and $\phi\colon [0, +\infty[\, \to [0, +\infty[$ an increasing function such that $\phi(0) = 0$ and, for every $x, y \in [0, +\infty[$, $|\phi(x) - \phi(y)| \leq k\phi\bigl(|x - y|\bigr)$. Then the function $\lambda\colon L \to [0, +\infty[$ defined by $\lambda(a) = \phi\bigl(\mu(a)\bigr)$ for $a \in L$ is a k-submeasure.

The following result generalizes the equivalence for a real-valued measure on L between modularity and subadditivity (see [R]).

PROPOSITION (2.4). *Let $\mu\colon L \to [0,+\infty[$ be a measure and $k \geq 1$. Then μ is a k-submeasure if and only if, for every $a, b \in L$,*

$$\mu(a \vee b) + k\mu(a \wedge b) \leq k\mu(a) + \mu(b) \leq k\mu(a \vee b) + \mu(a \wedge b).$$

Proof. The proof of $\Longleftarrow$ is trivial by (2.3)(2), since $\mu(a \vee b) \leq \mu(a \vee b) + k\mu(a \wedge b) \leq k\mu(a) + \mu(b)$.

$\Longrightarrow$: Let $a, b \in L$. Since

$$\begin{aligned} a \vee b &= (a \wedge b) \vee (a \triangle b) \qquad \text{with} \quad a \wedge b \perp a \triangle b, \\ a \triangle b &= \big[a \wedge (a \wedge b)'\big] \vee \big[b \wedge (a \wedge b)'\big], \\ a = (a \wedge b) \vee \big(a \wedge (a \wedge b)'\big) &\quad \text{and} \quad b = (a \wedge b) \vee \big(b \wedge (a \wedge b)'\big), \end{aligned}$$

we get

$$\mu(a \vee b) = \mu(a \wedge b) + \mu(a \triangle b) \leq \mu(a \wedge b) + k\mu(a) - k\mu(a \wedge b) + \mu(b) - \mu(a \wedge b),$$

from which $\mu(a \vee b) + k\mu(a \wedge b) \leq k\mu(a) + \mu(b)$. Moreover, since

$$\begin{aligned} a \vee b &= a \vee \big[(a \vee b) \wedge a'\big] = b \vee \big[(a \vee b) \wedge b'\big], \\ a \triangle b &\leq \big[a \triangle (a \vee b)\big] \vee \big[(a \vee b) \triangle b\big] = \big[(a \vee b) \wedge a'\big] \vee \big[(a \vee b) \wedge b'\big], \end{aligned}$$

we get

$$\mu(a \vee b) = \mu(a \wedge b) + \mu(a \triangle b) \leq \mu(a \wedge b) + k\mu(a \vee b) - k\mu(a) + \mu(a \vee b) - \mu(b),$$

from which $k\mu(a \vee b) + \mu(a \wedge b) \geq k\mu(a) + \mu(b)$. □

We use the following result of [W_3; (1.1)].

THEOREM (2.5). *Let $\mathcal{F}$ be a filter on L with the following properties:*

(1) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that $a \in L$, $b, c \in G$ and $a \leq b \vee c$ imply $a \in F$.*

(2) *For every $F \in \mathcal{F}$, there is $G \in \mathcal{F}$ such that $a \in G$ implies $(a \vee b) \wedge b' \in F$ for each $b \in L$.*

Then there exists a unique lattice uniformity $\mathcal{U}$ on L which has $\mathcal{F}$ as base of 0-neighbourhoods and a base for $\mathcal{U}$ is the family consisting of the sets $\{(a,b) \in L \times L :\ a \triangle b \in F\}$ with $F \in \mathcal{F}$.

PROPOSITION (2.6). *If η is a k-submeasure, there exists the weakest lattice uniformity $\mathcal{U}(\eta)$ which makes η uniformly continuous.*

Proof. By (2.5), the family consisting of the sets $\{(a,b) \in L \times L :\ \eta(a \triangle b) < \varepsilon\}$, where $\varepsilon > 0$, is a base for a lattice uniformity $\mathcal{U}(\eta)$ on L.

(i) We prove that η is uniformly continuous with respect to $\mathcal{U}(\eta)$.

Let $\varepsilon > 0$ and $a, b \in L$ such that $\eta(a\Delta b) < \varepsilon/k$. Since $a \vee b = (a \wedge b) \vee (a\Delta b)$, then $\eta(a \vee b) < \eta(a \wedge b) + k\eta(a\Delta b) < \eta(a \wedge b) + \varepsilon$. Then, if $\eta(a \vee b) = +\infty$, we get $\eta(a \wedge b) = +\infty$, from which $\eta(a) = \eta(b) = +\infty$. If $\eta(a \vee b) < +\infty$, then $\eta(a) < +\infty$ and $\eta(b) < +\infty$, from which $\eta(a) - \eta(b) \leq \eta(a \vee b) - \eta(a \wedge b) < \varepsilon$ and $\eta(b) - \eta(a) < \varepsilon$. In both the cases, $d_\infty(\eta(a), \eta(b)) < \varepsilon$.

(ii) Let $\mathcal{V}$ be a lattice uniformity which makes η uniformly continuous. We prove that $\mathcal{U}(\eta) \leq \mathcal{V}$.

