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BAIRE FUNCTIONS AND THEIR RESTRICTIONS TO SPECIAL SETS

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(*Communicated by Ladilav Mišík*)

ABSTRACT. In this paper I compare some families of functions whose restrictions to special sets have continuity points or intervals of continuity points. Moreover, I investigate the Darboux property in some of these families.

Notations

Let $\mathbb{R}$ denote the set of all reals. A function $f\colon X \to \mathbb{R}$ $(\emptyset \neq X \subset \mathbb{R})$ is said to be *quasicontinuous* (*cliquish*) *at a point* $x \in X$ ([6] and respectively ([1])) if for every open neighbourhood U of x and for every positive number r there is an open set $V \subset U$ such that $V \cap X \neq \emptyset$ and $|f(t) - f(x)| < r$ for every point $t \in V \cap X$ (and $\operatorname{osc}_{V\cap X} f \leq r$).

If $f\colon X \to \mathbb{R}$ $(\emptyset \neq X \subset \mathbb{R})$ is an arbitrary function, then $C(f)$, $C_q(f)$ and respectively $C_c(f)$ denote the sets of all continuity points of f, of all quasicontinuity points of f, and of all points at which f is cliquish. For a nonempty set $Y \subset X$ the symbol $f|_Y$ denotes the restriction of f to Y. $\operatorname{Int}_X A$ denotes the relative interior of A in X and $\operatorname{cl} A$ denotes the closure of A. Let

$$K_1 = \{X \subset \mathbb{R};\ X \neq \emptyset\},$$

$$K_2 = \{X \subset \mathbb{R};\ X \neq \emptyset \text{ and } X \text{ is countable}\},$$

$$K_3 = \{X \in K_1;\ X \text{ is perfect}\},$$

$$K_4 = \{X \in K_1;\ X \text{ is the sum of a sequence of perfect sets}\},$$

$$A_i = \big\{f\colon \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ C(f|_X) \neq \emptyset\big\},$$

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(see [7] for $i = 4$ and [5] for $i = 1, 2$),

$$A_{iq} = \{f: \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ C_q(f|_X) \neq \emptyset\},$$
$$A_{ic} = \{f: \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ C_c(f|_X) \neq \emptyset\},$$
$$D_{ic} = \{f: \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ \operatorname{Int}_X C_c(f|_X) \neq \emptyset\},$$
$$D_{iq} = \{f: \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ \operatorname{Int}_X C_q(f|_X) \neq \emptyset\},$$
$$D_i = \{f: \mathbb{R} \to \mathbb{R};\ \text{for every } X \in K_i,\ \operatorname{Int}_X C(f|_X) \neq \emptyset\},$$

(see [3] for $i = 3$ and [4] for $i = 1, 2, 4$),

$$B_1 = \{f: \mathbb{R} \to \mathbb{R};\ f \text{ is of Baire } 1\},$$
$$Q = \{f: \mathbb{R} \to \mathbb{R};\ C_q(f) = \mathbb{R}\},$$
$$D = \{f: \mathbb{R} \to \mathbb{R};\ f \text{ has the Darboux property}\}.$$

In this paper I compare the above families A_i, A_{iq}, A_{ic}, D_i, D_{iq}, D_{ic} and I investigate the Darboux property in some of them.

Remark 1. The following inclusions are evident:

$$A_1 \subset A_2; \quad A_1 \subset A_4 \subset A_3; \quad D_1 \subset D_2; \quad D_1 \subset D_4 \subset D_3;$$
$$A_{1q} \subset A_{2q}; \quad A_{1q} \subset A_{4q} \subset A_{3q}; \quad D_{1q} \subset D_{2q}; \quad D_{1q} \subset D_{4q} \subset D_{3q};$$
$$A_{1c} \subset A_{2c}; \quad A_{1c} \subset A_{4c} \subset A_{3c}; \quad D_{1c} \subset D_{2c}; \quad D_{1c} \subset D_{4c} \subset D_{3c};$$

$$\begin{matrix} D_i \subset A_i; \\ D_{iq} \subset A_{iq}; \\ D_{ic} \subset A_{ic}; \end{matrix} \quad \begin{matrix} A_i \subset A_{iq} \subset A_{ic}; \\ D_i \subset D_{iq} \subset D_{ic}; \end{matrix} \quad (i = 1, 2, 3, 4).$$

THEOREM 1. $A_{3c} = A_{3q} = A_3 = B_1$.

