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ON THE IDENTITY OF MINIMAL AND MAXIMAL REALIZATIONS RELATED TO FOURIER SERIES OPERATORS

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ABSTRACT. The identity of the maximal and minimal realizations of the linear Fourier series operators

$$(L(x,\mathrm{D})\varphi)(x) := (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} L(x,l)\varphi_l\, \mathrm{e}^{\mathrm{i}(l,x)}$$

in the appropriate subspaces of periodic distributions are studied. Specifically, criteria for the equality of the realizations from $B^{\pi}_{p,k}$ into $B^{\pi}_{p,k}$ are established. Here $B^{\pi}_{p,k}$ is the subspace of D'_{π} for whose elements u one has $(u_l k(l))_{l\in\mathbf{Z}^n} \in l_p$ (D'_{π} denotes the space of all periodic distributions). In the case when $p=2$ and $k\equiv 1$, one observes that $B^{\pi}_{p,k}$ is the space of all periodic $L_2(W)$-functions (where $W := \{x\in\mathbf{R}^n \mid x_j \in\,]-\pi,\pi[\}$). The equality of the realizations from $B^{\pi}_{p,k}$ into $L_{p'}(W)\cap D'_{\pi}$ is also examined, where $p\in\,]1,2]$ and $p'\in\mathbf{R}$ so that $1/p+1/p'=1$.

1. Introduction

Denote by $L(x,\mathrm{D})$ the *linear Fourier series operator* defined in the space C^{∞}_{π} of all *smooth periodic functions* $\varphi\colon \mathbf{R}^n \to \mathbf{C}$ by the requirement

$$(L(x,\mathrm{D})\varphi)(x) = (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} L(x,l)\varphi_l\, \mathrm{e}^{\mathrm{i}(l,x)}\,. \tag{1.1}$$

Here φ_l is the *Fourier coefficient* of φ. $L(\cdot,\cdot)$ is a *mapping* $\mathbb{R}^n\times\mathbb{Z}^n\to\mathbf{C}$ so that $L(\cdot,l)\in \mathrm{C}^{\infty}_{\pi}$ for any $l\in\mathbb{Z}^n$ and that with the constants $C_\alpha>0$ and $\mu_\alpha\in\mathbf{R}$ the estimate

$$\sup_{x\in W} |(\mathrm{D}^{\alpha}_x L)(x,l)| \le C_\alpha k_{\mu_\alpha}(l) := C_\alpha(1+|l|^2)^{\mu_\alpha/2} \tag{1.2}$$

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holds (in (1.2) W denotes the *cube* $\{x \in \mathbf{R}^n \mid x_j \in]-\pi, \pi[\}$).

This contribution deals with the equality of the *minimal* and *maximal realizations*, say $L^{\sim}_{p,k,h}$ and $L'^{\#}_{p,k,h}$, from $B^{\pi}_{p,k}$ into $B^{\pi}_{p,h}$. The *spaces* $B^{\pi}_{p,k}$ (where $p \in [1, \infty[$ and k lies in the class K'_{π} of certain *weight functions*) are appropriate *scales of the space* D'_{π} of all *periodic distributions*. The equality of the *realizations* $\boldsymbol{L}^{\sim}_{p,p',k}$ and $\boldsymbol{L}'^{\#}_{p,p',k}$ from $B^{\pi}_{p,k}$ into $L_{p'}(W) \bigcap D'_{\pi}$ ($p' \in \mathbf{R}$; $1/p + 1/p' = 1$) are also studied, when $p \in]1, 2]$ and $k \in K'_{\pi}$.

The best known example of the operators, which can be defined by (1.1), are linear partial differential operators with C^{∞}_{π}-coefficients (cf. [4], [3], [1], [6] and [7]). It follows from the well-known regularity results of solutions (cf. [4], pp. 90–119) that smooth periodic elliptic operators are essentially maximal in $H^{\pi}_{k_s} := B^{\pi}_{2,k_s}$, $s \in \mathbf{R}$, that is the equality $L^{\sim}_{2,k_s,k_s} = L'^{\#}_{2,k_s,k_s}$ holds. Some criteria for the essential maximality in $H^{\pi}_{k_0} := L_2(W) \bigcap D'_{\pi}$ can also be found in [8], pp. 28–38.

Suppose that in (1.2) for any $\alpha \in \mathbf{N}^n_0$, $\mu_\alpha = \mu + \delta|\alpha|$ with $\mu \in \mathbf{R}$ and $\delta < 1$ and that for any $|\alpha| \leq [N_h + n + \varepsilon] + n + 3$ one has

$$\sup_{x \in W} |(\mathrm{D}^{\alpha}_x L)(x, l)| \leq C_\alpha k(l)/h(l). \tag{1.3}$$

Here N_h is a constant depending only on $h \in K'_{\pi}$. We show that these assumptions are sufficient to guarantee the equality $L^{\sim}_{p,kk_{-1},h} = L'^{\#}_{p,kk_{-1},h}$ (cf. Theorem 3.5). Specially, this equality implies that for any smooth periodic partial differential operator $L(x, \mathrm{D}) = \sum_{|\sigma| \leq m} a_\sigma(x) \mathrm{D}^{\sigma}$, $m \in \mathbf{N}$ the equality $L^{\sim}_{p,kk_{m-1},k} = L'^{\#}_{p,kk_{m-1},k}$ holds. Hence any first order partial differential operator with C_{π}-coefficients is essentially maximal in $L_2(W) \bigcap D'_{\pi}$ (cf. Corollaries 3.6–3.8). In the case when $\mu_\alpha = \mu + \delta|\alpha|$; $\mu \in \mathbf{R}$, $\delta < 1$, the estimate

$$\sup_{x \in W} |(\mathrm{D}^{\alpha}_x L)(x, l)| \leq C_\alpha k(l) k_1(l) \tag{1.4}$$

holds for $|\alpha| \leq [n + \varepsilon] + n + 3$ and when $p \in]1, 2]$, we establish the identity $\boldsymbol{L}^{\sim}_{p,p',k} = \boldsymbol{L}'^{\#}_{p,p',k}$ (cf. Theorem 4.2).

2. Notations and definitions of realizations

2.1. *Denote by* W *the open cube* $\{x \in \mathbf{R}^n \mid -\pi < x_j < \pi$ for $j = 1, \ldots, n\}$. *By* C^{∞}_{π} *we denote the space of all smooth (with respect to* W*) periodic functions* $\varphi \colon \mathbf{R}^n \to \mathbf{C}$.

In C^∞_π we set a standard *Frechet space topology* defined by the semi-norms $q_\sigma(\varphi) := \sup_{x \in W} |(\mathrm{D}^\sigma \varphi)(x)|, \quad \sigma \in \mathbf{N}^n_0$. The dual of C^∞_π is denoted by D'_π and its elements are periodic distributions. In D'_π one uses the weak dual topology.

For $u \in D'_\pi$ and $l \in \mathbf{Z}^n$ we define $u_l \in \mathbf{C}$ by

$$u_l = u(\mathrm{e}^{-\mathrm{i}(l,\cdot)}). \tag{2.1}$$

Then one has for $u \in D'_\pi$ and $\varphi \in \mathrm{C}^\infty_\pi$

$$u(\varphi) = (2\pi)^{-n} \sum_l u_l \varphi_{-l}, \tag{2.2}$$

where

$$\varphi_l := \varphi(\mathrm{e}^{-\mathrm{i}(l,\cdot)}) := \int_W \varphi(x)\,\mathrm{e}^{-\mathrm{i}(l,x)}\,\mathrm{d}x. \tag{2.3}$$

For φ and $\psi \in \mathrm{C}^\infty_\pi$ we denote

$$\varphi(\psi) := \int_W \varphi(x)\psi(x)\,\mathrm{d}x,$$

and so specifically one gets $\varphi(\psi) = (2\pi)^{-n} \sum_l \varphi_l \psi_{-l}$.