Let $U \in \mathcal{U}(\eta)$ and $\varepsilon > 0$ such that $U_\varepsilon = \{(a, b) \in L \times L : \eta(a\Delta b) < \varepsilon\} \subseteq U$. Since η is $\mathcal{V}$-uniformly continuous, we can choose $V \in \mathcal{V}$ such that

$$(a, b) \in V \implies d_\infty(\eta(a), \eta(b)) < \varepsilon. \tag{$*$}$$

By [W$_1$; 1.1.2, 1.1.3], we can choose V', $V'' \in \mathcal{V}$ such that $V'' \subseteq V' \subseteq V$, $V' \wedge \Delta \subseteq V$ and

$$(a, b) \in V'' \implies [a \wedge b, a \vee b] \times [a \wedge b, a \vee b] \subseteq V'. \tag{$**$}$$

We prove that $V'' \subseteq U$. Let $(a, b) \in V''$. By $(**)$, $(a \wedge b, a \vee b) \in V'$. Then

$$\begin{aligned}(0, a\Delta b) &= ((a \wedge b) \wedge (a \wedge b)', (a \vee b) \wedge (a \wedge b)') \\ &= (a \wedge b, a \vee b) \wedge ((a \wedge b)', (a \wedge b)') \in V' \wedge \Delta \subseteq V.\end{aligned}$$

By $(*)$, we get $\eta(a\Delta b) < \varepsilon$, from which $(a, b) \in U_\varepsilon \subseteq U$. □

COROLLARY (2.7). *Let η be a k-submeasure and $\mathcal{U}$ a lattice uniformity. Then η is $\mathcal{U}$-uniformly continuous if and only if $\mathcal{U}(\eta) \leq \mathcal{U}$.*

We say that a k-submeasure η is a *uniform k-submeasure* if the following conditions hold:

(1) There exists $M > 0$ such that, for every $a, b, c \in L$ with $b \wedge c = 0$, $\eta((a \vee b) \wedge c) \leq Mk\eta(a)$.

(2) There exist $M_1, M_2 > 0$ such that, for every $a, b, c \in L$, $\eta((a \vee c)\Delta(b \vee c)) \leq M_1 k\eta(a\Delta b)$ and $\eta((a \wedge c)\Delta(b \wedge c)) \leq M_2 k\eta(a\Delta b)$.

We want to prove the following result.

THEOREM (2.8). *Let $\mathcal{U}$ be a lattice uniformity on L. Then:*

(1) *For every $k > 1$, there exists a family $\{\tilde{\eta}_\alpha\}$ of uniform k-submeasures such that $\mathcal{U} = \sup_\alpha \mathcal{U}(\tilde{\eta}_\alpha)$.*

(2) *If $\mathcal{U}$ has a countable base, for every $k > 1$ there exists a uniform k-submeasure $\tilde{\eta}$ such that $\mathcal{U} = \mathcal{U}(\tilde{\eta})$.*

(3) *If $\mathcal{U}$ is generated by a modular function $\mu: L \to G$ where G is a topological Abelian group, then there exists a family $\{\tilde{\mu}_\alpha\}$ of uniform submeasures such that $\mathcal{U} = \sup_\alpha \mathcal{U}(\tilde{\mu}_\alpha)$.*

To prove (2.8), essential tools are the following two results, which hold in any lattice.

THEOREM (2.9). ([W_3; 1.4]) *Let $k > 1$ and $\mathcal{U}$ a lattice uniformity. Then $\mathcal{U}$ is generated by a family $\{d_\alpha : \alpha \in A\}$ of pseudometrics with the following properties:*

(i) *For every $a, b, c \in L$, $d_\alpha(a \vee c, b \vee c) \le d_\alpha(a, b)$.*

(ii) *For every $a, b, c \in L$, $d_\alpha(a \wedge c, b \wedge c) \le k d_\alpha(a, b)$.*

Moreover, if $\mathcal{U}$ has a countable base, we can choose $|A| = 1$.

THEOREM (2.10). ([F-T_2; Theorem 3]) *Let μ be a modular function with values in a topological Abelian group G. Then $\mathcal{U}(\mu)$ is generated by a family of pseudometrics d_α defined by*

$$d_\alpha(a, b) = \sup\{p_\alpha\big(\mu(c) - \mu(d)\big) : \ c, d \in [a \wedge b, a \vee b],\ c \le d\}, \qquad a, b \in L,\ \alpha \in A,$$

where $\{p_\alpha : \alpha \in A\}$ is a family of group seminorms which generate the topology of G, and d_α have the properties (i) *and* (ii) *of* (2.9) *with $k = 1$.*

LEMMA (2.11). *Let $k \ge 1$ and d a pseudometric with the following properties:*

(i) *$d(a \vee c, b \vee c) \le d(a, b)$ for every $a, b, c \in L$.*

(ii) *$d(a \wedge c, b \wedge c) \le k d(a, b)$ for every $a, b, c \in L$.*

Then d has the following properties:

(1) *If $c \le a \le b \le d$, then $d(a, b) \le k d(c, d)$.*

(2) *$d(a \wedge b, a) \le k d(b, a \vee b)$.*

(3) *$d(b, a \vee b) \le d(a \wedge b, a)$.*

(4) *$d(b, a \vee b) \le d(a, b) \le 2k d(a \wedge b, a \vee b)$.*

(5) *$d(a \wedge b, a \vee b) \le 2k d(a, b)$.*

(6) *$d(a \triangle b, 0) \le k d(a \wedge b, a \vee b)$.*

(7) *$d(a \wedge b, a \vee b) \le d(a \triangle b, 0)$.*

(8) *If $b \wedge c = 0$, then $d\big((a \vee b) \wedge c, 0\big) \le k d(a \wedge b, a) \le k^2 d(a, 0)$.*

Proof. The proof of (1) – (5) can be obtained in similar way as in [W_3; 1.7].

