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Mathematica Slovaca, Vol. 51 (2001), No. 1, 75--91

Persistent URL: http://dml.cz/dmlcz/131890

Math. Slovaca, **51** (2001), No. 1, 75–91

Mathematica Slovaca

MAPS PRESERVING CONVERGENCE OF SERIES

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(Communicated by Michal Zajac)

ABSTRACT. For fairly broad classes of topological vector spaces X and Y, there are given complete characterizations of those maps $f\colon X \to Y$ for which the induced transformation of series $\sum_n x_n \rightsquigarrow \sum_n f(x_n)$ preserves properties such as convergence, boundedness, absolute convergence, and unconditional convergence. For example, the following extension of a result of R. Rado for the case of normed spaces is shown: If X is metrizable, and Y is sequentially complete and contains no isomorphic copy of the space ω of all scalar sequences, then f preserves convergence of series if and only if f is additive and continuous in a neighborhood of zero.

0. Introduction

It was shown by Rado [Ra; Theorem 3] that a map f from one Banach space into another preserves convergence of series if and only if f is continuous and $\mathbb{R}$-linear in a neighborhood of zero. He also mentioned that his research had originated from a problem raised by D. J. White, who also found an alternative proof for the case of maps $f\colon \mathbb{R} \to \mathbb{R}$. The result for this latter case was rediscovered by Wildenberg [W] with quite a complicated proof, but a much simpler proof was soon provided by Smith [S]. In fact, Smith's proof is easily adaptable to the more general case treated by Rado and, essentially, is very close to Rado's proof. Also, apparently unaware of [Ra], the case of maps $f\colon \mathbb{R}^n \to \mathbb{R}^n$ has recently been considered by Kostyrko [K]. In another recent paper of Borsík, Červeňanský and Šalát [BČŠ], maps f preserving absolute convergence of real series have been characterized as those for which $|f(x)| \leqslant \mathrm{const} \cdot |x|$ in a neighborhood of zero. (The referee has pointed

2000 Mathematics Subject Classification: Primary 40A05, 46A16, 47H99.
Key words: topological vector space, convergent series, absolutely convergent series, unconditionally convergent series, map preserving convergence of series.

This research was partially supported by Komitet Badań Naukowych (State Committee for Scientific Research, Poland), Grant no. 2 P03A 05115.

out that also two newer publications, [DMŠ] and [G], are somewhat related to the subject matter of the present paper.)

The purpose of this paper is to extend characterizations of the above type to maps f between fairly general topological vector spaces (or even topological abelian groups, see the Final remark at the end of the paper). This is done in Sections 3, 4, and 5 which are concerned, roughly speaking, with arbitrary convergent series, absolutely convergent series, and unconditionally convergent series, respectively. Sections 1 and 2 are of auxiliary character. In particular, in Section 2 we discuss a so-called Property $(\overline{\omega})$ that has to be imposed on the range space in order for a Rado's type result to hold. We show that it is intimately related to the noncontainment of a copy of the space ω of all scalar sequences.

We refer the reader to [J] and [Ro] for basic information about topological vector spaces, F-seminorms, F-norms, etc. In what follows, we use the abbreviation TVS for *Hausdorff topological vector space(s)*.

Let X and Y be TVS. A map $f\colon X \to Y$ is said to *preserve convergence of series*, or to *send convergent series* (in X) *to convergent series* (in Y), if whenever a series $\sum_n x_n$ in X converges, so does its f*-transform*, i.e., the series $\sum_n f(x_n)$ in Y. Similar, self-explanatory terminology will also be used when series of other types, like Cauchy series, or absolutely convergent series, or unconditionally convergent (or unconditionally Cauchy) series are considered as "inputs" or "outputs" of the map f.

1. Locally additive maps

Let X and Y be TVS. We shall say that a map $f\colon X \to Y$ is *locally additive* if it is additive in a neighborhood U of zero in X in the sense that $f(u+v) = f(u) + f(v)$ whenever $u, v \in U$. Then, clearly, $f(0) = 0$ and

$$f(u_1 + \cdots + u_k) = f(u_1) + \cdots + f(u_k) \qquad (*)$$

whenever $u_1, \ldots, u_k \in U$ and $u_1 + \cdots + u_i \in U$ for $i = 1, \ldots, k-1$.

Moreover, assuming (as we may) that U is symmetric, we have $f(-u) = -f(u)$ and $f(u-v) = f(u) - f(v)$ for $u, v \in U$. In consequence, if f is continuous at zero, then it is (uniformly) continuous in U. Likewise, if f is sequentially continuous at zero, then it is sequentially continuous in U.

PROPOSITION 1. *Let X and Y be TVS, and let a map $f\colon X \to Y$ be locally additive and sequentially continuous at zero. Then f sends convergent series to convergent series, and Cauchy series to Cauchy series.*

Proof. Let U be a closed symmetric neighborhood of zero in X in which f is additive. Let $\sum_n x_n$ be a Cauchy series in X. Then there exists k such that for $m \geqslant l \geqslant k$

$$\sum_{n=l}^{m} x_n \in U, \qquad \text{whence} \quad \sum_{n=l}^{m} f(x_n) = f\Big(\sum_{n=l}^{m} x_n\Big). \tag{†}$$

Take any two sequences of indices (l_j) and (m_j) such that $l_j \leqslant m_j$ and $l_j \to \infty$ as $j \to \infty$. Then

$$\sum_{n=l_j}^{m_j} x_n \to 0 \qquad \text{as} \quad j \to \infty$$

and, for large j,

$$\sum_{n=l_j}^{m_j} f(x_n) = f\Big(\sum_{n=l_j}^{m_j} x_n\Big).$$

Hence, by the sequential continuity of f at zero,

$$\sum_{n=l_j}^{m_j} f(x_n) \to 0 \qquad \text{as} \quad j \to \infty.$$

This proves that the series $\sum_n f(x_n)$ is Cauchy in Y.