Denote by K_π the totality of all positive functions $k\colon \mathbf{Z}^n \to \mathbf{R}$ such that for any $k \in K_\pi$ there exist constants
$c > 0$, $C > 0$, $m, M \in \mathbf{N}$ such that

$$c k_{-m}(l) \le k(l) \le C k_M(l) \qquad \textit{for all} \quad l \in \mathbf{Z}^n,$$

where $k_s(l) := (1 + |l|^2)^{s/2}$, $s \in \mathbf{R}$. Choose $p \in [1,\infty[$. A subspace $B^\pi_{p,k}$ of D'_π is defined as follows:

A distribution $u \in D'_\pi$ belongs to $B^\pi_{p,k}$ if and only if

$$\|u\|_{p,k} := \left((2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} |u_l k(l)|^p \right)^{1/p} < \infty. \tag{2.4}$$

One sees that the mapping $u \to \|u\|_{p,k}$ is a norm in $B^\pi_{p,k}$. The linear space $B^\pi_{p,k}$ equipped with the $\|\cdot\|_{p,k}$-norm is a Banach space.

Define $S_\pi := \left\{ \varphi \in C^\infty_\pi \mid \varphi(x) = (2\pi)^{-n} \sum_{|l| \le n_\varphi} \varphi_l \mathrm{e}^{\mathrm{i}(l,x)} \text{ with some } n_\varphi \in \mathbf{N} \right\}$, that is, S_π *is the space of all trigonometric polynomials*. One sees that S_π is a dense subspace of $B^\pi_{p,k}$ and so $B^\pi_{p,k}$ is (essentially) a completion of S_π with respect to the norm $\|\varphi\|_{p,k} := \left((2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} |\varphi_l k(l)|^p \right)^{1/p}$.

2.2. *Let L be a linear operator $S_\pi \to C_\pi^\infty$ such that the formal transpose $L'\colon S_\pi \to C_\pi^\infty$ exists, in other words, there exists a linear operator $L'\colon S_\pi \to C_\pi^\infty$ so that*

$$(L\varphi)(\psi) = \varphi(L'\psi) \qquad \textit{for all} \quad \varphi, \psi \in S_\pi. \tag{2.5}$$

Define linear dense operators $L_{p,k,h}$ and $L'^{\#}_{p,k,h}$; $p \in [1,\infty[$, $k, h \in K_\pi$ by the requirements

$$\left.\begin{aligned} &\mathrm{D}(L_{p,k,h}) = S_\pi \\ &L_{p,k,h}\varphi = L\varphi \qquad \textit{for} \quad \varphi \in S_\pi \end{aligned}\right\} \tag{2.6}$$

and

$$\left.\begin{aligned} \mathrm{D}\big(L'^{\#}_{p,k,h}\big) = \{\, & u \in B^\pi_{p,k} \mid \textit{there exists } f \in B^\pi_{p,h} \textit{ such that} \\ & u(L'\varphi) = f(\varphi) \textit{ for all } \varphi \in S_\pi \,\} \\ L'^{\#}_{p,k,h} u = f&. \end{aligned}\right\} \tag{2.7}$$

Let $p' \in\,]1,\infty]$ so that $1/p + 1/p' = 1$ and let $k^\vee \in K_\pi$ so that $k^\vee(l) = k(-l)$. Since the inequality

$$|\varphi(\psi)| \le \|\varphi\|_{p,k}\|\psi\|_{p',1/k^\vee} \qquad \text{for} \quad \varphi, \psi \in C_\pi^\infty \tag{2.8}$$

holds, one gets by (2.5) that $L_{p,k,h}$ is a closable operator $B^\pi_{p,k} \to B^\pi_{p,h}$, $L'^{\#}_{p,k,h}$ is a closed operator $B^\pi_{p,k} \to B^\pi_{p,h}$ and that $L_{p,k,h} \subset L'^{\#}_{p,k,h}$. Let $L^\sim_{p,k,h}$ be the smallest closed extension of $L_{p,k,h}$. Then one has $L^\sim_{p,k,h} \subset L'^{\#}_{p,k,h}$. The operator $L^\sim_{p,k,h}$ is called the *minimal realization* and the operator $L'^{\#}_{p,k,h}$, is called the *maximal realization* of L from $B^\pi_{p,k}$ to $B^\pi_{p,h}$.

Similarly, we are able to define minimal and maximal realizations, say $L^\sim_{p,q,k}$ and $L'^{\#}_{p,q,k}$ from $B^\pi_{p,k}$ to $L_q(W) \bigcap D'_\pi$, where $p \in [1,\infty[$, $q \in [1,\infty[$ and $k \in K_\pi$.

2.3. Let $L(\cdot,\cdot)$ be a function from $\mathbb{R}^n \times \mathbb{Z}^n$ to $\mathbb{C}$ such that $L(\cdot, l) \in C_\pi^\infty$ for any $l \in \mathbb{Z}^n$ and that with some constants $C_\alpha > 0$ and $\mu_\alpha \in \mathbb{R}$ one has

$$\sup_{x\in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha k_{\mu_\alpha}(l) \qquad \text{for all} \quad l \in \mathbb{Z}^n. \tag{2.9}$$

Then the Fourier series operator $L(x,\mathrm{D})$ defined by

$$(L(x,\mathrm{D})\varphi)(x) = (2\pi)^{-n} \sum_l L(x,l)\varphi_l\, \mathrm{e}^{\mathrm{i}(l,x)}, \qquad \varphi \in C_\pi^\infty \tag{2.10}$$

maps C_π^∞ continuously into C_π^∞ (cf. [9]). Hence, specifically, the inclusion

$$C_\pi^\infty \subset \mathrm{D}(L^\sim_{p,k,h}) \bigcap \mathrm{D}(L^\sim_{p,q,k})$$

holds. In the case when $\mu_\alpha = \mu + \delta|\alpha|$ with some $\mu \in \mathbf{R}$ and $\delta < 1$ we know that the continuous formal transpose $L'(x, \mathrm{D})\colon C_\pi^\infty \to C_\pi^\infty$ of $L(x, \mathrm{D})$ exists (cf. [9]). When $L'(x, \mathrm{D})\colon C_\pi^\infty \to C_\pi^\infty$ exists, then $L'(x, \mathrm{D})$ is always continuous. This follows from the Closed Graph Theorem.

Suppose that $L'(x, \mathrm{D})\colon C_\pi^\infty \to C_\pi^\infty$ exists. Then we are able to define the *continuous extension* $\bar{L}\colon \mathrm{D}'_\pi \to \mathrm{D}'_\pi$ of $L(x, \mathrm{D})$ by

$$(\bar{L}u)(\varphi) = u(L'(x, \mathrm{D})\varphi) \qquad \text{for} \quad \varphi \in C_\pi^\infty. \tag{2.11}$$

Denote by A_π the *space of mappings* $L(\cdot,\cdot)\colon \mathbf{R}^n \times \mathbf{Z}^n \to \mathbf{C}$ such that $L(\cdot, l) \in C_\pi^\infty$ for any $l \in \mathbf{Z}^n$ and that for each $L(\cdot,\cdot) \in A_\pi$ there exists $\mu \in \mathbf{R}$ and $\delta < 1$ such that

$$\sup_{x \in W} |(\mathrm{D}_x^\alpha L)(x, l)| \le C_\alpha k_{\mu + \delta|\alpha|}(l) \qquad \text{for} \quad l \in \mathbf{Z}^n. \tag{2.12}$$

The *space of operators* $\{L(x, \mathrm{D}) \mid L(x, \mathrm{D})$ is defined by (2.10), where $L(\cdot,\cdot) \in A_\pi\}$ is denoted by $\mathcal{A}_\pi$. Then for any $L(x, \mathrm{D}) \in \mathcal{A}_\pi$ the formal transpose $L'(x, \mathrm{D})\colon C_\pi^\infty \to C_\pi^\infty$ exists.

