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STOCHASTIC PROCESSES IN NUCLEAR SPACES: QUASI-MARTINGALES AND DECOMPOSITIONS

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ABSTRACT. We prove a Radon-Nikodým theorem for measures ranging in a special class of locally convex vector spaces, and deduce from it several decompositions for quasi martingales taking values in the same class of spaces.

1. Introduction

The modern approach to Stochastic integration follows the lines of the Theory of vector integration, as outlined in [2], [9], [10]. Indeed, given a "good" process X taking values in a Banach space E, its *stochastic measure* I_X is a L^1_E-valued σ-additive measure, defined in the predictable σ-field $\mathcal{P}$; thus, integrating any process Y with respect to X reduces to integrate Y (in a bilinear sense, in general) against the measure I_X.

Of course, a number of questions arises in this respect:
For instance, when does a process X have a stochastic measure I_X on the whole predictable σ-field $\mathcal{P}$?

Another question is:
Can a "good" process X be decomposed into "nice" processes in such a way that the integration with respect to X becomes "easier"?

A further investigation concerns the possibility of extending the results to processes ranging in a locally convex space, in particular nuclear-valued ones.

In the case of Banach-valued processes X, there are basic results in [1], [2], [3], though the space E is often assumed containing no copies of c_0. Other useful results concerning the existence of decompositions are found in [5], where the *weak* (namely Pettis) integral is involved.

As to nuclear-valued processes, some interesting results are due to Ustunel [12], and to Brooks [3].

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The purpose of this paper is to extend some of the results of [5] to the locally convex case, and then to deduce the decompositions for some kind of nuclear-valued processes. To this aim, a basic role is played by a Radon-Nikodým theorem, already established in [4], and here adapted.

In the next section, we define the basic class of processes we are interested in, namely the weak Quasi-Martingales (w.q.MG), and give a characterization for them in terms of their Doléans measure, in a fashion similar to that of [9].

In the third section, we establish some Radon-Nikodým theorems, and deduce from them a Riesz-type decomposition for a general w.q.MG. In the subsequent sections we restrict our attention to nuclear-valued stochastic processes, obtaining for them a number of "strong" results, and some decomposition theorems.

2. Generalities

We shall denote by E any quasi-complete locally convex Hausdorff linear space, and by $(\Omega, \mathcal{F}, P)$ any probability space.

We refer to the books [1] for the notations concerning locally convex spaces, and [9] for definitions and notations relative to stochastic processes.

Let $X : \Omega \times]0, +\infty[\to E$ be any stochastic process, adapted to some filtration $\{\mathcal{F}_t\}_{t>0}$, $\mathcal{F}_t \subset \mathcal{F}$.

We shall assume that $X_t = X(\cdot, t)$ is Pettis integrable with respect to P (namely there exists a P-continuous measure $\varphi_t : \mathcal{F}_t \to E$ satisfying

$$x^*(\varphi_t(A)) = \int_A x^*(X_t) \, \mathrm{d}P$$

for every $A \in \mathcal{F}_t$ and for every $x^* \in E^*$: in this case we write $\varphi_t(A) := \widetilde{\int_A} X_t \, \mathrm{d}P$).

Throughout this paper the symbol $\widetilde{\int}$ will denote the Pettis integral.

Finally, we require that $X_\infty = 0$.

Definition 2.1. We say that X is a *weak Quasi-Martingale* (w.q.MG) if, for any neighbourhood U of 0 in E, there exists a constant $H = H(U) > 0$ such that

$$\sum_{i=1}^{n-1} \left\| E\big(x^*(X_{t_{i+1}} - X_{t_i}) | \mathcal{F}_{t_i}\big) \right\|_1 < H \tag{1}$$

whenever x^* belongs to the polar U^0 of U, and for every decomposition of $]0, +\infty[$, say $0 < t_1 < \cdots < t_n \leq +\infty$.