Proof. The equality $A_3 = B_1$ follows from the well-known Baire Theorem. The inclusion $A_3 \subset A_{3q} \subset A_{3c}$ follows from Remark 1. We shall prove that $A_{3c} \subset B_1$. If $f \in A_{3c}$ and $X \subset \mathbb{R}$ is a perfect set then there is a sequence of open intervals I_n, $n = 1, 2, \ldots$, such that:

$\operatorname{cl} I_{n+1} \subset I_n$, $n = 1, 2, \ldots$;
the diameter $d(I_n)$ of the interval I_n is less than $1/n$, $n = 1, 2, \ldots$;
$I_n \cap X \neq \emptyset$, $n = 1, 2, \ldots$;
$\operatorname{osc}_{I_n \cap X} f < 1/n$, $n = 1, 2, \ldots$.

The intersection $X \cap \bigcap_{n=1}^{\infty} I_n = \bigcap_{n=1}^{\infty} (X \cap \mathrm{cl}\, I_n)$ is a singleton set $\{x\} \subset X$ and the restricted functions $f|_X$ is continuous at x. So, $C(f|_X) \neq \emptyset$ and $f \in B_1$. This completes the proof.

In [5] it is proved that $A_1 = A_2$. The following theorem is true:

THEOREM 2. $A_{1q} = A_{2q}$; $D_{1q} = D_{2q}$; $D_1 = D_2$.

Proof. The inclusions $A_{1q} \subset A_{2q}$, $D_{1q} \subset D_{2q}$, and $D_1 \subset D_2$ follow from Remark 1. We shall show the inclusion $A_{2q} \subset A_{1q}$. Let $f \in A_{2q}$ and let $X \subset \mathbb{R}$ be a nonempty set. If there is an isolated point x in X then $C_q(f|_X) \neq \emptyset$. Suppose that X is dense in itself. There is a countable set $Y \subset X$ such that

(i) $\mathrm{cl}\{(t, f(t));\ t \in Y\} \supset \{(t, f(t));\ t \in X\}$.

Since $f \in A_{2q}$, there is a point $x \in Y \subset X$ at which the restricted function $f|_Y$ is quasicontinuous. We shall show that the restricted function $f|_X$ is quasicontinuous at x. Let $r > 0$ be a number and let $U \ni x$ be an open set. Since $f|_Y$ is quasicontinuous at x there is an open interval $I \subset U$ such that $I \cap Y \neq \emptyset$ and

(ii) $|f(t) - f(x)| < r/2$ for each $t \in I \cap Y$.

If there is a point $u \in I \cap X$ with $|f(u) - f(x)| \geq r$, then it follows from (i) that there is a point $v \in I \cap Y$ such that $|f(v) - f(x)| > r/2$, in contradiction with (i). So,

$$|f(t) - f(x)| < r \qquad \text{for every} \quad t \in I \cap X,$$

and $x \in C_q(f|_X)$. Thus, $A_{2q} \subset A_{1q}$ and consequently, $A_{1q} = A_{2q}$. Now, we will show the inclusion $D_{2q} \subset D_{1q}$. Let $f \in D_{2q}$ and let $X \subset \mathbb{R}$ be a nonempty set. As above we can suppose that X is dense in itself and we can define a countable set $Y \subset X$ such that (i). Since $f \in D_{2q}$, there is an open interval I such that $\emptyset \neq I \cap Y \subset C_q(f|_Y)$. The same as in the proof of the inclusion $A_{2q} \subset A_{1q}$ we show that $I \cap Y \subset C_q(f|_X)$. Fix a point $x \in I \cap X$, $r > 0$, and an open set $U \ni x$. It follows from (i) that there is a point $u \in I \cap U \cap Y$ such that $|f(u) - f(x)| < r/2$. Then the restricted function $f|_X$ is quasicontinuous at u and there is an open interval $J \subset I \cap U$ such that $J \cap X \neq \emptyset$ and $|f(t) - f(u)| < r/2$ for each $t \in J \cap X$.