We denote by K'_π the subset of K_π such that for any $k \in K'_\pi$ there exist con tants $C_k > 0$ and $N_k \ge 0$ with which

$$k(l + z) \le C_k k_{N_k}(l) k(z) \qquad \text{for} \quad l, z \in \mathbf{Z}^n. \tag{2.13}$$

The smallest integer, which is greater or equal to $a \in \mathbf{R}$ is denoted by $[a]$. Choose h from K'_π. We denote $C_{n,\varepsilon,h} := C_h \cdot \gamma_{n,\varepsilon,h}^{-1} \sum_l k_{-(n+\varepsilon)}(l)$, where $\gamma_{n,\varepsilon,h} \in \mathbf{R}$ so that

$$\sum_{|\alpha| \le [N_h + n + \varepsilon]} l^{2\alpha} \ge \gamma_{n,\varepsilon,h}^2 k_{[N_h + n + \varepsilon]}^2(l) \qquad \text{for} \quad l \in \mathbf{Z}^n.$$

Theorem 2.1. *Suppose that $k, h \in K'_\pi$ and that $\mathcal{R}$ is a subset of A_π such that*

$$\sup_{x \in W} |(\mathrm{D}_x^\alpha R)(x, l)| \le C_\alpha k(l)/h(l) \qquad \textit{for all}$$
$$|\alpha| \le [N_h + n + \varepsilon] \textit{ and } R(\cdot,\cdot) \in \mathcal{R}. \tag{2.14}$$

Then one has

$$\| R(x, \mathrm{D})\varphi\|_{p,h} \le C_{n,\varepsilon,h} \Big[\sum_{|\alpha| \le [N_h + n + \varepsilon]} C_\alpha^2 \Big]^{1/2} \|\varphi\|_{p,k} \qquad \textit{for all}$$
$$\varphi \in C_\pi^\infty, \quad R(\cdot,\cdot) \in \mathcal{R} \quad \textit{and} \quad p \in [1, \infty[\tag{2.15}$$

P r o o f. A. We shall show that

$$\sum_{l\in\mathbf{Z}^n} |(R(\cdot,-l))_{l-z}|(1/k^\vee(l)) \quad \le \quad (2\pi)^n C_{n,\varepsilon,h}\Big[\sum_{|\alpha|\le[N_h+n+\varepsilon]} C_\alpha^2\Big]^{1/2}(1/h^\vee(z)) \tag{2.16}$$

and that

$$\sum_{z\in\mathbf{Z}^n} |(R(\cdot,-l))_{l-z}|h^\vee(z) \le (2\pi)^n C_{n,\varepsilon,h}\Big[\sum_{|\alpha|\le[N_h+n+\varepsilon]} C_\alpha^2\Big]^{1/2} k^\vee(l). \tag{2.17}$$

Then the Theorem 4.4 in [9] (cf. also the relation (4.17) in [9]) implies that (choose $k \leftrightarrow 1/k^\vee$ and $k^\sim \leftrightarrow (k/h)^\vee$)

$$\begin{aligned}
&\|R'(x,\mathrm{D})\varphi\|_{p',1/k^\vee}\\
&\qquad \le \Big(C_{n,\varepsilon,h}\Big[\sum_{|\alpha|\le[N_h+n+\varepsilon]} C_\alpha^2\Big]^{1/2}\Big)^{1/p+1/p'}\|\varphi\|_{p',1/h^\vee}\\
&\qquad = C_{n,\varepsilon,h}\Big[\sum_{|\alpha|\le[N_h+n+\varepsilon]} C_\alpha^2\Big]^{1/2}\|\varphi\|_{p',1/h^\vee}
\end{aligned} \tag{2.18}$$

for any $p' \in]1,\infty[$.

From (2.18) one gets that for any $p \in]1,\infty[$ (cf. [9], Lemma 4.3)

$$\|R(x,\mathrm{D})\varphi\|_{p,h} \le C_{n,\varepsilon,h}\Big[\sum_{|\alpha|\le[N_h+n+\varepsilon]} C_\alpha^2\Big]^{1/2}\|\varphi\|_{p,k}. \tag{2.19}$$

Since for any $\varphi \in C_\pi^\infty$ one has

$$\|\varphi\|_{p,k} \to \|\varphi\|_{1,k} \qquad \text{with} \quad p\to 1,$$

we see that the inequality (2.15) holds also in the case when $p=1$.

B. We show the estimates (2.16)–(2.17). In virtue of (2.13) one gets

$$h^\vee(z) \le C_h k_{N_h}(z-l)h^\vee(l) \tag{2.20}$$

and

$$1/h^\vee(l) \le C_h k_{N_h}(z-l)(1/h^\vee(z)). \tag{2.21}$$

For any $|\alpha| \le [N_h+n+\varepsilon]$ and $R(\cdot,\cdot)\in\mathcal{R}$ we obtain

$$|(l-z)^\alpha(R(\cdot,-l))_{l-z}| = |((\mathrm{D}_x^\alpha R)(\cdot,-l))_{l-z}| \le (2\pi)^n C_\alpha(k^\vee(l)/h^\vee(l)) \tag{2.22}$$

and so

$$\gamma_{n,\varepsilon,h}|(R(\cdot,-l))_{l-z}| \leq (2\pi)^n \Big[\sum_{|\alpha|\leq[N_h+n+\varepsilon]} C_\alpha^2 \Big]^{1/2} (k^\vee(l)/h^\vee(l)) k_{-[N_h+n+\varepsilon]}(z-l). \tag{2.23}$$

Here we used the inequality

$$\sum_{|\alpha|\leq[N_h+n+\varepsilon]} l^{2\alpha} \geq \gamma_{n,\varepsilon,h}^2 k_{[N_h+n+\varepsilon]}^2(l),$$

which implies by (2.22) that

$$\begin{aligned}
&\gamma_{n,\varepsilon,h}^2 k_{[N_h+n+\varepsilon]}^2(z-l)|(R(\cdot,-l))_{l-z}|^2 \\
&\qquad \leq \sum_{|\alpha|\leq[N_h+n+\varepsilon]} |(l-z)^\alpha (R(\cdot,-l))_{l-z}|^2 \\
&\qquad \leq (2\pi)^{2n} \Big[\sum_{|\alpha|\leq[N_h+n+\varepsilon]} C_\alpha^2 \Big] (k^\vee(l)/h^\vee(l))^2,
\end{aligned}$$

and so we get (2.23).

In virtue of (2.20), (2.21) and (2.23) we obtain that

$$\begin{aligned}
&\sum_l |(R(\cdot,-l))_{l-z}|(1/k^\vee(l)) \\
&\qquad \leq \gamma_{n,\varepsilon,h}^{-1}(2\pi)^n \Big[\sum_{|\alpha|\leq[N_h+n+\varepsilon]} C_\alpha^2 \Big]^{1/2} \sum_l (1/h^\vee(l)) k_{-[N_h+n+\varepsilon]}(z-l) \\
&\qquad \leq \gamma_{n,\varepsilon,h}^{-1}(2\pi)^n C_h \Big[\sum_{|\alpha|\leq[N_h+n+\varepsilon]} C_\alpha^2 \Big]^{1/2} \Big(\sum_l k_{-(n+\varepsilon)}(l) \Big) (1/h^\vee(z))
\end{aligned} \tag{2.24}$$

and then (2.16) holds.

Similarly, we get

$$\begin{aligned}
&\sum_z |(R(\cdot,-l))_{l-z}| h^\vee(z) \\
&\qquad \leq \gamma_{n,\varepsilon,h}^{-1}(2\pi)^n \Big[\sum_{|\alpha|\leq[N_h+n+\varepsilon]} C_\alpha^2 \Big]^{1/2} C_h \sum_z k^\vee(l) k_{-(n+\varepsilon)}(z-l),
\end{aligned} \tag{2.25}$$

which implies (2.17). This completes the proof.