(6): By (ii) and (5), we get

$$\begin{aligned} d(a \triangle b, 0) &= d\big((a \vee b) \wedge (a \wedge b)', (a \wedge b) \wedge (a \wedge b)'\big) \\ &\le k d(a \wedge b, a \vee b). \end{aligned}$$

(7): By (i), $d(a \wedge b, a \vee b) = d\big((a \wedge b) \vee 0, (a \wedge b) \vee (a \triangle b)\big) \le d(a \triangle b, 0)$.

(8): By (ii), (1), (3) and (7), we get

$$\begin{aligned} d\big((a \vee b) \wedge c, 0\big) &= d\big((a \vee b) \wedge c, b \wedge c\big) \\ &\le k d(a \vee b, b) \le k d(a \wedge b, a) \\ &= k d\big(a \wedge (a \wedge b), a \vee (a \wedge b)\big) \le k d\big(a \triangle (a \wedge b), 0\big) \\ &= k d\big(a \wedge (a \wedge b)', 0\big) \le k^2 d(a, 0). \end{aligned}$$

□

PROPOSITION (2.12). *Let $k \ge 1$ and d be as in (2.11). For $a \in L$, let*

$$\tilde{\eta}(a) = \sup\{d(b,0) : \ b \in [0,a]\}.$$

Then $\tilde{\eta}$ is a uniform k^2-submeasure and $\mathcal{U}(\tilde{\eta})$ coincides with the uniformity generated by d.

Proof.

(i) First we prove that, for every $a \in L$,

$$\tilde{\eta}(a) = \sup\{d(b,c) : \ b,c \in [0,a], \ b \le c\}. \tag{$*$}$$

Let $a \in L$ and denote by $\overline{\eta}(a)$ the right side of $(*)$. The inequality $\tilde{\eta}(a) \le \overline{\eta}(a)$ is trivial. Let $b,c \in [0,a]$ with $b \le c$, and set $d = c \wedge b'$. Since $c = b \vee d$ and $b \wedge d = 0$, by (3) of (2.11) we get

$$d(b,c) = d(b, b \vee d) \le d(b \wedge d, d) = d(0,d) \le \tilde{\eta}(a),$$

since $d \le c \le a$. Hence $\overline{\eta}(a) \le \tilde{\eta}(a)$.

(ii) We prove that $\tilde{\eta}$ is a uniform k-submeasure. Trivially $\tilde{\eta}$ is monotone and $\tilde{\eta}(0) = 0$. Let $c \in [0, a \vee b]$. By (1) and by (1), (2), (3) of (2.11), we get

$$\begin{aligned} d(c,0) &\le d(c, c \wedge a) + d(c \wedge a, 0) \\ &\le kd(a, a \vee c) + \tilde{\eta}(a) \le k^2 d(a, a \vee b) + \tilde{\eta}(a) \\ &\le k^2 d(a \wedge b, b) + \tilde{\eta}(a) \le k^2 \tilde{\eta}(b) + \tilde{\eta}(a), \end{aligned}$$

from which $\tilde{\eta}(a \vee b) \le k^2 \tilde{\eta}(b) + \tilde{\eta}(a)$.

Now let $a,b,c \in L$ with $b \wedge c = 0$ and choose $e,f \in \big[0, (a \vee b) \wedge c\big]$ with $e \le f$. By (ii), (1) and (3) of (2.11), we get

$$\begin{aligned} d(e,f) &\le kd\big(0, (a \vee b) \wedge c\big) = kd(b \wedge c, (a \vee b) \wedge c\big) \\ &\le k^2 d(b, a \vee b) \le k^2 d(a \wedge b, a) \le k^2 \tilde{\eta}(a). \end{aligned}$$

Hence $\tilde{\eta}\big((a \vee b) \wedge c\big) \le k^2 \tilde{\eta}(a)$.

Let $a,b,c \in L$ and $d \le (a \vee c) \triangle (b \vee c)$. By (i), (1), (4), (5), (6) and (7) of (2.11), we get

$$\begin{aligned} d(0,d) &\le kd\big(0, (a \vee c) \triangle (b \vee c)\big) \\ &\le 2k^3 d(a \vee c, b \vee c) \le 2k^3 d(a,b) \\ &\le 4k^4 d(a \wedge b, a \vee b) \le 4k^4 d(a \triangle b, 0) \le 4k^4 \tilde{\eta}(a \triangle b), \end{aligned}$$

from which $\tilde{\eta}\big((a \vee c) \triangle (b \vee c)\big) \le 4k^4 \tilde{\eta}(a \triangle b)$.