DEFINITION 2.2. Let $\mathcal{R}$ denote the set of all predictable rectangles. Set

$$\mu_X\big(]s,t] \times F\big) = \widetilde{\int_F} (X_t - X_s)\,\mathrm{d}P$$

for all $s,t \in]0,+\infty]$, and $F \in \mathcal{F}_s$. It is well known that μ_X extends uniquely to the *predictable algebra* $\mathcal{U}$ generated by $\mathcal{R}$, thus becoming an E-valued finitely additive measure: the *Doléans measure* of X.

In particular, $\mu_X\big(]t,+\infty] \times F\big) = -\widetilde{\int_F} X_t\,\mathrm{d}P$ for all $F \in \mathcal{F}_t$ and $t > 0$.

In case $\mu_X = 0$, then X will be said to be a weak *martingale* (w.MG).

Our first result allows us to characterize weak Quasi-Martingales in terms of their Doléans measures.

THEOREM 2.3. *The following are equivalent:*

1. *X is a w.q.MG.*
2. *μ_X is bounded on $\mathcal{U}$.*

Furthermore, if 1. *or* 2. *occur, for every neighbourhood U of* 0 *in E such that $U = U^{00}$, we have:*

$$\sup\{p_U(x):\ x \in \mu_X(\mathcal{U})\} \le H(U) \le 4\sup\{p_U(x):\ x \in \mu_X(\mathcal{U})\},$$

(Here, $\mu_X(\mathcal{U})$ denotes the range of μ_X, and p_U is the Minkowski functional of U).

Proof. First, we prove the implication 1. $\Longrightarrow$ 2.

Let us denote by R any element of $\mathcal{U}$, and set $R = \bigcup_{i=1}^{n}]s_i,t_i] \times F_i$, where $F_i \in \mathcal{F}_{s_i}$ for all i.

Without loss of generality, we shall assume that the rectangles $]s_i,t_i] \times F_i$ are pairwise disjoint. Now, let us denote by $\mathcal{I}_E(0)$ the set of all neighbourhoods U of 0 in E satisfying $U = U^{00}$. If X is any w.q.MG, for any $U \in \mathcal{I}_E(0)$ we find

$$\begin{aligned}
\left|x^*(\mu_X(R))\right| &= \left|\sum \int_{F_i} x^*(X_{t_i} - X_{s_i})\,\mathrm{d}P\right| \\
&= \left|\sum \int_{F_i} E(x^*(X_{t_i} - X_{s_i})|\mathcal{F}_{s_i})\,\mathrm{d}P\right| \\
&\le \sum \left\|E(x^*(X_{t_i} - X_{s_i})|\mathcal{F}_{s_i})\right\|_1 \le H(U)
\end{aligned}$$

for every $x^* \in U^0$. Hence $\mu_X(R) \in H(U)U^{00}$ for each $R \in \mathcal{U}$, and so $\mu_X(\mathcal{U}) \subset H(U)U$. This shows that μ_X is bounded, and $p_U(x) \le H(U)$ for all $x \in \mu_X(\mathcal{U})$.

We now turn to the converse implication.

Let $U \in \mathcal{I}_E(0)$ be fixed. Then, there exists $K \in \mathbb{R}^+$ such that $\mu_X(U) \subset KU$. Fix $x^* \in U^0$, and let $0 < t_0 < t_1 < \cdots < t_{n-1} < t_n$ be a fixed decomposition of $\mathbb{R}^+$. For any index i, we have

$$\begin{aligned} S_i := \left\|E\left(x^*(X_{t_{i+1}} - X_{t_i})|\mathcal{F}_{t_i}\right)\right\|_1 &\leq 2\sup\left\{ \left| \int_F x^*(X_{t_{i+1}} - X_{t_i}) \, \mathrm{d}P \right| : \ F \in \mathcal{F}_{t_i} \right\} \\ &= 2\sup\left\{ \left|x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F\right)\right)\right| : \ F \in \mathcal{F}_{t_i} \right\} \\ &\leq 2K , \end{aligned}$$

as $x^* \in U^0$. So, if we fix $\varepsilon > 0$, there exists $F_i \in \mathcal{F}_{t_i}$ such that