2.4. Let $\tilde{\Theta}$ be in $C_0^\infty(B(0,1))$ so that $\int_W \tilde{\mathrm{O}}(x)\,\mathrm{d}x = 1$.

Define $\tilde{\Theta}_m \in C_0^\infty := C_0^\infty(\mathbf{R}^n)$ by

$$\tilde{\Theta}_m(x) = m^n \tilde{\Theta}(mx), \quad m \in \mathbf{N}.$$

Furthermore, define $\Theta_m \in S$ (here S denotes the *Schwartz class*) by

$$\Theta_m = (2\pi)^n F^{-1}(\tilde{\Theta}_m^\vee),$$

where $F\colon S \to S$ is the *Fourier transform*. Define a *Fourier series op rator* $\Theta_m(\mathrm{D})$ by

$$(\Theta_m(\mathrm{D})\varphi)(x) = (2\pi)^{-n} \sum_{l \in \mathbf{Z}^n} \Theta_m(l)\varphi_l \,\mathrm{e}^{\mathrm{i}(l,x)}. \tag{2.26}$$

Let $\Theta_m\colon D'_\pi \to D'_\pi$ be the continuous extension of $\mathrm{O}_m(\mathrm{D})$ (cf. (2.11); note that $\Theta'_m(\mathrm{D})$ exists). Then one sees that for any $u \in D'_\pi$ one has

$$(\bar{\Theta}_m u)_l = (\bar{\Theta}_m u)(\mathrm{e}^{-\mathrm{i}(l,\cdot)}) = u(\Theta'_m(\mathrm{D})(\mathrm{e}^{-\mathrm{i}(l,\cdot)})) = u(\Theta_m(l)\,\mathrm{e}^{-\mathrm{i}(l,\cdot)}) = \Theta_m(l)u_l. \tag{2.27}$$

Thus we obtain for $p < \infty$

Lemma 2.2. *Let u be in $B^\pi_{p,k}$. Then one has*

$$\bar{\mathrm{O}}_m u \in C_\pi \qquad \textit{and} \qquad \|\Theta_m u - u\|_{p,k} \to 0 \qquad \textit{with} \quad m \to \infty. \tag{2.28}$$

P r o o f. One has (recall that $F^{-1}\phi = (2\pi)^{-n} F\phi^\vee$)

$$\Theta_m(l) = (F\tilde{\mathrm{O}}_m)(l) = \int_{\mathbf{R}} m^n \tilde{\mathrm{O}}(my)\,\mathrm{e}^{-\mathrm{i}(l,y)}\,\mathrm{d}y = (F\tilde{\mathrm{O}})\,(l/m).$$

Furthermore, we obtain for any $\varphi \in C_\pi^\infty$ (cf. (2.2) and (2.27))

$$(\Theta_m u)(\varphi) - (2\pi)^{-n} \sum_l \Theta_m(l) u_l \varphi_{-l} = \Big[(2\pi)^{-n} \sum_l \Theta_m(l) u_l \,\mathrm{e}^{\mathrm{i}(l,\cdot)}\Big](\varphi).$$

Thus $\bar{\mathrm{O}}_m u - (2\pi)^{-n} \sum_l (F\tilde{\Theta})(l/m) u_l \,\mathrm{e}^{\mathrm{i}(l,\,)} \in C_\pi^\infty$. In addition, one gets

$$|(\bar{\mathrm{O}}_m u)_l| k(l) = |(F\tilde{\mathrm{O}})(l/m) u_l k(l)| \leq |\tilde{\Theta}\,|_{L_1(W)} |u_l k(l)|$$

and

$$(\mathrm{O}_m u)_l k(l) \to (F\tilde{\mathrm{O}})(0) u_l k(l) \quad \Big(\int_W \tilde{\Theta}\; x)\,\mathrm{d}x\Big) u_l k(l) \quad u_l k(l).$$

Thus

$$\|\bar{\mathrm{O}}_m u - u\,|^p_{p\;k} \quad (2\pi)^{-n} \sum_l |((\mathrm{O}_m u)_l - u_l) k(l)| \;\to 0 \qquad \text{with} \quad m \to \infty,$$

which finishes the proof.

3. On the equality $L^{\sim}_{p,k,h} = L'^{\#}_{p,k,h}$

3.1. For the first instance we shall deal with the composition $(\Theta_m \circ L)(x,\mathrm{D}) := \Theta_m(\mathrm{D}) \circ L(x,\mathrm{D})$.

Lemma 3.1. *Let $L(\cdot,\cdot)$ be a mapping $\mathbb{R}^n \times \mathbb{Z}^n \to \mathbb{C}$ so that $L(\cdot,l) \in C^\infty_\pi$ for any $l \in \mathbb{Z}^n$ and that (with $C_\alpha > 0$ and $\mu_\alpha \in \mathbb{R}$) the estimate*

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x L)(x,l)| \le C_\alpha k_{\mu_\alpha}(l) \qquad \textit{for} \quad l \in \mathbb{Z}^n \tag{3.1}$$

holds. Then one has

$$\Theta_m(\mathrm{D}) \circ L(x,\mathrm{D}) = L(x,\mathrm{D}) \circ \Theta_m(\mathrm{D}) + R_m(x,\mathrm{D}), \tag{3.2}$$

where

$$R_m(x,l) = \sum_{|\gamma|=1} \int_0^1 \sum_{z\in\mathbf{Z}^n} (\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}^\gamma_x L)(\cdot,l))_z \, \mathrm{e}^{\mathrm{i}(z,x)} \, \mathrm{d}t \tag{3.3}$$

P r o o f. For any $\varphi \in C^\infty_\pi$ we obtain

$$\begin{aligned}
[(\Theta_m \circ L)(x,\mathrm{D})\varphi](x) &= (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \Theta_m(z)(L(x,\mathrm{D})\varphi)_z \, \mathrm{e}^{\mathrm{i}(x,z)} \\
&= (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \Theta_m(z) \Big[(2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} (L(\cdot,l))_{z-l}\varphi_l\Big] \, \mathrm{e}^{\mathrm{i}(z,x)} \\
&= (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \Theta_m(z)(L(\cdot,l))_{z-l} \, \mathrm{e}^{\mathrm{i}(z-l,x)} \, \varphi_l \, \mathrm{e}^{\mathrm{i}(l,x)},
\end{aligned} \tag{3.4}$$

where the order of summation is legitimate to change, since $\Theta_m \in S$. In the third step we used the relation

$$\begin{aligned}
(L(x,\mathrm{D})\varphi)_z &= (2\pi)^{-n} \int_W \sum_{l\in\mathbf{Z}^n} L(x,l)\varphi_l \, \mathrm{e}^{\mathrm{i}(l-z,x)} \, \mathrm{d}x \\
&= (2\pi)^{-n} \sum_{l\in\mathbf{Z}^n} \int_W L(x,l)\varphi_l \, \mathrm{e}^{\mathrm{i}(l-z,x)} \, \mathrm{d}x,
\end{aligned}$$

which is valid, since the sum $\sum_{l\in\mathbf{Z}^n} L(x,l)\varphi_l \, \mathrm{e}^{\mathrm{i}(z-l,x)}$ is by (3.1) uniformly convergent in $\mathbb{R}^n$.