If the series $\sum_n x_n$ is convergent, let x denote the sum of the series $\sum_{n=k}^{\infty} x_n$. By (†) and since U is closed, $x \in U$. Moreover, $\sum_{n=k}^{m} f(x_n) = f\Big(\sum_{n=k}^{m} x_n\Big) \to f(x)$ as $m \to \infty$. Hence the series $\sum_n f(x_n)$ is convergent in Y. □

The following should be treated as a "folklore" result.

PROPOSITION 2. *Let X and Y be* TVS. *If a map $f\colon X \to Y$ is locally additive and continuous at zero, then there exists a unique continuous $\mathbb{R}$-linear operator $T\colon X \to Y$ such that f and T coincide in a neighborhood of zero in X.*

Proof. Let U be a balanced neighborhood of zero in X in which f is additive.

Fix any $x \in U$. From property $(*)$ above it follows that $f(x) = nf(x/n)$, or $f(x/n) = f(x)/n$ for all $n \in \mathbb{N}$. Since f is odd in U, applying $(*)$ one more time gives $f(rx) = rf(x)$ for any rational r with $|r| \leqslant 1$. In consequence, by the continuity of f,

$$f(rx) = rf(x) \qquad \text{whenever} \quad x \in U \text{ and } |r| \leqslant 1.$$

Now, take any $x \in X$ and let $a, b > 0$ be such that both x/a and x/b are in U. Then, assuming as we may that $c := b/a \leqslant 1$ and using the last property of f,

$$af(a^{-1}x) = af(cb^{-1}x) = bf(b^{-1}x).$$

Hence the formula

$$T(x) = af(a^{-1}x), \qquad \text{where} \quad a > 0 \text{ is any number such that } x/a \in U,$$

well-defines a map $T\colon X \to Y$, and it is obvious that $T(x) = f(x)$ for $x \in U$. Moreover, T is additive. In fact, let $u, v \in X$ and choose any number $a > 0$ such that all three elements $a^{-1}u$, $a^{-1}v$, and $a^{-1}(u+v)$ are in U. Then

$$T(u+v) = af\big(a^{-1}(u+v)\big) = af(a^{-1}u) + af(a^{-1}v) = T(u) + T(v).$$

Similarly, T is $\mathbb{R}$-homogeneous, thus $\mathbb{R}$-linear. The uniqueness of T is obvious. □

Remark. It is well known that a continuous additive map (or operator) between TVS is always $\mathbb{R}$-homogeneous, and thus $\mathbb{R}$-linear.

2. Spaces with Property $(\overline{\omega})$

We shall say that a TVS Y has *Property* $(\overline{\omega})$ if there is no sequence (y_n) in Y with nonzero terms such that the series $\sum_n a_n y_n$ is Cauchy for every sequence (a_n) of positive integers. The reason for considering such "strange" sequences will become apparent in the proof of Theorem 1 below.

An obvious example of a TVS that lacks Property $(\overline{\omega})$ is ω, the space of all scalar sequences with its usual product (or coordinatewise convergence) topology: Here the sequence (e_n) of unit vectors has the property that the series $\sum_n a_n e_n$ converges for every sequence (a_n) of scalars. Hence, if a TVS Y has Property $(\overline{\omega})$, then it contains no isomorphic copy of ω, nor even a copy of the subspace $\omega_0 = \operatorname{lin}(e_n)$ of ω. (In particular, no infinite product of nonzero TVS can have Property $(\overline{\omega})$.) We are going to show below that also the converse is true.

LEMMA 1. *Let (y_n) be a sequence in a TVS Y such that the series $\sum_n a_n y_n$ is Cauchy for every sequence (a_n) of positive integers. Then it is Cauchy for every sequence (a_n) of scalars.*

Proof. First note that the series $\sum_n a_n y_n$ must be Cauchy for every sequence (a_n) of integers. Next observe that this is equivalent to the following:

(cc) For every sequence (a_n) of integers and every sequence (I_j) of disjoint consecutive intervals in $\mathbb{N}$,
$$\sum_{n\in I_j} a_n y_n \to 0 \qquad \text{as} \quad j\to\infty .$$

We have to show that (cc) holds true if "integers" is replaced by "reals". Suppose it is not so. Thus there is a sequence (c_n) of reals for which one can find a sequence (I_j) of disjoint consecutive intervals in $\mathbb{N}$ such that
$$p\Big(\sum_{n\in I_j} c_n y_n\Big) > \varepsilon \qquad (j=1,2,\dots)$$
for some continuous F-seminorm p on Y and some number $\varepsilon > 0$. Then we can choose a sequence (b_n) of rationals with each b_n close enough to c_n to ensure that
$$p\Big(\sum_{n\in I_j} b_n y_n\Big) > \varepsilon \qquad (j=1,2,\dots).$$
For each j let $m_j \in \mathbb{N}$ be such that $a_n := m_j b_n$ is an integer for $n \in I_j$. Then
$$p\Big(\frac{1}{m_j}\sum_{n\in I_j} a_n y_n\Big) > \varepsilon \qquad (j=1,2,\dots)$$
so that
$$\frac{1}{m_j}\sum_{n\in I_j} a_n y_n \not\to 0 \qquad \text{as} \quad j\to\infty ,$$
which clearly contradicts (cc). □

We will need the following properties of the space ω; see [BPR] and [D].