Now let $d \le (a \wedge c) \triangle (b \wedge c)$. By (ii), (1), (4), (5), (6) and (7) of (2.11), we get

$$\begin{aligned} d(0,d) &\le kd\big(0, (a \wedge c) \triangle (b \wedge c)\big) \\ &\le 2k^3 d(a \wedge c, b \wedge c) \le 2k^4 d(a,b) \\ &\le 4k^5 d(a \wedge b, a \vee b) \le 4k^5 d(a \triangle b, 0) \le 4k^5 \tilde{\eta}(a \triangle b), \end{aligned}$$

from which $\tilde{\eta}\big((a \wedge c)\Delta(b \wedge c)\big) \leq 4k^5\tilde{\eta}(a\Delta b)$.

(iii) Denote by $\mathcal{V}$ the uniformity generated by d. Since, for every $a \in L$, $d(0,a) \leq \tilde{\eta}(a)$ and, by (1) of (2.11), $d(0,a) < \varepsilon$ implies $\tilde{\eta}(a) < k\varepsilon$, then $\mathcal{U}(\tilde{\eta})$ and $\mathcal{V}$ have the same base of 0-neighbourhoods. Hence, by (2.5), $\mathcal{U}(\tilde{\eta}) = \mathcal{V}$. □

Proof of (2.8). By (2.12), if $\mathcal{U}$ is generated by a family $\{d_\alpha\}$ of pseudometrics with the properties (i) and (ii) of (2.9) with $k \geq 1$, then there exists a family $\{\tilde{\eta}_\alpha\}$ of uniform k^2-submeasures such that $\mathcal{U} = \sup_\alpha \mathcal{U}(\tilde{\eta}_\alpha)$. Then (1) and (2) follow from (2.9), and (3) follows from (2.10). □

Remark. By (2.10) and (2.12), we get that the submeasures $\tilde{\mu}_\alpha$ in (2.8)(3) are defined by

$$\begin{aligned}\tilde{\mu}_\alpha(a) = d_\alpha(a,0) &= \sup\{p_\alpha\big(\mu(c) - \mu(d)\big) :\ c, d \in [0,a],\ \ c \leq d\} \\ &= \sup\{p_\alpha\big(\mu(b) - \mu(0)\big) :\ b \in [0,a]\}\end{aligned}$$

for every $a \in L$.

Remark. In $[W_2]$ it is proved that, in general, the conclusions of (2.9) fail if $k = 1$.

A consequence of (2.8) is a characterization of lattice uniformities with (σ) by means of the family of k-submeasures which generate them.

Property (σ) has been introduced in $[W_2$; (3.1)] for arbitrary lattices and it is an essential tool for many results, for example to obtain that a uniform lattice is a Baire space ($[W_2$; 3.15)], or to obtain extension theorems (see $[W_2$; 8.2.1] and its applications in [A-D]).

We say that a lattice uniformity $\mathcal{U}$ *has* (σ) if, for every $U \in \mathcal{U}$, there exists a sequence $\{U_n\}$ in $\mathcal{U}$ with the following property: if $a_n \uparrow a$ or $a_n \downarrow a$ and $(a_i, a_j) \in U_n$ for $i, j \geq n$, then $(a_1, a) \in U$.

By $[W_2$; 3.3], if $\mathcal{U}$ has a countable base, then $\mathcal{U}$ has (σ) if and only if every monotone Cauchy sequence $\{a_n\}$ in $(L, \mathcal{U})$, with $a_n \uparrow a$ or $a_n \downarrow a$, converges to a in $(L, \mathcal{U})$.

Property (σ) is connected with the σ-order continuity by the following result of $[W_2$; (8.1.2)], which holds in any lattice.

PROPOSITION (2.13).

(1) *If $\mathcal{U}$ is exhaustive and has (σ), then $\mathcal{U}$ is σ-o.c.*
(2) *If $\mathcal{U}$ is σ-o.c., then $\mathcal{U}$ has (σ).*
(3) *If $(L, \leq)$ is σ-complete, then $\mathcal{U}$ is σ-o.c. if and only if $\mathcal{U}$ is exhaustive and has (σ).*

If L is a Boolean algebra, the uniformity generated by a submeasure η has (σ) if and only if η is σ-subadditive. Moreover the uniformity generated by a

Fréchet-Nikodym topology τ has (σ) if and only if τ is generated by a family of σ-subadditive submeasures (see [W_2; 3.17]).

We prove that a similar characterization holds for lattice uniformities on orthomodular lattices.

We need the following definitions.

We say that a function $\eta\colon L \to [0,+\infty]$ *has* (σ) if $a_n \uparrow a$ or $a_n \downarrow a$ and $\lim\limits_{n,m} \eta(a_n \Delta a_m) = 0$ imply $\lim\limits_{n,m} \eta(a_n \Delta a) = 0$.

Then, if η is a k-submeasure, η has (σ) if and only if $\mathcal{U}(\eta)$ has (σ).

If $k \geq 1$, we say that a function $\eta\colon L \to [0,+\infty]$ is *σ-k-subadditive* if, for every sequence $\{a_n\}$ in L such that $\bigvee\limits_n a_n$ exists in L, $\eta\Big(\bigvee\limits_n a_n\Big) \leq k \sum\limits_{n=1}^{\infty} \eta(a_n)$. A σ-k-subadditive k-submeasure is called *σ-k-submeasure*.

The following result has been proved in [A-D; (2.2)] for σ-subadditive submeasures and the proof is the same for σ-k-submeasures.