From (3.4) we see that

$$(\mathrm{O}_m \circ L)(x,l) = (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \Theta_m(l+z)(L(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}$$

(note that $(\Theta_m \circ L)(\cdot,\cdot)$ is a function $\mathbb{R}^n \times \mathbb{Z}^n \to \mathbb{C}$ so that $(\mathrm{O}_m \circ L)(\cdot,l) \in C^\infty_\pi$ for any $l \in \mathbb{Z}^n$ and that $|\mathrm{D}^\alpha_x(\Theta_m \circ L)(x,l)| \le C'_\alpha k_{\mu'_\alpha}(l))$. Due to the Taylor formula we obtain

$$(\Theta_m \circ L)(x,l) = (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \mathrm{O}_m(l)(L(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}$$

$$+ (2\pi)^{-n} \sum_{z\in\mathbf{Z}^n} \Big[\sum_{|\gamma|=1} \int_0^1 (\partial^\gamma \Theta_m)(l+tz)\Big] z^\gamma (L(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}\, \mathrm{d}t$$

$$= L(x,l)\Theta_m(l) + (2\pi)^{-n} \sum_{|\gamma|=1} \int_0^1 \sum_{z\in\mathbf{Z}^n} (\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}^\gamma_x L)(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}\, \mathrm{d}t$$

$$= (L \circ \Theta_m)(x,l) + R_m(x,l),$$

as required.

From (3.3) one sees easily that $R_m(\cdot,\cdot)$ is a function $\mathbb{R}^n \times \mathbb{Z}^n \to \mathbb{C}$, $R_m(\cdot,l) \in C^\infty_\pi$ for any $l \in \mathbb{Z}^n$ and that

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x R_m)(x,l)| \le C''_\alpha k_{\mu''_\alpha}(l).$$

A more careful study of the rest operator $R_m(x,\mathrm{D})$ yields

Lemma 3.2. *Suppose that for any $\alpha \in \mathbf{N}^n_0$ there exists a function $k_\alpha \in K_\pi$ so that*

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x L)(x,l)| \le C_\alpha k_\alpha(l) \qquad \text{for} \quad l \in \mathbb{Z}^n \tag{3 5}$$

and that $R_m(\cdot,\cdot)$ is defined by (3.3). Then one has

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x R_m)(x,l)| \le C'_\alpha(\bar{k}_\alpha k_{-1})(l) \qquad \text{for} \quad l \in \mathbb{Z}^n, \tag{3.6}$$

where

$$\bar{k}_\alpha := \max_{\substack{|\beta|\le n+2 \\ |\gamma|=1}} \{k_{\alpha+\beta+\gamma}\} \tag{3.7}$$

P r o o f. A. Define $g^\gamma(x,l,t) := \sum\limits_{z\in\mathbf{Z}^n} (\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^\gamma L)(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}$.
We shall establish that

$$|(\mathrm{D}_x^\alpha g^\gamma)(x,l,t)| \le C_{\alpha,\gamma}(k_{\alpha,\gamma}k_{-1})(l), \tag{3.8}$$

where $k_{\alpha,\gamma} := \max\limits_{|\beta|\le n+2}\{k_{\alpha+\beta+\gamma}\}$. (3.8) implies immediately the estimate (3.6).

Since $\Theta := F\tilde{\Theta} \in S$ we obtain that with some $C''_\gamma > 0$

$$|(\partial^\gamma\Theta)(x)| \le C''_\gamma k_{-1}(x) \qquad \text{for all} \quad x\in\mathbf{R}^n$$

and so one has (note that $\Theta_m = \Theta(l/m)$)

$$\begin{aligned} |(\partial^\gamma\Theta_m)(l+tz)| &= (1/m)|(\partial^\gamma\Theta)((l+tz)/m)| \\ &\le (C''_\gamma/m)(1+|(l+tz)/m|^2)^{-1/2} = C''_\gamma(m^2+|1+tz|^2)^{-1/2} \\ &\le C''_\gamma k_{-1}(l+tz). \end{aligned} \tag{3.9}$$

B. For any $|\beta| \le n+2$ one gets

$$\begin{aligned} &|z^\beta[(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z| \\ &\quad = |(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha+\beta} L)(\cdot,l))_z| \\ &\quad \le C''_\gamma(2\pi)^n C_{\alpha+\beta+\gamma}k_{-1}(l+tz)k_{\alpha+\beta+\gamma}(l) \\ &\quad \le C_{\alpha,\beta,\gamma}k_{-1}(l+tz)k_{\alpha,\gamma}(l) \end{aligned}$$

and so with a suitable constant $C'_{\alpha,\gamma} > 0$

$$\begin{aligned} &|(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z| \\ &\qquad \le C'_{\alpha,\gamma}k_{-1}(l+tz)k_{\alpha,\gamma}(l)k_{-(n+2)}(z). \end{aligned} \tag{3.10}$$

Specifically, the estimate (3.10) implies that the series (note that $k_{-1}(l+tz) \le 1$)

$$\begin{aligned} &\sum_z \mathrm{D}_x^\alpha\left[(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^\gamma L)(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)}\right] \\ &\quad = \sum_z(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^\gamma L)(\cdot,l))_z z^\alpha\, \mathrm{e}^{\mathrm{i}(z,x)} \\ &\quad = \sum_z(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)} \end{aligned}$$

is (absolutely) and uniformly (in $\mathbf{R}^n$) convergent for any $\alpha\in\mathbf{N}_0^n$. Hence $g^\gamma(\cdot,l,t)\in C_\pi^\infty$ for any $l\in\mathbb{Z}^n$ and $t\in[0,1]$ and $(\mathrm{D}_x^\alpha g^\gamma)(\cdot,l,t)$ is given by

$$(\mathrm{D}_x^\alpha g^\gamma)(x,l,t) = \sum_z(\partial^\gamma\Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z\, \mathrm{e}^{\mathrm{i}(z,x)} \tag{3.11}$$

C. To obtain the estimate (3.6) we decompose the sum in (3.11) as follows

$$\sum_z (\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z \, e^{i(z,x)} =$$

$$\sum_{2|z|>|l|} (\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z \, e^{i(z,x)}$$

$$+ \sum_{2|z|\leq|l|} (\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z \, e^{i(z,x)}$$

$$=: S_{\alpha,1}^\gamma(x,l,t) + S_{\alpha,2}^\gamma(x,l,t). \quad (3.12)$$

C_1. In the case when $l \leq 2|z|$ one gets by (3.10)

$$|(\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z|$$
$$\leq C'_{\alpha,\gamma} k_{\alpha,\gamma}(l) k_{-1}(z) k_{-(n+1)}(z) \leq 2C'_{\alpha,\gamma}(k_{\alpha,\gamma}k_{-1})(l) k_{-(n+1)}(z)$$

and so

$$S_{\alpha,1}(x,l,t) \leq 2C'_{\alpha,\gamma}\Big(\sum_z k_{-(n+1)}(z)\Big)(k_{\alpha,\gamma}k_{-1})(l). \quad (3.13)$$

C_2. In the case when $|l| \geq 2|z|$ one finds that

$$|l+tz| \geq |l| - |z| \geq (1/2)|l|$$

and so for $|l| \geq 2|z|$ we have by (3.10)

$$|(\partial^\gamma \Theta_m)(l+tz)((\mathrm{D}_x^{\gamma+\alpha} L)(\cdot,l))_z| \leq 2C'_{\alpha,\gamma} k_{-1}(l) k_{\alpha,\gamma}(l) k_{-(n+1)}(z).$$

This yields the estimate

$$\left|S_{\alpha,2}^\gamma(x,l,t)\right| \leq 2C'_{\alpha,\gamma}\Big(\sum_z k_{-(n+1)}(z)\Big)(k_{\alpha,\gamma}k_{-1})(l) \quad (3.14)$$

and so by (3.11)–(3.13) we get

$$|(\mathrm{D}_x^\alpha g^\gamma)(x,l,t)| \leq C_{\alpha,\gamma}(k_{\alpha,\gamma}k_{-1})(l),$$

as desired.