(A) The general form of a continuous linear operator T from ω to a TVS Y is $T\big((a_n)\big) = \sum_n a_n y_n$, where (y_n) is a sequence in Y such that the series $\sum_n a_n y_n$ is convergent for every $(a_n) \in \omega$.

(B) If a continuous linear operator T from ω to a TVS Y has infinite dimensional range, then $T(\omega) \approx \omega$.

(C) ω is a minimal TVS; hence if a continuous linear operator T from ω to a TVS Y is one-to-one, then it is an isomorphic embedding.

LEMMA 2. *Let T be a continuous linear operator from ω to a* TVS *Y having an infinitely dimensional range. Then there exists an infinite subset M of $\mathbb{N}$ such that $T|_{\omega(M)}$ is an isomorphic embedding, where $\omega(M) := \{(a_n) \in \omega : (\forall n \notin M)(a_n = 0)\} \approx \omega$.*

Proof. Clearly, $y_n := T(e_n) \neq 0$ for infinitely many n. By (B), the subspace $W = T(\omega)$ of Y is isomorphic to ω. Fix a basis (w_k) of W (equivalent to the basis (e_k) of ω) and let (w_k^*) be the associated sequence of coefficient functionals. For each $w \in W$, let $s(w) = \{k : w_k^*(w) \neq 0\}$, the support of w relative to (w_k). Since the series $\sum_n a_n y_n$ converges in W for all scalar sequences (a_n), it is easy to see that

$$|\{n : k \in s(y_n)\}| < \infty \qquad \text{for all} \quad k \in \mathbb{N}.$$

Making use of this fact and proceeding by induction, one can easily construct infinite sequences $n_1 < n_2 < \ldots$ and $k_1 < k_2 < \ldots$ in $\mathbb{N}$ such that, denoting $z_j = y_{n_j}$,

$$k_j \in s(z_j) \quad \text{and} \quad \{1, \ldots, k_j - 1\} \cap s(z_j) = \emptyset \qquad \text{for} \quad j = 1, 2, \ldots.$$

Using these two properties it is not hard to see that whenever $\sum_j b_j z_j = 0$, then $b_j = 0$ for all j. In consequence, the set $M = \{n_j : j \in \mathbb{N}\}$ is as required: If $(a_n) \in \omega(M)$ and $T((a_n)) = 0$, then $(a_n) = 0$ so that $T|_{\omega(M)}$ is one-to-one, and it remains to apply (C). □

COROLLARY 1. *Let Y be a* TVS, *and let (y_n) be a sequence in Y with all $y_n \neq 0$ and such that the series $\sum_n a_n y_n$ is convergent for every sequence (a_n) of scalars. Then (y_n) has a subsequence (z_n) which is equivalent to the unit vector basis (e_n) of ω. That is, the linear operator $J\colon \omega \to Y$ defined by $J((a_n)) = \sum_n a_n z_n$ is an isomorphic embedding.*

Proof. First note that the linear span of (y_n) must be of infinite dimension. Next, define a continuous linear operator $T\colon \omega \to Y$ by $T((a_n)) = \sum_n a_n y_n$. Then $\dim T(\omega) = \infty$, and the assertion follows from Lemma 2. □

Remark. An infinite dimensional subspace Z of ω always contains a sequence (z_j) equivalent to the basis (e_n) of ω. In fact, since $\dim Z = \infty$, for every $k \in \mathbb{N}$ there must exist $0 \neq z \in Z$ such that $\{1, \ldots, k\} \cap s(z) = \emptyset$. (Here the support $s(z)$ is taken relative to (e_n).) From this, one can easily deduce the existence of a sequence $k_1 < k_2 < \ldots$ in $\mathbb{N}$ and (z_j) in Z satisfying the two conditions displayed above. Then $\sum_j a_j z_j$ converges for every $(a_j) \in \omega$, and $\sum_j a_j z_j = 0$ if and only if $(a_j) = 0$. Thus (z_j) is as required.

PROPOSITION 3. *For a* TVS Y *the following are equivalent.*

(a) Y *has Property* $(\overline{\omega})$.
(b) Y *has no subspace isomorphic to a dense (or just infinite dimensional) subspace of* ω.
(c) Y *has no subspace isomorphic to the subspace* ω_0 *of* ω.

Proof.
(a) $\Longrightarrow$ (b) holds by the Remark above, and (b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): Assuming (a) is false and making use of Lemma 1, we find a sequence $(y_n) \subset Y$ with nonzero terms such that the series $\sum_n a_n y_n$ is Cauchy for every sequence (a_n) of scalars. Viewing (y_n) as a sequence in $\widetilde{Y}$, the completion of Y, and applying Corollary 1, we get a subsequence (z_n) equivalent to the unit vector basis (e_n) of ω. Then the subspace $\operatorname{lin}(z_n) \subset Y$ is isomorphic to ω_0, contradicting (c). □

COROLLARY 2. *A sequentially complete* TVS Y *has Property* $(\overline{\omega})$ *if and only if* Y *contains no isomorphic copy of* ω.