$$S_i \leq 2\left|x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F_i\right)\right)\right| + \frac{\varepsilon}{n}$$

and thus

$$\begin{aligned} \sum_{i=1}^n S_i &\leq 2\sum_{i=1}^n \left|x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F_i\right)\right)\right| + \varepsilon \\ &= 2\sum_{i\in J} x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F_i\right)\right) - 2\sum_{i \in J'} x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F_i\right)\right) + \varepsilon , \end{aligned}$$

where J is the set of all indexes i, such that $x^*\left(\mu_X\left(]t_i, t_{i+1}] \times F_i\right)\right)$ is positive, and J' is the set of remaining indexes. Setting

$$G_1 = \bigcup_{i\in J}\left(]t_i, t_{i+1}] \times F_i\right), \qquad G_2 = \bigcup_{i\in J'}\left(]t_i, t_{i+1}] \times F_i\right),$$

we find

$$\sum_{i=1}^n S_i \leq 2\left|x^*\left(\mu_X(G_1)\right)\right| + 2\left|x^*\left(\mu_X(G_2)\right)\right| \leq 4K + \varepsilon .$$

Since ε is arbitrary, we get $\sum\limits_{i=1}^n S_i \leq 4K$ for all $x^* \in U^0$, where K satisfies $\mu_X(\mathcal{U}\ ) \subset KU$.

Therefore $H(U) \leq 4K$, whenever $K > \sup\{p_U(x) : x \in \mu_X(\mathcal{U})\}$, and so

$$H(U) \leq 4\sup\{p_U(x) : \ x \in \mu_X(\mathcal{U})\} .$$

This shows that X is a w.q.MG, and the last inequality of the assertion. □

3. Riesz decomposition

In this section we shall find a Riesz-type decomposition theorem for an E-valued w.q.MG X.

Proposition 3.1. *Assume that E is weakly sequentially complete, and let X be a w.q.MG taking values in E. Then there exists a finitely additive measure*

$$\mu^*: \bigcup_{s>0} \mathcal{F}_s \to E,$$

such that:

1. $\lim\limits_{t\to\infty} \mu_X\big(]t,\infty] \times F\big) = \mu^*(F)$ *weakly in E for any $F \in \bigcup \mathcal{F}_s$;*
2. $\mu^*\big|_{\mathcal{F}_s}$ *is σ-additive for each $s > 0$, and P-continuous in the weak sense.*

Proof. Let us fix $F \in \bigcup\limits_{s>0} \mathcal{F}_s$, and let $\tau > 0$ be such that $F \in \mathcal{F}_\tau$. As X is a w.q.MG, for each $x^* \in E^*$ the mapping

$$t \mapsto |x^*(\mu_X)|\big(]t,+\infty] \times F\big)$$

is Cauchy, as $t \to +\infty$. Then the limit

$$\lim_{t\to+\infty} x^*(\mu_X)\big(]t,+\infty] \times F\big) = \alpha(x^*, F)$$

exists in $\mathbb{R}$.

Since E is weakly sequentially complete, this yields a map $\mu^*: \mathcal{F}_\tau \to E$ such that:

$$\alpha(x^*, F) = x^*(\mu^*(F)) \qquad \text{for all} \quad x^* \in E^*, \text{ and } F \in \mathcal{F}_\tau.$$

As τ is arbitrary, this shows the first part of the assertion. The second part follows from the Vitali-Hahn-Saks theorem, when one restricts to any σ-field $\mathcal{F}_s$, because $\mu^*(F) = \lim\limits_{n\to\infty} \Big(-\int_F X_n \, dP\Big)$, in the weak sense, for all $F \in \mathcal{F}_s$. □

In order to get a Riesz-type decomposition, we shall obtain a density of $\mu^*\big|_{\mathcal{F}_s}$ with respect to P for all $s > 0$. An existence theorem for such densities can be found, strengthening the hypotheses on E.

Definition 3.2. We shall say that the space E enjoys *property* (SP)* if, for every bounded set $B \subset E$, the space $E_b^*(B^0)$ is separable (see [11] for notations).