Remark 3.5. From the proof of Lemma 3.2 one sees that the constants C'_α in (3.6) obey

$$C'_\alpha \leq \sum_{|\gamma|=1}\Big(\sum_{|\beta|\leq n+2}(C''_\gamma C_{\alpha+\beta+\gamma})^2\Big)^{1/2}\Big(\sum_z k_{-(n+1)}(z)\Big).$$

Combining Theorem 2.1 and Lemma 3.2. we get

Theorem 3.4. *Suppose that* $L(\cdot,\cdot)$ *belongs to* A_π *and that for any* $|\alpha| \le [N_h + n + \varepsilon] + n + 3$ *the estimate*

$$\sup_{x\in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha k(l)/h(l) \tag{3.15}$$

holds. Let $R_m(\cdot,\cdot)$ *be defined by* (3.3). *Then one has*

$$\|R_m(x,\mathrm{D})\varphi\|_{p,h} \le C\|\varphi\|_{p,kk_{-1}} \qquad \textit{for all} \quad \varphi \in C_\pi^\infty, \tag{3.16}$$

where C *does not depend on* $m \in \mathbf{N}$ *and* $p \in [1,\infty]$.

Proof. A. Any $R_m(\cdot,\cdot)$ belongs to A_π: In virtue of (3.6) one sees that

$$\sup_{x\in W} |(\mathrm{D}_x^\alpha R_m)(x,l)| \le C'_\alpha(\bar{k}_\alpha k_{-1})(l). \tag{3.17}$$

Since

$$\sup_{x\in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha k_{\mu+\delta|\alpha|}(l),$$

we can choose $k_\alpha = k_{\mu+\delta|\alpha|}$ and so

$$\bar{k}_\alpha \le k_{\mu+\delta(n+3)+\delta|\alpha|}.$$

Thus $R_m(\cdot,\cdot) \in A_\pi$.

B. For any $|\alpha| \le [N_h + n + \varepsilon] + n + 3$ we can choose $k_\alpha = k/h$ and so

$$k_{\alpha+\beta+\gamma} \le k/h \qquad \text{for any} \quad |\alpha| \le [N_h + n + \varepsilon], \quad |\beta| \le n+2, \quad |\gamma| = 1$$

This implies that

$$\bar{k}_\alpha \le k/h \qquad \text{for any} \quad |\alpha| \le [N_h + n + \varepsilon]$$

and so by (3.17)

$$\sup_{x\in W} |(\mathrm{D}_x^\alpha R_m)(x,l)| \le C'(kk_{-1}/h)(l), \qquad \text{for} \quad |\alpha| \le [N_h + n + \varepsilon].$$

Applying Theorem 2.1 to the set $\mathcal{R} := \{R_m(\cdot,\cdot) \mid m \in \mathbf{N}\}$ one gets that

$$\|R_m(x,\mathrm{D})\varphi\|_{p,h} \le C\|\varphi\|_{p,kk_{-1}} \qquad \text{for} \quad \varphi \in C_\pi^\infty,$$

where C does not depend on m and p. This finishes the proof.

3.2. Suppose that $L(\cdot,\cdot)$ belongs to A_π. Then the formal transpose of $L(x,\mathrm{D})$ and $R_m(x,\mathrm{D})$ exists (cf. the proof of Theorem 3.4). Furthermore, the formal transpose $\Theta'_m(\mathrm{D})$ of $\Theta_m(\mathrm{D})$ exists. Thus we can define the continuous extensions $\bar{\Theta}_m$, $\bar{L}$ and $\bar{R}_m : D'_\pi \to D'_\pi$. From (3.2) one sees that

$$R'_m(x,\mathrm{D}) = L'(x,\mathrm{D}) \circ \Theta'_m(\mathrm{D}) - \Theta'_m(\mathrm{D}) \circ L'(x,\mathrm{D}) \tag{3.18}$$

and so

$$\bar{R}_m u = \bar{\Theta}_m(\bar{L}u) - \bar{L}(\bar{\Theta}_m u) \qquad \text{for} \quad u \in D'_\pi. \tag{3.19}$$

We are ready to establish

Theorem 3.5. *Suppose that $L(\cdot,\cdot)$ belongs to A_π and that for any $|\alpha| \leq [N_h + n + \varepsilon] + n + 3$ the estimate* (3.15) *holds. Then the equality*

$$L^{\sim}_{p,kk_{-1},h} = L'^{\#}_{p,kk_{-1},h}, \qquad p \in [1,\infty[, \quad k,h \in K'_\pi \tag{3.20}$$

holds.

Proof. Let u be in $\mathrm{D}(L'^{\#}_{p,kk_{-1},h}) \subset B^{\pi}_{p,kk_{-1}}$. Due to Lemma 2.2 one has $\Theta_m u \in C^\infty_\pi$ and so

$$\bar{L}(\bar{\Theta}_m u) = L(x,\mathrm{D})(\bar{\Theta}_m u) = L^{\sim}_{p,kk_{-1},h}(\bar{\Theta}_m u).$$

Furthermore, in virtue of (3.16) we get

$$\|\bar{R}_m u\|_{p,h} \leq C\|u\|_{p,kk_{-1}} \tag{3.21}$$

and so by (3.19) one has (note that $\bar{L}u = L'^{\#}_{p,kk_{-1},h}u$)

$$\begin{aligned} &\|L^{\sim}_{p,kk_{-1},h}(\bar{\Theta}_m u) - L'^{\#}_{p,kk_{-1},h}u\|_{p,h} \\ &\qquad \leq \|\bar{\Theta}_m(L'^{\#}_{p,kk_{-1},h}u) - L'^{\#}_{p,kk_{-1},h}u\|_{p,h} + C\|u\|_{p,kk_{-1}} \end{aligned} \tag{3.22}$$

for all $m \in \mathbf{N}$ and $u \in \mathrm{D}(L'^{\#}_{p,kk_{-1},h})$.

Let ε be a positive number. Choose $\varphi \in S_\pi$ so that $\|u - \varphi\|_{p,kk_{-1}} < \varepsilon$. Furthermore, choose $m_0 \in \mathbf{N}$ such that (cf. Lemma 2.2)

$$\|\bar{\Theta}_m(L'^{\#}_{p,kk_{-1},h}(u-\varphi)) - L'^{\#}_{p,kk_{-1},h}(u-\varphi)\|_{p,h} < \varepsilon \tag{3.23}$$

and that

$$\|\Theta_m\varphi - \varphi\|_{p,k} < \varepsilon \qquad \text{for} \quad m \geq m_0.$$

Due to Theorem 2.1 one has with some constant $C' > 0$

$$\|L(x,\mathrm{D})\varphi\|_{p,h} \leq C'\|\varphi\|_{p,k} \qquad \text{for all} \quad \varphi \in C^\infty_\pi$$

and so

$$\|L(x,\mathrm{D}(\Theta_m\varphi) - L(x,\mathrm{D})\varphi\|_{p,h} \leq C'\varepsilon \qquad \text{for} \quad m \geq m_0. \tag{3.24}$$

Using (3.22)–(3.24) we observe that

$$\begin{aligned} &\|L^{\sim}_{p,kk_{-1},h}(\Theta_m u) - L'^{\#}_{p,kk_{-1},h}u\|_{p,h} \\ &\qquad \leq \|\bar{\Theta}_m(L'^{\#}_{p,kk_{-1},h}(u-\varphi) - L'^{\#}_{p,kk_{-1},h}(u-\varphi)\|_{p,h} \\ &\qquad + \|L(x,\mathrm{D})(\bar{\mathrm{O}}_m\varphi) - L(x,\mathrm{D})\rho\|_{p,h} + C\|u-\varphi\|_{p,kk_{-1}} \\ &\qquad\qquad \leq \varepsilon + (C+C')\varepsilon \qquad \text{for} \quad m \geq m_0. \end{aligned}$$

Hence

$$\|L^{\sim}_{p,kk_{-1},h}(\bar{\Theta}_m u) - L'^{\#}_{p,kk_{-1},h}u\|_{p,h} \to 0 \qquad \text{with} \quad m \to \infty$$

and since (cf. Lemma 2.2)

$$\|\bar{\Theta}_m u - u\|_{p,kk_{-1}} \to 0 \qquad \text{with} \quad m \to \infty,$$

one sees that $u \in \mathrm{D}(L_{p,kk_{-1},h})$ and that $L^{\sim}_{p,kk_{-1},h}u = L'^{\#}_{p,kk_{-1},h}u$, as required.