Remark. A property of a TVS Y apparently stronger than Property $(\overline{\omega})$ is the following: For every sequence (y_n) in Y with nonzero terms there exists a continuous F-seminorm q on Y such that $\inf_n \sup_t q(ty_n) > 0$ or, equivalently, there is a balanced neighborhood V of zero in Y which contains none of the lines passing through 0 and y_n for $n = 1, 2, \ldots$. If $Y = (Y, \|\cdot\|)$ is an F-normed space, then this holds if and only if $\inf_{y \neq 0} \sup_t \|ty\| > 0$ (i.e., Y does not have "arbitrarily short lines"). When Y is an F-space, the latter condition is in turn equivalent to the noncontainment by Y of copies of ω (by an old result of Bessaga, Pełczyński and Rolewicz [BPR], see also [Ro]).

3. Maps preserving convergent or Cauchy series

PROPOSITION 4. *Let* X *and* Y *be* TVS. *If a map* $f: X \to Y$ *sends convergent series to Cauchy series, then* $f(0) = 0$ *and* f *is sequentially continuous at zero.*

Proof. It is obvious that $f(0) = 0$. To show that f is sequentially continuous at 0, take any null sequence (u_n). Then the series $u_1 + (-u_1) + \ldots + u_n + (-u_n) + \ldots$ converges in X (to 0), hence its f-transform is a Cauchy series in Y. In consequence $f(u_n) \to 0$. □

Recall that the topology of a metrizable TVS X can always be defined by an F-norm $\|\cdot\|$.

We now come to the first of our main results. The proof below is a modification of the arguments used in [Ra] and [S].

THEOREM 1. *Let X be a metrizable* TVS *and Y any* TVS *having Property* (ϖ). *Then for a map $f\colon X \to Y$ the following are equivalent:*

(a) *f sends convergent series to convergent series.*
(b) *f sends convergent series to Cauchy series.*
(c) *f is locally additive and continuous at zero.*

Proof.

(b) $\Longrightarrow$ (c): In view of Proposition 4 it remains to show that f is locally additive. Equivalently, we need to show that for the function

$$g(u,v) := f(u+v) + f(-u) + f(-v), \qquad u, v \in X,$$

there is a neighborhood U of zero in X such that $g(u,v) = 0$ for all $u, v \in U$. Suppose it is not so. Then there exist null sequences (u_j) and (v_j) in X such that $y_j := g(u_j, v_j) \neq 0$ for each j.

Choose any sequence (a_j) in $\mathbb{N}$, and consider the series $\sum\limits_n x_n$ in X whose terms come up in consecutive blocks A_j each of which in turn consists of a_j consecutive subblocks, each of the form

$$(u_j + v_j) + (-u_j) + (-v_j).$$

Note. Here, and in similar constructions below, no parentheses are used around "blocks".

Then the series $\sum\limits_n x_n$ converges (to 0) in X. By assumption (b), the series $\sum\limits_n f(x_n)$ is Cauchy in Y, hence so is the series $\sum\limits_j z_j$, where z_j denotes the sum of the block of the series $\sum\limits_n f(x_n)$ corresponding to A_j. Since $z_j = a_j y_j$, we see that the series $\sum\limits_n a_j y_j$ is Cauchy for every choice of $(a_j) \subset \mathbb{N}$, contradicting Property (ϖ) of Y.

(c) $\Longrightarrow$ (a) is contained in Proposition 1, and (a) $\Longrightarrow$ (b) is trivial. $\square$

Remark. Let $X = l_1$ with its weak topology, $Y = l_1$ with its norm topology, and let $f\colon X \to Y$ be the identity map. By the Schur property of l_1, f is a sequentially continuous linear map. Hence it sends convergent (Cauchy) series to convergent (Cauchy) series. However, it is not continuous at zero. Furthermore, also the map $g\colon X \to Y$ which coincides with f on the unit ball of l_1 and is zero elsewhere preserves convergent (Cauchy) series, although it is not locally additive. Thus the assumption of metrizability of X in Theorem 1 is essential.

COROLLARY 3. *Let X be a metrizable* TVS, *and let $Y = \prod_{i\in I} Y_i$ be a product of* TVS, *each having Property $(\overline{\omega})$. Then for a map $f = (f_i)\colon X \to Y$ the following are equivalent.*

(a) *f sends convergent series to convergent (or Cauchy) series.*

(b) *Each of the component maps $f_i\colon X \to Y_i$ is locally additive and continuous at zero.*

Remark. Note that a neighborhood of zero in which f_i is additive may, in general, depend on i. Hence f itself need not be locally additive. This happens, for example, when $f = (f_k)\colon X = (X, \|\cdot\|) \to \omega$, where each f_k vanishes for $\|x\| \leqslant 1/k$, and assumes arbitrary nonzero values for $\|x\| > 1/k$. In fact, in this case for every *null* sequence (x_n) in X the series $\sum_n f(x_n)$ is unconditionally convergent in ω. This follows easily from the fact that for each k only a finite number of vectors $f(x_n)$ may have a nonzero kth component.