Beyond (SP)* we shall also require that E is semireflexive, a condition stronger than the weak sequential completeness.

THEOREM 3.3 (RADON-NIKODÝM). *Let us assume that E is semireflexive, and enjoys property* (SP)*. *Let $\lambda: \mathcal{A} \to [0, +\infty[$ be any σ-additive measure on some σ-algebra $\mathcal{A}$, and let $\mu: \mathcal{A} \to E$ denote any E-valued measure, weakly absolutely continuous with respect to λ. A sufficient condition for the existence of a Pettis-type derivative $\frac{d\mu}{d\lambda}$ is that there exists an increasing sequence $(\Omega_n)_{n-1}^{\infty}$ of elements in $\mathcal{A}$ such that $\lambda(\Omega \setminus \Omega_n) \downarrow 0$, and the sets*

$$S_n = \left\{ \tfrac{\mu(A)}{\lambda(A)} : \ \lambda(A) > 0, \ A \subset \Omega_n, \ A \in \mathcal{A} \right\}$$

are bounded in E for all n.

Proof. In view of the hypotheses concerning $\lambda(\Omega_n)$, it will suffice to prove the theorem just restricting λ and μ to each Ω_n. In other words, without loss of generality we can assume that

$$S := \left\{ \tfrac{\mu(A)}{\lambda(A)} : \ \lambda(A) > 0, \ A \subset \Omega, \ A \in \mathcal{A} \right\}$$

is bounded. Set $U = S^0$. Then, U is a neighbourhood of 0 in E_b^*. By (SP)*, we deduce that $E_b^*(U)$ is separable, and it is the dual space of $(E_{U^0} . p_{U^0})$. From the definition of S we deduce

$$p_{U^0}(\mu(A)) \le \lambda(A)$$

for all $A \in \mathcal{A}$, and therefore $\mu \ll \lambda$ in the space (E_{U^0}, p_{U^0}). We can then apply [6; Proposition 3.2] and deduce the existence of a Gel'fand derivative $\frac{d\mu}{d\lambda} : \Omega \to (E_b^*(U))^*$.

Now, if $z \in (E_{U^0}, p_{U^0})$, z can be viewed as a linear functional on E^* by putting $z(x^*) = z([x^*])$ for all $x^* \in E^*$, because $[x^*] = \{y \in E^* : \ p_U(x^* - y^*) = 0\}$.

We claim that $z \in (E_b^*)^*$. Indeed, we have

$$|z(x^*)| = |z([x^*])| \le \|z\| p_U(x^*) \le \|z\|$$

for all $x^* \in U$. Hence, z is bounded on U, and therefore it is continuous in the strong topology of E^*. As E^* is semireflexive, then $z \in E$. Thus, $\frac{d\mu}{d\lambda}$ ranges in E, which implies the assertion. □

THEOREM 3.4. *Let us assume that E is semireflexive, and enjoys property* (SP)*. *Let X be any E-valued w.q.MG satisfying the following condition:*

(2) *For every $s > 0$ there exists an increasing sequence $(\Omega_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $P(\Omega \setminus \Omega_n) \downarrow 0$, and such that the random variables X_t, $t > s$, are uniformly bounded in each Ω_n.*

Then, there exists a weak martingale (Y_t) *taking values in* E, *such that*

$$\lim_{t\to+\infty} \widetilde{\int_F} (X_t - Y_t)\, \mathrm{d}P = 0$$

weakly in E, *for every* $F \in \bigcup_{t>0} \mathcal{F}_t$.