We obtain the next corollaries

Corollary 3.6. *Suppose that* $L(\cdot,\cdot)$ *belongs to* A_π *and that for any* $|\alpha| \le [N_k + n + \varepsilon] + n + 3$ *the estimate*

$$\sup_{x \in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha k_m(l) \qquad \textit{for} \quad l \in \mathbb{Z}^n \tag{3.25}$$

holds, where $m \in \mathbb{R}$. *Then one has*

$$L^{\sim}_{p,kk_{m-1},k} = L'^{\#}_{p,kk_{m-1},k} \qquad \textit{for} \quad p \in [1,\infty[\quad k \in K'_\pi. \tag{3.26}$$

P r o o f. In view of (3.25) one sees that

$$\sup_{x \in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha (kk_m)(l)/k(l)$$

for $|\alpha| \le [N_k + n + \varepsilon] + n + 3$. Hence by Theorem 3.5 we obtain $L^{\sim}_{p,kk_m k_{-1},k} = L'^{\#}_{p,kk_m k_{-1},k}$, as we asserted.

Corollary 3.7. *Let* $m \in \mathbf{N}$ *and let*

$$L(x,\mathrm{D}) = \sum_{|\sigma| \le m} a_\sigma(x)\,\mathrm{D}^\sigma$$

be a linear partial differential operator with smooth periodic coefficients (that is, $a_\sigma \in C_\pi^\infty$*). Then for any* $p \in [1,\infty[$, $k \in K'_\pi$ *one has*

$$L^{\sim}_{p,kk_{m-1},k} = L'^{\#}_{p,kk_{m-1},k}. \tag{3.27}$$

P r o o f. The mapping $L(\cdot,\cdot)$ obeys

$$\sup_{x \in W} |(\mathrm{D}_x^\alpha L)(x,l)| \le C_\alpha k_m(l)$$

for any $\alpha \in \mathbf{N}_0^n$. Hence the proof follows from Corollary 3.6.

Corollary 3.8. *Let $L(x,\mathrm{D})$ be the first order linear partial differential operator with coefficients $a_\sigma \in C_\pi^\infty$. Then the equality*

$$L^{\sim}_{p,k,k} = L'^{\#}_{p,k,k} \qquad \text{for} \quad p \in [1,\infty[\quad k \in K'_\pi \tag{3.28}$$

holds.

Apply Corollary 3.7 with $m = 1$.

Remark 3.9. We have $B^\pi_{2,k_0} = L_2(W) \bigcap D'_\pi = \{u \in L_2(W) \mid u \text{ is periodic}\}$ Due to Corollary 3.8 for any first order smooth periodic partial differential operator $L(x,\mathrm{D})$ the relation $L^{\sim} = L'^{\#}$ holds, where $L^{\sim} = L^{\sim}_{2,k_0,k}$ and $L'^{\#} = L'^{\#}_{2,k_0,k_0}$. Hence for any weak solution of $L(x,\mathrm{D})u = f$; $u,f \in B^\pi_{2,k}$ there exists a sequence $\{\varphi_n\} \subset S_\pi$ so that

$$\|\varphi_n - u\| + \|L(x,\mathrm{D})\varphi_n - f\| \to 0 \qquad \text{with} \quad n \to \infty,$$

where $\|\cdot\| := \|\cdot\|_{2,k_0} = \|\cdot\|_{L_2(W)}$.

4. On the identity $L^{\sim}_{p,p',k} = L'^{\#}_{p,p',k}$

We recall that $L^{\sim}_{p,p',k}$ and $L'^{\#}_{p,p',k}$ denotes the minimal and respective the maximal realization of $L(x,\mathrm{D})$ from $B^\pi_{p,k}$ into $L_{p'}(W) \bigcap D'_\pi$. We need the following lemma

Lemma 4.1. *Suppose that $L(\cdot,\cdot) \in A_\pi$ such that*

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x L)(x,l)| \le C_\alpha k(l) k_1(l) \qquad \text{for} \quad l \in \mathbb{Z}^n \tag{4.1}$$

for any $|\alpha| \le [n+\varepsilon] + n + 3$. Then one has for $p \in [1,2]$, $1/p + 1/p' = 1$,

$$\|R_m(x,\mathrm{D})\varphi\|_{p'} \le C\|\varphi\|_{p,k} \qquad \text{for all} \quad \varphi \in C^\infty_\pi, \tag{4.2}$$

where C does not depend on p and m. Here $R_m(\cdot,\cdot)$ is defined by (3.3) and we denote $\|\cdot\|_{p'} = \|\cdot\|_{L_{p'}(W)}$.

Proof. A. In virtue of (4.1) one sees that

$$\sup_{x\in W} |(\mathrm{D}^\alpha_x L)(x,l)| \le C_\alpha (kk_1)(l)/k_0(l)$$

for any $|\alpha| \le [N_{k_0} + n + \varepsilon] + n + 3$ (note that $N_{k_0} = 0$). Hence we obtain by Theorem 3.4 that

$$\|R_m(x,\mathrm{D})\varphi\|_{L_2(W)} = \|R_m(x,\mathrm{D})\varphi\|_{2,k_0} \le C_1\|\varphi\|_{2,k}, \tag{4.3}$$

where C_1 does not depend on m.

B. Furthermore, we get by (3.6)

$$\begin{aligned}\|R_m(x,\mathrm{D})\varphi\|_{L_\infty(W)} &= \sup_{x\in W} |[R_m(x,\mathrm{D})\varphi](x)| \\ &\le \sum_l |R_m(x,l)|\,\varphi_l| \le C_0' \sum_l |\varphi_l k(l)| = C_0'\|\varphi\|_{1,k} \quad (4.4)\end{aligned}$$

(since $k_0 = \max\limits_{\substack{|\gamma|=1\\|\beta|\le n+2}} \{k_{\beta+\gamma}\} = \max\limits_{\substack{|\gamma|=1\\|\beta|\le n+2}} \{k_1 k\} = k_1 k$). Hence one obtains that the operators

$$R_m(x,\mathrm{D})\circ T^{-1}\colon L_2(\mathbb{Z}^n, d\nu) \to L_2(W, dm)$$

and

$$R_m(x,\mathrm{D})\circ T^{-1}\colon L_1(\mathbb{Z}^n, d\nu) \to L_\infty(W, dm)$$

are bounded. Here dm denotes the *Lebesque measure in* W and $d\nu$ denotes the *counting measure in* $\mathbb{Z}^n$. The operator T is an injection

$$C_\pi^\infty \to L_1(\mathbb{Z}^n, d\nu) \bigcap L_2(\mathbb{Z}^n, d\nu)$$

such that

$$(T\varphi)(l) = \varphi_l k(l).$$

The application of the *Riesz-Thorin Theorem* (cf. [2], p. 2) yields that the operator

$$R_m(x,\mathrm{D})\circ T^{-1}\colon L_p(\mathbb{Z}^n, d\nu) \to L_{p'}(W, dm)$$

is bounded and that with $0 < \Theta < 1$ one has

$$\|R_m(x,\mathrm{D})\circ T^{-1}\| \le C_1^{1-\Theta}(C_0')^\Theta \le \max\{C_1, C_0'\}$$

(note that when $1/p = (1-\Theta)/2 + \Theta/1$ and $1/q = (1-\Theta)/2 + \Theta/\infty$, then $q = p'$ and $1 < p < 2$). Thus we obtain

$$\begin{aligned}\|R_m(x,\mathrm{D})\varphi\|_{p'} &= \|R_m(x,\mathrm{D})\circ T^{-1})(T\varphi)\|_{p'} \\ &\le \max\{C_1, C_0'\}\,\|T\varphi\|_p = \max\{C_1, C_0'\}\,\|\varphi\|_{p,k}, \quad (4.5)\end{aligned}$$

where $C := \max\{C_1, C_0'\}$ does not depend on m and p. This proves the assertion.