We strengthen the above remark by showing that the implication (b) $\Longrightarrow$ (c) of Theorem 1 is simply false for each TVS Y without Property $(\overline{\omega})$.

PROPOSITION 5. *If a* TVS *Y does not have Property $(\overline{\omega})$, then for every nonzero* TVS *X there exists a map $f\colon X \to Y$ which is continuous at zero but not locally additive, and yet for every null sequence (x_k) in X the series $\sum_k f(x_k)$ is unconditionally Cauchy.*

Proof. By assumption, there is a sequence (y_n) in Y with nonzero terms such that

(cc_+) the series $\sum_n a_n y_n$ is Cauchy for every sequence (a_n) in $\mathbb{N}$.

We may assume that (y_n) is linearly independent. Fix a null sequence (u_n) in X with pairwise distinct nonzero terms. Define $f\colon X \to Y$ by setting $f(x) = y_n$ if $x = u_n$ $(n = 1, 2, \dots)$, and 0 otherwise. Clearly, f is continuous at zero, but it is not locally additive. Take any null sequence (x_k) in X. Without loss of generality it can be assumed that $\{x_k : k \in \mathbb{N}\} \subset \{u_n : n \in \mathbb{N}\}$. For each n let $m_n = |\{k : x_k = u_n\}|$. Fix any $N \in \mathbb{N}$ and denote by $r(N)$ the largest integer such that

$$|\{k :\ k \leqslant N \ \&\ x_k = u_n\}| = m_n \qquad \text{for} \quad 1 \leqslant n \leqslant r(N)\,.$$

Also, let

$$s(N) = \max\{n :\ x_k = u_n \ \text{for some}\ 1 \leqslant k \leqslant N\}\,.$$

Then $r(N) \leqslant s(N)$ and

$$\sum_{k=1}^{N} f(x_k) = \sum_{n=1}^{r(N)} m_n y_n + \sum_{n=r(N)+1}^{s(N)} a_n(N) y_n\,,$$

where $a_n(N)$ are some integers with $0 \leqslant a_n(N) < m_n$ for $r(N) < n \leqslant s(N)$. Evidently, both $r(N)$ and $s(N)$ tend nondecreasingly to infinity as $N \to \infty$. From (cc_+) it follows easily that

$$\sum_{n=r(N)+1}^{s(N)} a_n(N) y_n \to 0 \qquad \text{as} \quad N \to \infty .$$

Finally, by (cc_+) again, the series $\sum_n m_n y_n$ is Cauchy, hence so is the series $\sum_k f(x_k)$. □

Thus there seems to be no hope for any characterization of maps preserving convergence of series when the range space lacks Property $(\overline{\omega})$. Nonetheless, we have the following general result. We say that a series is *bounded* if the sequence of its partial sums is bounded.

THEOREM 2. *Let X be a metrizable* TVS *and Y any* TVS. *Then for a map $f\colon X \to Y$ the following are equivalent.*

(a) *f sends Cauchy series to bounded series.*
(b) *f sends Cauchy series to Cauchy series.*
(c) *f sends convergent series to Cauchy series.*
(d) *f sends bounded series to bounded series.*

Proof. Let $\|\cdot\|$ be an F-norm defining the topology of X.

(a) $\Longrightarrow$ (b): Suppose there is a Cauchy series $\sum_n x_n$ in X for which the series $\sum_n f(x_n)$ is not Cauchy in Y. Then there are strictly increasing sequences (k_j) and (m_j) in $\mathbb{N}$ such that $k_j \leqslant m_j < k_{j+1}$ and

$$p\Big(\sum_{n=k_j}^{m_j} f(x_n)\Big) > \varepsilon$$

for some continuous F-seminorm p on Y and some $\varepsilon > 0$. Since the series $\sum_n x_n$ is Cauchy, we may assume that

$$\Big\|\sum_{n=k}^{l} x_n\Big\| \leqslant \frac{1}{j^2} \qquad \text{whenever} \quad k_j \leqslant k \leqslant l\,, \quad j = 1, 2, \ldots . \tag{o}$$

Now, consider the series $\sum_{r=1}^{\infty} u_r$ whose terms come up in consecutive blocks A_j so that, for $j = 1, 2, \ldots$,

- A_{2j-1} consists of j consecutive subblocks, each of the form

$$x_{k_j} + x_{k_j+1} + \cdots + x_{m_j}\,,$$

- A_{2j} consists of j consecutive subblocks, each of the form

$$(-x_{k_j}) + (-x_{k_j+1}) + \cdots + (-x_{m_j}).$$

Using (o) it is easy to see that the series $\sum_r u_r$ converges to 0. However, denoting by z_j the sum of the block of the series $\sum_r f(u_r)$ corresponding to A_j, we have

$$p(j^{-1} z_{2j-1}) = p\Bigl(\sum_{n=k_j}^{m_j} f(x_n)\Bigr) > \varepsilon.$$

Hence the series $\sum_r f(z_r)$ is not bounded in Y, contradicting the assumption.

(b) $\Longrightarrow$ (c): Obvious.