PROPOSITION (2.14). *Every σ-k-submeasure has (σ).*

LEMMA (2.15). *Let k, d and $\tilde{\eta}$ be as in* (2.12). *Then, if $\tilde{\eta}$ has (σ), $\tilde{\eta}$ is a σ-k^2-submeasure.*

Proof. By (2.12), $\mathcal{U}(\tilde{\eta})$ coincides with the uniformity generated by d. Since $\mathcal{U}(\tilde{\eta})$ has (σ), by [W_2; 3.3] we get that, for every sequence $\{a_n\}$ in L such that $a = \bigvee\limits_n a_n$ exists in L, $d(a_1, a) \leq k \sum\limits_{n=1}^{\infty} d(a_n, a_{n+1})$. Let $\{a_n\} \subseteq L$ be such that $a = \bigvee\limits_n a_n$ exists in L. Set $b_n = \bigvee\limits_{i=0}^{n} a_i$, where $a_0 = 0$. Then $a = \bigvee\limits_{n=0}^{\infty} b_n$. If $b \leq a$, by (2.11) we get

$$\begin{aligned} d(b,0) &\leq k d(a, b_0) \\ &\leq k^2 \sum_{n=0}^{\infty} d(b_n, b_{n+1}) = k^2 \sum_{n=0}^{\infty} d\Big(\bigvee_{i=0}^{n} a_i, \bigvee_{i=0}^{n} a_i \vee a_{n+1}\Big) \\ &\leq k^2 \sum_{n=0}^{\infty} d(0, a_{n+1}) \leq k^2 \sum_{n=0}^{\infty} \tilde{\eta}(a_{n+1}), \end{aligned}$$

from which $\tilde{\eta}(a) \leq k^2 \sum\limits_{n=1}^{\infty} \tilde{\eta}(a_n)$. □

By (2.8)(3) and (2.15), we get:

COROLLARY (2.16). *Let $k > 1$. Then every k-submeasure with (σ) is equivalent (i.e. generates the same uniformity) to a σ-k-submeasure.*

Now we can give a characterization of lattice uniformities with (σ).

COROLLARY (2.17). *Let $k > 1$ and $\mathcal{U}$ be a lattice uniformity. Then the following conditions are equivalent:*

(1) *$\mathcal{U}$ has (σ).*

(2) *There exists a family of σ-k-submeasures which generates $\mathcal{U}$.*

Proof.

(1) $\Longrightarrow$ (2): By [W_2; 3.2], there exists a family $\{\mathcal{U}_\alpha : \alpha \in A\}$ of pseudometrizable lattice uniformities with (σ) such that $\mathcal{U} = \sup_{\alpha\in A} \mathcal{U}_\alpha$. By (2.8) and (2.15), for each $\alpha \in A$ we can choose a σ-k-submeasure λ_α such that $\mathcal{U}_\alpha = \mathcal{U}(\lambda_\alpha)$.

(2) $\Longrightarrow$ (1): By (2.14), $\mathcal{U}$ is the supremum of a family of lattice uniformities with (σ). Hence $\mathcal{U}$ has (σ) by [W_2; 3.2]. □

3. Uniform D-lattices

In this section, L is a D-lattice and G is a topological Abelian group.

A measure $\mu\colon L \to G$ is called *modular measure* if it is a modular function.

We prove that the lattice uniformity generated by every modular measure on L makes uniformly continuous the difference operation of L. As an example of consequence of this result, we obtain the equivalence between Vitali-Hahn-Saks and Brooks-Jewett theorems for modular measures on D-lattices.

Following the terminology of [P_1], a lattice uniformity on L is called *D-lattice uniformity* if $\ominus$ is uniformly continuous. By the definition of $\oplus$, it is clear that a D-lattice uniformity makes $\oplus$ uniformly continuous, too.

For $U, V \subseteq L \times L$, we set

$$U \ominus V = \{(a \ominus c, b \ominus d) : \ c \le a, \ d \le b, \ (a,b) \in U, \ (c,d) \in V\},$$
$$U \oplus V = \{(a \oplus c, b \oplus d) : \ a \perp c, \ b \perp d, \ (a,b) \in U, \ (c,d) \in V\}.$$

Then $\ominus$ is uniformly continuous if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \ominus V \subseteq U$, and $\oplus$ is uniformly continuous if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \oplus V \subseteq U$.

PROPOSITION (3.1). *Let $\mathcal{U}$ be a lattice uniformity. Then $\mathcal{U}$ is a D-lattice uniformity if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \ominus \Delta \subseteq U$ and $\Delta \ominus V \subseteq U$.*

Proof. It is clear that the condition is necessary. We prove that it is sufficient, too. Let $U \in \mathcal{U}$ and choose $V, V_1, V_2 \in \mathcal{U}$ such that

$$V \circ V \circ V \subseteq U, \qquad V_1 \ominus \Delta \subseteq V, \qquad \Delta \ominus V_1 \subseteq V, \qquad V_2 \wedge V_2 \subseteq V_1.$$

We prove that $V_2 \ominus V_2 \subseteq U$. Let $(a,b),(c,d) \in V_2$ such that $c \leq a$ and $d \leq b$. By $(c, c \wedge d) \in V_1$, we get

$$\big(a \ominus c, a \ominus (c \wedge d)\big) \in \Delta \ominus V_1 \subseteq V. \tag{1}$$

By $(c \wedge d, d) \in V_1$, we get

$$\big(b \ominus (c \wedge d), b \ominus d\big) \in \Delta \ominus V_1 \subseteq V. \tag{2}$$