P r o o f. By Proposition 3.1, the limit

$$\lim_{t\to+\infty} -\widetilde{\int_A} X_t\, \mathrm{d}P = \mu^*(A)$$

exists in the weak sense for every $A \in \bigcup_{t>0} \mathcal{F}_t$. Furthermore, we have also $\mu^*\big|_{\mathcal{F}_s} \ll P\big|_{\mathcal{F}_s}$ for all $s > 0$. Let $s > 0$ be fixed, and let $(\Omega_n)_{n=1}^{\infty}$ be the sequence given by (2). For every integer n, choose $A \subset \Omega_n$, $A \in \mathcal{F}_s$, with $P(A) > 0$. We have

$$\mu^*(A) = \lim_{t\to+\infty} -\widetilde{\int_A} X_t\, \mathrm{d}P.$$

Now, if $U \in \mathcal{I}_E(0)$, let $K > 0$ be any integer satisfying $X_t(\omega) \in KU$ for all $t > s$, $\omega \in \Omega_n$. Then

$$-\widetilde{\int_A} X_t\, \mathrm{d}P \in P(A)KU$$

as U is closed and convex.

Therefore, $\frac{\mu^*(A)}{P(A)} \in KU$. As A is arbitrary, we find that the hypotheses of Theorem 3.3 are satisfied by the measures $\mu^*\big|_{\mathcal{F}}$ and $P\big|_{\mathcal{F}}$, and so there exists a Pettis-type derivative $Y_s = -\frac{\mathrm{d}\mu^*|_{\mathcal{F}}}{\mathrm{d}P|_{\mathcal{F}}}$ for all $s > 0$. Of course, the process Y satisfies the condition $\mu_Y(F) = 0$ whenever $F \in \mathcal{U}$, and therefore it is a (weak) martingale (and also a w.q.MG, thanks to Theorem 2.3). Finally, if $F \in \mathcal{F}_\tau$ we have

$$\lim_{t\to+\infty} \widetilde{\int_F} (X_t - Y_t)\, \mathrm{d}P = -\mu^*(F) - \widetilde{\int_F} Y_t\, \mathrm{d}P = -\mu^*(F) + \mu^*(F) = 0,$$

and this concludes the proof. □

The last theorem can be viewed as a Riesz decomposition theorem, by using the next definition.

DEFINITION 3.5. A w.q.MG (V_t) is said to be a *weak Quasi-Potential* (w.Q-P) if

$$\lim_{t\to\infty} \widetilde{\int_F} V_t \, \mathrm{d}P = 0$$

holds in the weak sense, for all $F \in \bigcup_{s>0} \mathcal{F}_s$.

If (V_t) is simultaneously a weak Quasi-Potential and a weak Martingale, then the process $\big(x^*(V_t)\big)$ is equal to 0, up to modifications, for all $x^* \in E^*$. This is easily seen, and will be used in the next result.

COROLLARY 3.6 (RIESZ DECOMPOSITION). *In the same hypotheses as in Theorem 3.4, there exist a w.MG Y and a w.Q-P V such that*

$$X_t = Y_t + V_t$$

for all t. Moreover, Y and V are unique, up to modifications.

Proof. Let Y be the process found in the proof of Theorem 3.4, and set $V_t = X_t - Y_t$. Then, (V_t) is a w.q.MG, since it is the difference of two w.q.MG's, and, again by Theorem 3.4, it is a w.Q-P.

Y and V are unique, because of (2): indeed, if Y' and V' give another decomposition of X, then $Y' - Y$ is a w.Q-P and a w.MG, hence for each t and n there exists a P-null set $N \in \mathcal{F}_t$ such that $x^*(Y_t) = x^*(Y'_t)$ for each x^* in a countable dense subset of B_n^0 (where B_n is some bounded set in E, containing the range of the variables $X_\tau|_{\Omega_n}$ for all $\tau > t$.) Hence, $Y_t = Y'_t$ a.e. in Ω_n, and hence $Y_t = Y'_t$ a.e. in Ω. This also shows that $V_t = V'_t$ a.e., and the theorem is proved. □

4. The dual-nuclear case

The previous results can be strongly improved if the space E is of nuclear type. This is really important, because many useful processes take their values in spaces of distributions.

We shall assume that E is quasi-complete and dual-nuclear (namely, E_b^* is nuclear). Under these hypotheses, E is automatically semi-reflexive, and enjoys property $(\mathrm{SP})^*$.