(c) $\Longrightarrow$ (d): Suppose there is a bounded series $\sum_n u_n$ in X for which the series $\sum_n f(u_n)$ is not bounded in Y. Then there are strictly increasing sequences (k_j) and (m_j) in $\mathbb{N}$ such that $k_j \leqslant m_j < k_{j+1}$ and the sequence

$$w_j := \sum_{n=k_j}^{m_j} f(u_n), \qquad j = 1, 2, \ldots,$$

is not bounded in Y. Hence there is a sequence $0 < a_j \to 0$ such that $a_j w_j \not\to 0$ in Y. Since the series $\sum_n u_n$ is bounded in X, so is the set $S := \Bigl\{ \sum_{n=k}^{m} u_n : k, m \in \mathbb{N},\ k \leqslant m \Bigr\}$. Hence

$$\sup\{\|ax\| : x \in S\} \to 0 \qquad \text{as} \quad a \to 0. \tag{oo}$$

Consider the series $\sum_n x_n$ in X whose terms come up in consecutive blocks A_j, where for $j = 1, 2, \ldots$,

$$A_{2j-1} = a_j u_{k_j} + a_j u_{k_j+1} + \cdots + a_j u_{m_j}$$

and

$$A_{2j} = (-a_j u_{k_j}) + (-a_j u_{k_j+1}) + \cdots + (-a_j u_{m_j}).$$

Using (oo) it is easy to see that the series $\sum_n x_n$ converges to 0. However, the series $\sum_n f(x_n)$ is not Cauchy in Y because the sums of its blocks corresponding to A_{2j-1} do not tend to zero.

(d) $\Longrightarrow$ (a): Obvious. $\square$

Remarks.

1. For X and Y Banach spaces, the equivalence of (a) and (b) in Theorem 2 was already noted by Rado [Ra: Theorem 5].

2. The condition "f sends convergent series to convergent series" cannot be added to the list of equivalent conditions in Theorem 2. To see this, construct a map f as in the proof of Proposition 5 for $X = \mathbb{R}$, $u_n = n^{-2}$, $Y = \omega_0 \subset \omega$, and $y_n = e_n$. Then the series $\sum_n u_n$ is convergent in $\mathbb{R}$, while the series $\sum_n f(u_n) \equiv \sum_n e_n$ does not converge in Y.

4. Maps preserving absolutely convergent series

If X is a metrizable TVS and the choice of an F-norm $\|\cdot\|$ defining the topology of X is of importance, we indicate this by writing $X = (X, \|\cdot\|)$, and refer to X as an F-normed space. It is so, for instance, when absolutely convergent series $\sum_n x_n$ in X, that is, those with $\sum_n \|x_n\| < \infty$, are considered.

THEOREM 3. *Let $X = (X, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be F-normed spaces. Then for a map $f \colon X \to Y$ the following are equivalent.*

(a) *f sends absolutely convergent series to absolutely convergent series.*

(b) *There is a constant K such that $\|f(x)\| \leqslant K\|x\|$ in a neighborhood of zero in X.*

Proof.

(a) $\Longrightarrow$ (b): Suppose condition (b) is not satisfied. Then there exists a sequence (u_j) in X such that

$$\|u_j\| \leqslant j^{-2} \quad \text{and} \quad \|f(u_j)\| > j^3\|u_j\|, \qquad j = 1, 2, \ldots .$$

For each j, let m_j be the least integer such that $m_j\|u_j\| \geqslant j^{-2}$. Note that then

$$m_j\|u_j\| = (m_j - 1)\|u_j\| + \|u_j\| < 2j^{-2} .$$

Now, consider the series $\sum_n x_n$ in X whose terms come up in consecutive blocks A_j, where $A_j = u_j + \cdots + u_j$ (m_j summands). Then $\sum_n \|x_n\| = \sum_j m_j \| u_j\| < \infty$, while

$$\sum_n \|f(x_n)\| = \sum_j m_j\|f(u_j)\| \geqslant \sum_j j^3 m_j\|u_j\| \geqslant \sum_j j = \infty ,$$

contradicting (a). Since the other implication is obvious, the proof is finished. □

THEOREM 4. *Let $X = (X, \|\cdot\|)$ be an F-normed space and $Y = (Y, \|\cdot\|)$ a normed space. Then for a map $f\colon X \to Y$ the following are equivalent.*

(a) *f sends absolutely convergent series to absolutely convergent series.*
(b) *f sends absolutely convergent series to unconditionally Cauchy series.*
(c) *f sends absolutely convergent series to Cauchy series.*
(d) *f sends absolutely convergent series to bounded series.*
(e) *There is a constant K such that $\|f(x)\| \leqslant K\|x\|$ in a neighborhood of zero in X.*

Proof. Only the implication (d) $\Longrightarrow$ (e) needs a proof: Assuming (e) fails, we proceed as in the preceding proof and finish as follows: ... Then $\sum_n \|x_n\| = \sum_j m_j\|u_j\| < \infty$. However, the series $\sum_n f(x_n)$ is not bounded because for the sum z_j of its block corresponding to A_j we have $\|z_j\| = m_j\|f(u_j)\| > m_j j^3\|u_j\| \geqslant j$, contradicting (d). $\square$

Remarks.

1. The above two proofs are modeled on the arguments used in [BČŠ; Theorem 2.6] to establish the equivalence of conditions (a), (c), and (e) in Theorem 4 for the case $X = Y = \mathbb{R}$. It is worth pointing out that the proof given there is unnecessarily complicated: In fact, in that particular case, (a) and (c) are evidently equivalent because (c) implies (b), and for scalar series unconditional and absolute convergence coincide.