Moreover, since $(a,b) \in V_2 \subseteq V_1$, we have

$$\big(a \ominus (c \wedge d), b \ominus (c \wedge d)\big) \in V_1 \ominus \Delta \subseteq V. \tag{3}$$

By (1), (2) and (3), we get

$$(a \ominus c, b \ominus d) \in V \circ V \circ V \subseteq U.$$

In similar way, we can prove that, if $\mathcal{U}$ is a lattice uniformity, then $\oplus$ is uniformly continuous if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \oplus \Delta \subseteq U$. □

THEOREM (3.2). *If $\mu\colon L \to G$ is a modular measure, then $\mathcal{U}(\mu)$ is a D-lattice uniformity and a base of $\mathcal{U}(\mu)$ is the family consisting of the sets $A_W = \{(a,b) \in L \times L : \ \mu(c) \in W$ for all $c \leq a\Delta b\}$, where W is a 0-neighbourhood in G.*

Proof. For every 0-neighbourhood W in G, set

$$U_W = \{(a,b) \in L \times L : \ \mu(c) - \mu(d) \in W \text{ for all } c,d \in [a \wedge b, a \vee b], \ c \geq d\}.$$

(i) First we prove that $A_W = U_W$.
Let $(a,b) \in A_W$ and $c,d \in [a \wedge b, a \vee b]$, with $c \geq d$. By the definition of $\ominus$ and (1.1)(3), we get $c \ominus d \leq (a \vee b) \ominus (a \wedge b) = a\Delta b$, from which $\mu(c \ominus d) \in W$. Since, by (1.1)(2), $c = d \oplus (c \ominus d)$, we get $\mu(c) - \mu(d) = \mu(c \ominus d) \in W$, from which $(a,b) \in U_W$.

Now let $(a,b) \in U_W$ and $c \leq a\Delta b$. By (1.1)(2), we can find $d \in L$ such that $a\Delta b = c \oplus d$ and therefore $a \vee b = (a \wedge b) \oplus (a\Delta b) = (a \wedge b) \oplus c \oplus d$. By (1.1)(4), we get $c = (a \vee b) \ominus \big((a \wedge b) \oplus d\big)$. Then $(a \wedge b) \oplus d \in [a \wedge b, a \vee b]$ and $\mu(c) = \mu(a \vee b) - \mu\big((a \wedge b) \oplus d\big) \in W$, from which $(a,b) \in A_W$.

(ii) Now we prove that $a\Delta b = (a \ominus c)\Delta(b \ominus c)$ for every $a,b \geq c$.
Set $d = a \ominus c$ and $e = b \ominus c$. By (1.1)(1), we get $d \vee e = (a \vee b) \ominus c$ and $d \wedge e = (a \wedge b) \ominus c$. Then $(a \ominus c)\Delta(b \ominus c) = d\Delta e = \big((a \vee b) \ominus c\big) \ominus \big((a \wedge b) \ominus c\big)$. Since $c \leq a \wedge b \leq a \vee b$, by (1.1)(5), we get $(a \ominus c)\Delta(b \ominus c) = (a \vee b) \ominus (a \wedge b) = a\Delta b$.

(iii) We prove that $a\Delta b = (c \ominus a)\Delta(c \ominus b)$ for every $a,b \leq c$.
By (1.1)(1),(6) and by the definition of $\ominus$, we get $(c \ominus a)\Delta(c \ominus b) = \big((c \ominus a) \vee (c \ominus b)\big) \ominus \big((c \ominus a) \wedge (c \ominus b)\big) = \big(c \ominus (a \wedge b)\big) \ominus \big(c \ominus (a \vee b)\big) = (a \vee b) \ominus (a \wedge b) = a\Delta b$.

(iv) By (ii) and (iii), we get $A_W \ominus \Delta \subseteq A_W$ and $\Delta \ominus A_W \subseteq A_W$. Then, by (i), $\mathcal{U}(\mu)$ is a D-lattice uniformity and $\{A_W : W$ is a 0-neighbourhood in $G\}$ is a base of $\mathcal{U}(\mu)$. □

As consequence of (3.2), we obtain — in a similar way as in [A-L] for modular functions on orthomodular lattices — the equivalence in any D-lattice between Brooks-Jewett and Vitali-Hahn-Saks theorems for modular measures.

First we recall the definitions which we need.

We say that a family K of G-valued measures on L is *uniformly exhaustive* if, for every orthogonal sequence $\{a_n\}$ in L, $\mu(a_n) \to 0$ in G uniformly for $\mu \in K$. If G' is another topological Abelian group and $\lambda \colon L \to G'$ is a measure, we say that K is λ*-equicontinuous* if, for every 0-neighbourhood W in G, there exists a 0-neighbourhood W' in G' such that, if $a \in L$ and $\lambda(b) \in W'$ for every $b \le a$, then $\mu(a) \in W$ for every $\mu \in K$. In particular, if $\mu \colon L \to G$ is a measure, we say that μ is exhaustive or λ-continuous if $K = \{\mu\}$ is exhaustive or λ-equicontinuous, respectively.