Consequently,

$$|f(t) - f(x)| \leq |f(t) - f(u)| + |f(u) - f(x)| < r/2 + r/2 = r$$

for each $t \in J \cap X$, and $x \in C_q(f|_X)$. So, $I \cap X \subset C_q(f|_X)$, and $D_{2q} \subset D_{1q}$.

The proof of the equality $D_1 = D_2$ is similar.

THEOREM 3. $A_{ic} = D_{ic} = B_1$ *for* $i = 1, 2, 3, 4$.

Proof. From Theorem 1 we have the equality $A_{3c} = B_1$. Moreover, by Remark 1,

$$D_{1c} \subset A_{1c} \subset A_{3c}; \quad D_{1c} \subset D_{4c} \subset D_{3c} \subset A_{3c};$$
$$D_{1c} \subset D_{4c} \subset A_{4c} \subset A_{3c}; \quad \text{and} \quad A_{1c} \subset A_{2c}.$$

Thus it suffices to prove that $A_{3c} \subset D_{1c}$ and $A_{2c} \subset A_{1c}$. We start from the proof of the inclusion $A_{3c} \subset D_{1c}$. Let $f \in A_{3c}$ and let $X \subset \mathbb{R}$ be a nonempty set. If there is an isolated point in X, then $\mathrm{Int}_X C_c(f|_X) \neq \emptyset$. So, we suppose that X is dense in itself. Then $\mathrm{cl}\, X$ is a perfect set and $f|_{\mathrm{cl}\, X}$ is cliquish at each point $x \in \mathrm{cl}\, X$. Consequently,

$$\mathrm{cl}\, X = C_c(f|_{\mathrm{cl}\, X}), \qquad X = C_c(f|_X),$$

and

$$\mathrm{Int}_X C_c(f|_X) \neq \emptyset.$$

Thus $f \in D_{1c}$.

The proof of the inclusion $A_{2c} \subset A_{1c}$ is similar to the proof of the inclusion $A_{2q} \subset A_{1q}$ in the proof of Theorem 2.

It is known that $A_1 \neq A_4 \neq B_1$ ([7]) and $D_1 \neq B_1$ ([3]).

THEOREM 4. $A_{1q} \neq A_4$; $A_{4q} \neq B_1$.

Proof. Let (w_n) be an enumeration of all rationals such that $w_n \neq w_m$ for $n \neq m$, $n, m = 1, 2, \dots$. The function

$$g(x) = \begin{cases} 1/n & \text{for } x \in w_n, \ n = 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $A_4 - A_{1q}$, since for $X = \{w_n; n = 1, 2, \dots\}$, $C_q(g|_X) = \emptyset$.

Let (X_n) be a sequence of nowhere dense perfect sets such that $X_n \cap X_m = \emptyset$ for $n \neq m$ $(n, m = 1, 2, \dots)$ and $X = \bigcup_n X_n$ is dense. The function

$$h(x) = \begin{cases} 1/n & \text{for } x \in X_n, \ n = 1, 2, \dots, \\ 0 & \text{otherwise} \end{cases}$$

belongs to $B_1 - A_{4q}$, since $C_q(h|_X) = \emptyset$.

THEOREM 5. $D_1 = D_2 = D_3 = D_4$; $D_{1q} = D_{2q} = D_{3q} = D_{4q}$.

Proof. By Remark 1, $D_1 \subset D_4 \subset D_3$, and $D_{1q} \subset D_{4q} \subset D_{3q}$. By Theorem 2, $D_1 = D_2$ and $D_{1q} = D_{2q}$. So, it suffices to prove that $D_3 \subset D_1$ and $D_{3q} \subset D_{1q}$. Let $f \in D_{3q}$ and let $X \subset \mathbb{R}$ be a nonempty set. If X contains an isolated point then $\operatorname{Int}_X C_q(f|_X) \neq \emptyset$. Suppose that X is dense in itself. Then $\operatorname{cl} X$ is a perfect set. Since $f \in D_{3q}$, there is an open interval I such that $I \cap X \neq \emptyset$ and $I \cap \operatorname{cl} X \subset C_q(f|_{\operatorname{cl} X})$. Evidently, $I \cap X \subset C_q(f|_X)$. Thus $\operatorname{Int}_X C_q(f|_X) \neq \emptyset$ and $f \in D_{1q}$. Consequently, $D_{3q} \subset D_{1q}$. The proof of the inclusion $D_3 \subset D_1$ is similar.