Actually, arguments like those above have been of constant use in the theory of Orlicz-type sequence spaces since its very beginning, see e.g. [BO; Kap. 1, Satz 1] and [MO; 1.13, 1.14]. Following this line one may observe that the same proof as for Theorem 3 gives also a more general result:

THEOREM 3A. *Let X be a vector space and let $\varphi, \psi\colon X \to \mathbb{R}_+$. Then $\sum_n \psi(x_n) < \infty$ whenever $\sum_n \varphi(x_n) < \infty$ if and only if there are constants $c > 0$ and $K \geqslant 0$ such that $\psi(x) \leqslant K\varphi(x)$ whenever $\varphi(x) \leqslant c$.*

Of course, to deduce Theorem 3 from Theorem 3a it is enough to set $\varphi(x) = \|x\|$ and $\psi(x) = \|f(x)\|$.

2. In Theorem 4, even if X were a Banach space but Y merely a p-normed space, where $0 < p < 1$, the implication (b) $\Longrightarrow$ (a) would not hold. To see this, take $X = \mathbb{R}$ with its usual norm $|\cdot|$; $Y = \mathbb{R}$ with the p-norm $|\cdot|^p$ $(0 < p < 1)$; f = the identity map; and consider the series $\sum_n n^{-1/p}$. Nevertheless, as we show below, conditions (b), (c) and (d) of Theorem 4 remain equivalent even for arbitrary TVS Y.

3. If both X and Y are normed spaces, and the map $f\colon X \to Y$ is positively homogeneous, then condition (e) in Theorem 4 can be replaced by the following:

There are constants $c > 0$ and $K \geqslant 0$ such that $\|f(x)\| \leqslant K\|x\|$ for every $x \in X$ with $\|x\| \leqslant c$. Similar remarks apply to Theorems 6 and 7 below.

THEOREM 5. *Let $X = (X, \|\cdot\|)$ be an F-normed space and Y any* TVS. *Then for a map $f\colon X \to Y$ the following are equivalent.*

(a) *f sends absolutely convergent series to unconditionally Cauchy series.*
(b) *f sends absolutely convergent series to Cauchy series.*
(c) *f sends absolutely convergent series to bounded series.*

Proof. Only the implication (c) $\Longrightarrow$ (a) needs a proof: Assuming (a) fails, we proceed almost as in the proof of the implication (a) $\Longrightarrow$ (b) in Theorem 2.

Suppose there is an absolutely convergent series $\sum_n x_n$ in X for which the series $\sum_n f(x_n)$ is not unconditionally Cauchy in Y. Then there exists a sequence (F_j) of finite subsets of $\mathbb{N}$ such that $\max F_j < \min F_{j+1}$,

$$\sum_{n\in F_j} \|x_n\| \leqslant \frac{1}{j^3} \quad \text{and} \quad p\Big(\sum_{n\in F_j} f(x_n)\Big) > \varepsilon \qquad (j = 1, 2, \dots) \qquad (+)$$

for some continuous F-seminorm p on Y and some $\varepsilon > 0$.

Now, consider the series $\sum_{r=1}^{\infty} u_r$ whose terms come up in consecutive blocks A_j, each in turn consisting of j identical subblocks $\sum_{n\in F_j} x_n$ $(j = 1, 2, \dots)$.

Using (+) it is easy to see that $\sum_r \|u_r\| < \infty$. However, denoting by z_j the sum of the block of the series $\sum_r f(u_r)$ corresponding to A_j, we have

$$p(j^{-1}z_j) = p\Big(\sum_{n\in F_j} f(x_n)\Big) > \varepsilon\,.$$

Hence the series $\sum_r f(u_r)$ is not bounded in Y, contradicting the assumption. $\square$

We conclude this section with the following.

PROPOSITION 6. *Let $X = (X, \|\cdot\|)$ be an F-normed space and Y an arbitrary* TVS. *If a map $f\colon X \to Y$ sends absolutely convergent series to bounded series, then $f(0) = 0$ and f is continuous at zero.*

Proof. Obviously, $f(0) = 0$. Suppose f is not continuous at 0. Then there is a sequence (u_j) in X such that $\|u_j\| < j^{-3}$ and $p(f(u_j)) > \varepsilon$ for some continuous F-seminorm p on Y and some $\varepsilon > 0$. Now, consider the series $\sum_n x_n$ in X whose terms come up in consecutive blocks $A_j = u_j + \cdots + u_j$

(j summands). Then $\sum_n \|x_n\| = \sum_j j\|u_j\| \leqslant \sum_j j^{-2} < \infty$. However, the series $\sum_n f(x_n)$ is not bounded in Y because, denoting by z_j the sum of its block corresponding to A_j, we have $p(j^{-1}z_j) = p(f(u_j)) > \varepsilon$ and thus $j^{-1}z_j \not\to 0$ as $j \to \infty$. □

5. Maps preserving unconditionally convergent series

From Theorem 4 it follows that if X is a finite dimensional Banach space and Y an arbitrary Banach space, then a map $f\colon X \to Y$ preserves unconditional convergence of series if and only if $\|f(x)\| \leqslant \text{const}\cdot\|x\|$ near the origin. This is no longer true if $\dim X = \infty$: In view of the Dvoretzky-Rogers theorem, the map $f\colon X \to \mathbb{R}$ defined by $f(x) = \|x\|$ does not preserve unconditional convergence.