If λ is a modular measure, by (3.2), a base of 0-neighbourhoods in $\mathcal{U}(\lambda)$ is the family consisting of the sets $\{a \in L : \lambda(b) \in W$ for all $b \le a\}$, where W is a 0-neighbourhood in G. Then, in this case, if we denote by τ_∞ the topology of the uniform convergence in G^K, we have that K is λ-equicontinuous if and only if the function $\nu = (\mu)_{\mu \in K} \colon \big(L, \mathcal{U}(\lambda)\big) \to (G^K, \tau_\infty)$ is continuous and K is uniformly exhaustive if and only if $\nu \colon L \to (G^K, \tau_\infty)$ is exhaustive.

The notion of exhaustive measure given here is a particular case of the notion of x_0-exhaustive measure given in [D-P] and we need it in the proof of (3.7). The following result allows to prove in a standard way (see (3.4)) that this notion is equivalent to that of H. Weber in [W_5], which we need in the proof of (3.6).

LEMMA (3.3). *Let $a_0, a_1, \dots, a_n$ be in L such that $a_0 \le a_1 \le \dots \le a_n$ and set $b_i = a_i \ominus a_{i-1}$ for every $i \in \{1, \dots, n\}$. Then $\{b_1, \dots, b_n\}$ is orthogonal and $b_1 \oplus \dots \oplus b_n = a_n \ominus a_0$.*

Proof. Since $a_1 \ominus a_0 \le a_1 \le a_2$, by (1.1)(9), $b_1 \oplus b_2 = (a_1 \ominus a_0) \oplus (a_2 \ominus a_1)$ exists and it is equal to $a_2 \ominus \big(a_1 \ominus (a_1 \ominus a_0)\big) = a_2 \ominus a_0$. Then the assertion is true for $n = 2$. Now suppose that the assertion is true for $n - 1$. Since $b_1 \oplus \dots \oplus b_{n-1} = a_{n-1} \ominus a_0 \le a_{n-1} \le a_n$, by (1.1)(7), we get that $b_1 \oplus \dots \oplus b_n = (b_1 \oplus \dots \oplus b_{n-1}) \oplus (a_n \ominus a_{n-1})$ exists and it is equal to $a_n \ominus \big(a_{n-1} \ominus (a_{n-1} \ominus (a_{n-1} \ominus a_0)\big)\big) = a_n \ominus a_0$. □

PROPOSITION (3.4). *Let $\mu \colon L \to G$ be a measure. Then the following conditions are equivalent:*

(1) *μ is exhaustive.*
(2) *For every monotone sequence $\{a_n\}$ in L, $\{\mu(a_n)\}$ is a Cauchy sequence in G.*

(3) *For every increasing sequence* $\{a_n\}$ *in* L, $\{\mu(a_n)\}$ *is a Cauchy sequence in* G.

Proof.

(1) $\Longrightarrow$ (2):

(i) Let $\{a_n\}$ be an increasing sequence in L and, for each $n \in \mathcal{N}$, set $b_n = a_n \ominus a_{n-1}$, where $a_0 = 0$. By (3.3), $\{b_n\}$ is orthogonal. Then $\mu(a_n) - \mu(a_{n-1}) = \mu(b_n) \to 0$.

(ii) Now let $\{a_n\}$ be a decreasing sequence in L and set $b_n = a_n^{\perp}$. Then $\{b_n\}$ is an increasing sequence. By (i), $\mu(a_n) - \mu(a_{n-1}) = \mu(b_{n-1}) - \mu(b_n) \to 0$.

(2) $\Longrightarrow$ (3) is trivial.

(3) $\Longrightarrow$ (1): Let $\{a_n\}$ be an orthogonal sequence in L and set $b_1 = 0$, $b_n = \bigoplus_{i \le n-1} a_i$ for every $n \geq 2$. Then $\{b_n\}$ is increasing and $a_n \oplus b_n = b_{n+1}$. Therefore $\mu(a_n) = \mu(b_{n+1}) - \mu(b_n) \to 0$. □

PROPOSITION (3.5). *Let* $\mu \colon L \to G$ *be a measure and* $\mathcal{U}$ *a D-lattice uniformity on* L. *Then* μ *is continuous in* 0 *if and only if* μ *is uniformly continuous.*

Proof. Let W, W' be 0-neighbourhoods in G with $W' - W' \subseteq W$, and choose $U \in \mathcal{U}$ such that

$$(a,0) \in U \implies \mu(a) \in W'. \tag{$*$}$$

Let $V, V' \in \mathcal{U}$ such that $V \ominus \Delta \subseteq U$ and, for every $(a,b) \in V'$, $[a \wedge b, a \vee b] \times [a \wedge b, a \vee b] \subseteq V$ (see [W$_2$; 1.1.3]). Let $(a,b) \in V'$. We prove that $\mu(a) - \mu(b) \in W$. Set $c = a \ominus (a \wedge b)$ and $d = b \ominus (a \wedge b)$. Then $(c,0) = (a, a\wedge b) \ominus (a \wedge b, a \wedge b) \in V \ominus \Delta \subseteq U$. By $(*)$, we get $\mu(c) \in W'$. In similar way we obtain $\mu(d) \in W'$. Then $\mu(a) - \mu(b) = \mu(c) - \mu(d) \in W$. □

We say that L has *the Vitali-Hahn-Saks property* (*VHS-property*) if, for every topological Abelian group G', for every G'-valued modular measure λ on L and for every sequence $\{\mu_n : n \in \mathcal{N}\}$ of exhaustive λ-continuous G-valued modular measures on L which is pointwise convergent on L to a function μ_0, $\{\mu_n : n \in \mathcal{N} \cup \{0\}\}$ is λ-equicontinuous.