LEMMA 4.1. *Assume that E is quasi-complete and dual-nuclear. Let Σ be any algebra on Ω, and $P\colon \Sigma \to [0, +\infty[$ any finitely additive measure. Assume also that $m\colon \Sigma \to E$ is any bounded finitely additive measure, weakly absolutely continuous with respect to P. Then, the absolute continuity holds in the strong sense, too.*

P r o o f. Fix any neighbourhood $U \in \mathcal{I}_E(0)$. As E_b^* is a Montel space (see [11]), U^0 is compact. Now, for any $x^* \in U^0$ there exists $\delta = \delta(x^*) > 0$ such that $P(H) < \delta \implies |x^*(m)|(H) < \frac{1}{2}$.

If we denote by R the (bounded) range of m, then the family

$$\left\{ \operatorname{int}\left(x^* + \tfrac{R^0}{2}\right) : x^* \in U^0 \right\}$$

is an open covering of U^0 (where by $\operatorname{int}(A)$ we denote the interior of A).

There exist hence some points $x_1^*, x_2^*, \dots, x_k^* \in U^0$ such that

$$U^0 \subset \bigcup_{i=1}^{k} \operatorname{int}\left(x_i^* + \frac{R^0}{2}\right).$$

Setting $\delta_0 = \min\{\delta(x_i^*) : 1 \le i \le k\}$, and choosing any $A \in \Sigma$, with $P(A) < \delta_0$, for each $x^* \in U^0$ we get $|x^*(m)|(A) < 1$, hence $m(A) \subset U^{00} = U$. As U is arbitrary, this shows that m is strongly absolutely continuous with respect to P. □

In view of this Lemma, and of [4; Theorem 6], one easily deduces the following results.

THEOREM 4.2. *Assume that E is quasi-complete and dual-nuclear. If $(\Omega, \mathcal{F}, P)$ is any σ-additive measure space, and $m\colon \mathcal{F} \to E$ is any measure weakly absolutely continuous with respect to P, then there exists a Bochner-integrable function $f\colon \Omega \to E$ satisfying:*

$$\int_A f \, \mathrm{d}P = m(A)$$

for all $A \in \mathcal{F}$.

COROLLARY 4.3. *Let E and $(\Omega, \mathcal{F}, P)$ be chosen as in Theorem 4.2. If $X\colon \Omega \to E$ is a Pettis-integrable function, then there exists a Bochner-integrable function $Y\colon \Omega \to E$ such that*

$$\int_F Y \, \mathrm{d}P = \widetilde{\int_F} X \, \mathrm{d}P,$$

for all $F \in \mathcal{F}$.

THEOREM 4.4 (RIESZ DECOMPOSITION). *Let E be as in Theorem 4.2, and let X be a w.q.MG. Then there exist a Bochner integrable martingale $M = (M_t)$, and a quasi-potential $V = (V_t)$, such that*

$$X_t = M_t + V_t$$

for all t.

5. Further decompositions

In this section, we shall deduce, for a general w.q.MG, decomposition theorems similar to those in [5], [9]. We need some preliminary results.

DEFINITION 5.1. Given an E-valued w.q.MG X on Ω, we say that X is (*weakly*) *of class* D if the set

$$\{X_T : \ T \text{ is a finite stopping time}\}$$

is (weakly) uniformly integrable.

X is a *weak local martingale* if there exists an increasing sequence of stopping times $(T_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} T_n(\omega) = +\infty$ and such that the stopped process (see [9]) X^{T_n} is a weak martingale for each n.

THEOREM 5.2. *Assume that μ_X is s-bounded. Then μ_X is σ-additive if and only if X is weakly of class D.*

P r o o f. We shall denote by $\mathcal{T}$ the set of all finite stopping times. Let us assume that μ_X is σ-additive. Then $x^*(\mu_X)$ is σ-additive for all $x^* \in E^*$. By [9; p. 92], it follows that $x^*(X)$ is of class D, hence

$$\{x^*(X_T) : \ T \in \mathcal{T}\}$$

is uniformly integrable for all x^*. Therefore, $\{X_T : \ T \in \mathcal{T}\}$ is weakly uniformly integrable.