Given a finite family $(x_n\colon n \in N)$ of vectors in an F-normed space $(X, \|\cdot\|)$, we define

$$\mu(x_n\colon n \in N) = \sup\left\{\Big\|\sum_{n\in F} x_n\Big\| : F \subset N\right\}.$$

Recall that a series $\sum_n x_n$ in X is unconditionally Cauchy if and only if $\mu(x_n\colon n \in N) \to 0$ as $\min N \to \infty$.

THEOREM 6. *Let $X = (X, \|\cdot\|)$ and $Y = (Y, \|\cdot\|)$ be F-normed spaces. A map $f\colon X \to Y$ sends unconditionally Cauchy series to absolutely convergent series if and only if there exist constants $c > 0$ and $K \geqslant 0$ such that*

$$\|f(x_1)\| + \cdots + \|f(x_n)\| \leqslant K\mu(x_1, \ldots, x_n) \qquad \textit{whenever} \quad \mu(x_1, \ldots, x_n) \leqslant c.$$

Proof. The "if" part is obvious.

The "only if" part: Suppose the condition does not hold. Then we can find a sequence (u_n) in X together with a sequence (N_j) of consecutive intervals in $\mathbb{N}$ such that for each j

$$\mu(u_n\colon n \in N_j) \leqslant j^{-2} \qquad \text{and} \qquad \sum_{n\in N_j} \|f(u_n)\| > j^3\mu(u_n\colon n \in N_j).$$

For each j, let m_j be an integer such that $j^{-2} \leqslant m_j\mu(u_n\colon n \in N_j) < 2j^{-2}$.

Now, consider the series $\sum_n x_n$ in X whose terms come up in consecutive blocks A_j, each consisting of m_j identical subblocks $\sum_{n\in N_j} u_n$. Since $\sum_j m_j\mu(u_n\colon n \in N_j) < \infty$, it follows easily that the series $\sum_n x_n$ is unconditionally Cauchy. However, as in the proof of Theorem 4, $\sum_n \|f(x_n)\| = \infty$. □

Remark. The following is a more general form of the above result (with the same proof).

THEOREM 6A. *Let $X = (X, \|\cdot\|)$ be an F-normed space. A map $\psi\colon X \to \mathbb{R}_+$ sends unconditionally Cauchy series to convergent series if and only if there are constants $c > 0$ and $K \geqslant 0$ such that $\psi(x_1) + \cdots + \psi(x_n) \leqslant K\mu(x_1, \ldots, x_n)$ whenever $\mu(x_1, \ldots, x_n) \leqslant c$.*

Of course, to deduce Theorem 6 from Theorem 6a it is enough to set $\psi(x) = \|f(x)\|$.

THEOREM 7. *Let $X = (X, \|\cdot\|)$ be an F-normed space and $Y = (Y, \|\cdot\|)$ a normed space. Then for a map $f\colon X \to Y$ the following are equivalent.*

(a) *f sends unconditionally Cauchy series to (unconditionally) Cauchy series.*

(b) *f sends unconditionally Cauchy series to bounded series.*

(c) *There are constants $c > 0$ and $K \geqslant 0$ such that*

$$\|f(x_1) + \cdots + f(x_n)\| \leqslant K\mu(x_1, \ldots, x_n) \qquad \textit{whenever} \quad \mu(x_1, \ldots, x_n) \leqslant c.$$

P r o o f.
(b) $\Longrightarrow$ (c): Assuming (c) is false, we proceed as in the previous proof, replacing $\sum_{n\in N_j} \|f(u_n)\|$ with $\|\sum_{n\in N_j} f(u_n)\|$. As before, we get an unconditionally Cauchy series $\sum_n x_n$ in X. However, the series $\sum_n f(x_n)$ is not bounded because for the sum z_j of its block corresponding to A_j we have $\|z_j\| = m_j \|\sum_{n\in N_j} u_n\| > m_j j^3 \mu(u_n\colon n \in N_j) \geqslant j$. □

Remarks.

1. The same simple choice of X and Y as in Remark 2 after Theorem 4 shows that the assumption in Theorem 7 that Y is a normed space is essential for the implication (a) $\Longrightarrow$ (c) to hold.

2. If a map f from a metrizable TVS X to an F-normed space Y (resp., a TVS Y with Property $(\overline{\omega})$) sends convergent series to absolutely convergent (resp., unconditionally Cauchy) series, then $f = 0$ in a neighborhood of zero in X (comp. [BČŠ; Theorem 2.10]). In fact, by Theorem 1 and Proposition 2, there is a balanced neighborhood U of zero in X such that $f|U = T|U$ for some continuous $\mathbb{R}$-linear operator $T\colon X \to Y$. Suppose $y := f(x) \neq 0$ for some $x \in U$. Then the series $\sum_n (-1)^n n^{-1} x$ is convergent in $\mathbb{R} \cdot x \subset X$, while its f-transform $\sum_n (-1)^n n^{-1} y$ is not absolutely (or unconditionally) convergent in $\mathbb{R} \cdot y \subset Y$.

6. Final remark

The following of our results have exact analogues in the setting of Hausdorff topological abelian groups: Propositions 1 and 4, Theorem 1 (with formally the same definition of Property $(\overline{\omega})$), Corollary 3, and Theorems 3 and 6. Their proofs need only minor terminological modifications.

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Received February 8. 1999
Revised December 11, 2000

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