THEOREM 6. $A_{1q} = D_{3q}$.

Proof. Since $D_{3q} = D_{1q} \subset A_{1q}$, the inclusion $D_{3q} \subset A_{1q}$ is proved.

Now, let $f \in A_{1q}$. If $f \notin D_{3q}$, then there is a perfect set $X \subset \mathbb{R}$ such that $\operatorname{Int}_X C_q(f|_X) = \emptyset$. Let $Y \subset X - C_q(f|_X)$ be a countable set dense in X. Since $f \in A_{1q}$, the restricted function $f|_Y$ is quasicontinuous at a point $u \in Y$. The function $f|_X$ is not quasicontinuous at u. There is a positive number r such that

(i) $\left(\operatorname{cl}\{(t, f(t));\ t \in C(f|_X)\}\right) \cap \left([u-r, u+r] \times [f(u)-2r, f(u)+2r]\right) = \emptyset$.

Since $u \in C_q(f|_Y)$, there is an open interval $I \subset (u - r, u + r)$ such that $I \cap Y \neq \emptyset$ and $|f(t) - f(u)| < r$ for every point $t \in I \cap Y$. The set $C(f|_X)$ is dense in X. Thus there is a point $v \in (I \cap X) \cap C(f|_X)$. Let $J \subset I$ be an open interval such that $v \in J$ and $|f(t) - f(v)| < r$ for each point $t \in J \cap X$. Since the set Y is dense in X, there is a point $w \in J \cap Y$. Then we have

$$|f(w) - f(u)| < r, \quad |f(w) - f(v)| < r, \quad \text{and}$$

$$|f(u) - f(v)| \le |f(u) - f(w)| + |f(w) - f(v)| < r + r = 2r,$$

in contradiction with (i). So, $f \in D_{3q}$, and the proof is complete.

Problem.

(1) $D_3 = D_{3q}$?

(2) $A_4 = A_{4q}$?

THEOREM 7. $DA_{4q} \subset Q$.

Proof. Suppose that there is a function $f \in DA_{4q} - Q$. Then there is a point $x \in \mathbb{R}$ such that

$$(x, f(x)) \notin \operatorname{cl}\{(t, f(t));\ t \in C(f)\}.$$

Let $r > 0$ be such that

$$\big([x-r,\ x+r] \times [f(x)-r,\ f(x)+r]\big) \cap \mathrm{cl}\big\{(t, f(t))\,;\ t \in C(f)\big\} = \emptyset\,.$$

Since $f \in B_1$, the set $C(f)$ is dense. Consequently, the sets $U = \big\{t \in \mathbb{R};\ |t-x| < r$ and $|f(t)-f(x)| < r/2\big\}$ and $\mathrm{cl}\,U$ are nowhere dense. Since $f \in DB_1$, the set U is c-dense in itself ([2]) and the set $\mathrm{cl}\,U$ is perfect. From the Darboux property of f it follows that $|f(u) - f(x)| \geq r/2$ for each point $u \in \mathrm{cl}\,U$ being the end of a component of the set $(x-r,\ x+r) - \mathrm{cl}\,U$. Since $f \in A_{4q}$ and $U \in K_4$, there is a point $w \in U$ at which the function $f|_U$ is quasicontinuous. Evidently, $|f(w) - f(x)| < r/2$. Thus, there is an interval $I \subset (x-r,\ x+r)$ such that $I \cap U \neq \emptyset$ and $|f(t) - f(x)| < r_1 < r/2$ for each $t \in I \cap U$.

But the set

$$V = \big\{t \in I \cap \mathrm{cl}\,U\,;\ t \text{ is the end of a component of the set } I - \mathrm{cl}\,U\big\}$$

is dense in $I \cap \mathrm{cl}\,U$ and $|f(t) - f(x)| \geq r/2$ for each $T \in V$, thus the restricted function $f|_{(I \cap \mathrm{cl}\,U)}$ is not continuous at a point $t \in I \cap \mathrm{cl}\,U$. This contradiction finishes the proof.

COROLLARY. *We have:*

$$DA_{1q} = DA_{2q} \subset DA_{4q} \subset Q\,;$$
$$DD_{iq} \subset DA_{4q} \subset Q\,, \qquad i = 1, 2, 3, 4\,.$$

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