We say that L has *the Brooks-Jewett property* (*BJ-property*) if, for every sequence $\{\mu_n : n \in \mathcal{N}\}$ of exhaustive G-valued modular measures on L which is pointwise convergent on L to a function μ_0, $\{\mu_n : n \in \mathcal{N} \cup \{0\}\}$ is uniformly exhaustive.

Then, if we denote by $c(G)$ the space of all convergent sequences in G, by λ_p the topology of the pointwise convergence in $c(G)$ and by λ_∞ the topology of the uniform convergence in $c(G)$, it is clear (see (3.5)) that L has the VHS-property if and only if, for every topological Abelian group G' and for every modular measure $\lambda \colon L \to G'$, every exhaustive λ-continuous modular measure $\mu\colon L \to (c(G), \lambda_p)$ is λ-continuous with respect to λ_∞, and L has the BJ-property if and only if every exhaustive modular measure $\mu\colon L \to (c(G), \lambda_p)$ is λ_∞-exhaustive.

Using (3.4) and (3.5), the equivalence between VHS and BJ properties can be proved in a similar way as in [A-L; (1.2.15)].

THEOREM (3.6). *L has the VHS-property if and only if L has the BJ-property.*

Proof.

$\Longleftarrow$: The VHS-property follows from the BJ-property as consequence of the following result of [W_5; (6.2)], which holds in an arbitrary lattice: if $\mathcal{U}$ is a lattice uniformity and K is a uniformly exhaustive family of $\mathcal{U}$-continuous modular functions, then K is $\mathcal{U}$-equicontinuous.

$\Longrightarrow$: Let $\mu\colon L \to (c(G), \lambda_p)$ be an exhaustive modular measure and set $\mathcal{U} = \mathcal{U}(\mu)$. By [$W_5$; 3.6], $\mathcal{U}$ is exhaustive since μ is exhaustive. Since $\mu\colon (L,\mathcal{U}) \to (c(G), \lambda_p)$ is uniformly continuous, by the VHS-property, $\mu\colon (L,\mathcal{U}) \to (c(G), \lambda_\infty)$ is continuous, too. By (3.5), $\mu\colon (L,\mathcal{U}) \to (c(G), \lambda_\infty)$ is uniformly continuous. Then, since $\mathcal{U}$ is exhaustive, by (3.4) we obtain that $\mu\colon L \to (c(G), \lambda_\infty)$ is exhaustive, too. Therefore L has the BJ-property. □

In [D-P; (12.4)], the Brooks-Jewett theorem has been proved for measures on quasi-σ-complete D-posets (i.e. on D-posets L such that, for every orthogonal sequence $\{a_n\}$ in L, there exists a subsequence $\{a_n : n \in M\}$ such that $\bigoplus_{i\in I} a_i$ exists for every $I \subseteq M$). Then, by [D-P] and (3.6), we obtain the Vitali-Hahn-Saks theorem for modular measures on quasi-σ-complete D-lattices.

COROLLARY (3.7). *If L is a quasi-σ-complete D-lattice, then L has BJ and VHS properties.*

We conclude remarking that we can derive by [W_5; (4.1)] a characterization of modular measures with weakly relatively compact range.

PROPOSITION (3.8). *Let G be a complete locally convex linear space and $\mu\colon L \to G$ a modular measure. Then $\mu(L)$ is weakly relatively compact if and only if μ is exhaustive. In particular, if $G = \mathbb{R}^n$, μ is exhaustive if and only if μ is bounded.*

Proof. In [W_5; (4.1)], it is proved that, if L_0 is an arbitrary lattice, the equivalence between exhaustivity and relative weak compactness of the range holds for any modular function $\mu\colon L_0 \to G$ which satisfies the following condition: for every $a_0 \le \dots \le a_n$ in L_0 and for every $I \subseteq \{1, \dots, n\}$,

$$\sum_{i\in I} [\mu(a_i) - \mu(a_{i-1})] \in \mu(L_0). \tag{$*$}$$

Therefore we have only to observe that $(*)$ holds for any measure on L. For every $i \leq n$, set $b_i = a_i \ominus a_{i-1}$. By (3.3), for every $i \leq n$, $\{b_1, \dots, b_i\}$ is orthogonal and $b_1 \oplus \cdots \oplus b_i = a_i \ominus a_0$. Let $I \subseteq \{1, \dots, n\}$. Then $\sum\limits_{i \in I} [\mu(a_i) - \mu(a_{i-1})] = \sum\limits_{i \in I} [\mu(a_i \ominus a_0) - \mu(a_{i-1} \ominus a_0)] = \sum\limits_{i \in I} \Big[\mu\Big(\bigoplus\limits_{j \leq i} b_i\Big) - \mu\Big(\bigoplus\limits_{j \leq i-1} b_j\Big)\Big] = \sum\limits_{i \in I} \Big[\sum\limits_{j \leq i} \mu(b_j) - \sum\limits_{j \leq i-1} \mu(b_j)\Big] = \sum\limits_{i \in I} \mu(b_i) = \mu\Big(\bigoplus\limits_{i \in I} b_i\Big) \in \mu(L)$. □

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