On the converse, assume that X is weakly of class D. Then $x^*(\mu_X)$ is σ-additive, and so μ_X is weakly σ-additive. As E is quasi-complete, μ_X is σ-additive also in the strong sense. □

THEOREM 5.3. *If μ_X is purely finitely additive and admits a Rybakov control, namely an equivalent scalar finitely additive measure of the form $|x^*\mu_x|$ for some $x^* \in X^*$, then X is a weak local martingale.*

P r o o f. Choose $x^* \in E^*$ in such a way that $|x^*(\mu_X)|$ is a control for μ_X. Then $x^*(\mu_X)$ is purely finitely additive, hence there exists a sequence $(T_n)_{n=1}^{\infty}$ of stopping times, $T_n \uparrow \infty$, such that $x^*(T_n)$ is a (uniformly integrable) martingale for all n.

Let us consider the σ-algebra $\mathcal{P}$ of predictable sets: μ_X is purely finitely additive on $\mathcal{P}$, and equivalent to $x^*(\mu_X)$. Let T denote any stopping time of the sequence T_n, and set

$$\mathcal{P}_T = \{A \cap]0,T] : \ A \in \mathcal{P}\}.$$

Then $\mathcal{P}_T$ is a σ-algebra on $]0,T]$ and $x^*(\mu_X)$ is null on it.

As $\mu_X \ll x^*(\mu_X)$, $\mu_X|_{\mathcal{P}_T}$ is also null, and therefore X^T is a martingale. It follows then that X is a local martingale. □

COROLLARY 5.4. *If E satisfies the hypotheses of Theorem 4.2, and if μ_X is purely finitely additive, then X is a local martingale.*

P r o o f . As μ_X is bounded, it follows from [4; Corollary 1] that μ_X admits a Rybakov control, and therefore the assertion follows from Theorem 5.3. □

THEOREM 5.5. *In the same hypotheses of Theorem 3.4, there exists a decomposition*

$$X = U + Z$$

where U is a w.q.MG weakly of class D, and Z is a (weak) local martingale.

P r o o f . Let $\mu_X = \mu_1 + \mu_2$ be the Yosida-Hewitt decomposition of μ_X. So, μ_1 is σ-additive, μ_2 is purely finitely additive, $\mu_1 \ll \mu_X$, $\mu_2 \ll \mu_X$. By the Radon-Nikodým theorem, there are two processes, U and Z, such that

$$X = U + Z\,, \qquad \mu_1 = \mu_U\,, \qquad \mu_2 = \mu_Z\,.$$

The properties of U and Z follow from Theorem 5.2 and Theorem 5.3. □

THEOREM 5.6 (DOOB-MEYER DECOMPOSITION). *In the same hypotheses of Theorem 3.4, let U be any process whose Doléans measure μ_U is σ-additive. Then there exist two processes, Y and V, such that*

$$U = Y + V\,,$$

Y is a martingale weakly uniformly integrable, and weakly right continuous;
V is a process with weakly integrable variation.

P r o o f . Let us apply to U the Riesz decomposition (namely Corollary 3.6): $U = Y + V$. Thanks to [9; p. 91, 9.12], it follows that $x^*(Y)$ is uniformly integrable, and right-continuous, for all $x^* \in E^*$. Moreover, by the same result, $x^*(V)$ is a process with integrable variation. □

THEOREM 5.7 (FINAL DECOMPOSITION). *In the same hypotheses as in Theorem 5.6, if X is a w.q.MG, there exist four processes, Y, V, M, W such that $X = Y + V + M + W$ and*

a) *Y is a weakly uniformly integrable and weakly right-continuous martingale;*
b) *V is a process with weakly integrable variation;*
c) *M is a martingale, whose Doléans measure is purely finitely additive;*
d) *W is a Quasi-Potential and a local martingale.*

P r o o f . It suffices to decompose $X = U + Z$ according to Theorem 5.5, $U = Y + V$ according to Theorem 5.6, and $Z = M + W$ in the Riesz sense. □

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