We establish the next theorem for the equality of realizations

Theorem 4.2. *Suppose that* $L(\cdot,\cdot) \in A_\pi$ *and that the estimate* (4.1) *holds for any* $|\alpha| \leq [n+\varepsilon] + n + 3$. *Then one has*

$$\boldsymbol{L}^{\sim}_{p,p',k} = \boldsymbol{L}'^{\#}_{p,p',k} \qquad \text{for} \quad p \in]1,2], \quad k \in K'_\pi. \tag{4.6}$$

P r o o f. A. From (3.19) one gets

$$\boldsymbol{L}^{\sim}_{p,p',k}(\bar{\Theta}_m u) = \Theta_m(\boldsymbol{L}'^{\#}_{p,p',k} u) - \bar{R}_m u \tag{4.7}$$

for any $u \in \mathrm{D}(\boldsymbol{L}'^{\#}_{p,p',k})$, since $\bar{\Theta}_m u \in C^\infty_\pi \subset \mathrm{D}(\boldsymbol{L}^{\sim}_{p,p',k})$. Similarly as in the proof of Lemma 4.1 one gets that

$$\|L(x,\mathrm{D})\varphi\|_{p'} \leq C\|\varphi\|_{p,kk_1} \qquad \text{for } \varphi \in C^\infty_\pi, \tag{4.8}$$

and by Lemma 4.1 we obtain

$$\|\bar{R}_m u\|_{p'} \leq C\|u\|_{p,k} \qquad \text{for all} \quad m \in \mathbf{N}. \tag{4.9}$$

We shall verify that for any $f \in L_{p'}(W) \bigcap D'_\pi$ the approximation

$$\|\bar{\Theta}_m(f) - f\|_{p'} \to 0 \qquad \text{with} \quad m \to \infty \tag{4.10}$$

holds. Then the assertion follows with the same kind of conclusion as we made in the proof of Theorem 3.5.

B. Let ϕ be in $C^\infty_0(W)$. Define a function $\phi^\pi \colon \mathbf{R}^n \to \mathbf{C}$ by the relation

$$\phi^\pi(x) = (2\pi)^{-n} \sum_l (F\phi)(l)\,\mathrm{e}^{\mathrm{i}(l,x)}. \tag{4.11}$$

Then one sees thst $\phi^\pi \in C^\infty_\pi$. Furthermore, for any $\varphi \in C^\infty_0(W)$ one has (cf. [10], pp. 86 88)

$$\begin{aligned}\int_W \phi^\pi(x)\varphi(x)\,\mathrm{d}x &= (2\pi)^{-n}\sum_l (F\phi)(l)(F\varphi)(-l) \\ &= (2\pi)^{-n}\sum_l \langle\phi, \mathrm{e}^{\mathrm{i}(l,\cdot)}\rangle_{L_2(W)}\langle\varphi, \mathrm{e}^{-\mathrm{i}(l,\cdot)}\rangle_{L_2(W)} \\ &\qquad - \int_W \phi(x)\varphi(x)\,\mathrm{d}x,\end{aligned}$$

and so $\phi^\pi\big|_W - \phi$. For any $l \in \mathbb{Z}^n$ one gets

$$(\mathrm{O}_m\phi^\pi)_l = \mathrm{O}_m(l)(\phi^\pi)_l = (F\tilde{\Theta}_m)(l)(F\phi)(l) - F(\tilde{\mathrm{O}}_m * \phi)(l), \tag{4.12}$$

where $*$ denotes the *convolution of functions* $\tilde{\Theta}_m$ and $\phi \in C_0^\infty(W)$. We find that $\operatorname{supp}(\tilde{\Theta}_m * \phi) \subset \operatorname{supp} \tilde{\Theta}_m + \operatorname{supp}\phi \subset \bar{B}(0, 1/m) + \operatorname{supp}\phi$ and so $\tilde{\Theta}_m * \phi \in C_0^\infty(W)$ for m large enough, say $m \geq m_0$. Thus we get by (4.12)

$$\begin{aligned} \|\tilde{\Theta}_m(\phi^\pi) - \phi^\pi\|_{p'} &= \|(\tilde{\Theta}_m * \phi)^\pi - \phi^\pi\|_{p'} \\ &= \|\tilde{\Theta}_m * \phi - \phi\|_{p'} = \|(\tilde{\Theta}_m * \phi)^\vee - \phi^\vee\|_{p'} \\ &= \|F(\tilde{\Theta}_m * \phi) - F\phi\|_{p',1} = \|\Theta_m F\phi - F\phi\|_{p',1} \to 0 \end{aligned} \tag{4.13}$$

with $m \to \infty$ (cf. [5], p. 42; the norm $\|\cdot\|_{p',1} = \|\cdot\|_{p',k_0}$ denotes the *Hörmander norm*). In addition, one has
for $m \geq m_0$

$$\|\bar{\Theta}_m(\phi^\pi)\|_{p'} = \|\tilde{\Theta}_m * \phi\|_{p'} \leq \|\tilde{\Theta}\|_{L_1(W)}\|\phi^\pi\|_{p'}. \tag{4.14}$$

Since $C_0^\infty(W)$ is dense in $L_{p'}(W)$ one gets from (4.14) that

$$\|\bar{\Theta}_m(f)\|_{p'} \leq \|\tilde{\Theta}\|_{L_1(W)}\|f\|_{p'} \qquad \text{for all} \quad f \in L_{p'}(W) \cap D'_\pi. \tag{4.15}$$

Let ε be a positive number. Choose $\phi \in C_0^\infty(W)$ so that

$$\|\phi^\pi - f\|_{p'} = \|\phi - f\|_{p'} < \varepsilon$$

and choose $m_\varepsilon \geq m_0$ such that

$$\|\bar{\Theta}_m(\phi^\pi) - \phi^\pi\|_{p'} < \varepsilon \qquad \text{for} \quad m \geq m_\varepsilon.$$

Then we obtain for $m \geq m_\varepsilon$

$$\begin{aligned} \|\bar{\Theta}_m(f) - f\|_{p'} &\leq \|\bar{\Theta}_m(\phi^\pi) - \phi^\pi\|_{p'} + \|\bar{\Theta}_m(f - \phi^\pi)\|_{p'} + \|\phi^\pi - f\|_{p'} \\ &< \varepsilon + \|\tilde{\Theta}\|_{L_1(W)}\varepsilon + \varepsilon. \end{aligned} \tag{4.16}$$

Thus $\|\bar{\Theta}_m(f) - f\|_{p'} \to 0$ with $m \to \infty$, which completes the proof of (4.10).

Theorem 4.2 yields immediately

Corollary 4.3. *Let* $L(x, \mathrm{D}) = \sum_{\sigma \leq m} a_\sigma(x)\,\mathrm{D}^\sigma$ *be a partial differential operator with coefficients* $a_\sigma \in C_\pi^\infty$. *Then the identity*

$$L^\sim_{p,p',k_{m-1}} = L'^{\#}_{p,p',k_{m-1}} \qquad \textit{for any} \quad p \in \,]1,2] \tag{4.17}$$

holds.

Remark 4.4. Let $L(x, \mathrm{D}) = \sum_{\sigma \leq 1} a_\sigma(x)\,\mathrm{D}^\sigma$ be the first order, partial differential operator with coefficients $a_\sigma \in C_\pi^\infty$. Then the identity $L^\sim_{p,p'} := L^\sim_{p,p',k_0} = L'^{\#}_{p,p'}$ holds ($p \in \,]1,2]$). Hence for any solution of $L(x, \mathrm{D})u = f$; $u \in B^\pi_{p,k_0}$, $f \in L_{p'}(W) \cap D'_\pi$ there exists a sequence $\{\varphi_n\} \subset S_\pi$ so that $\|\varphi_n - u\|_{p,k_0} + |\, L(x, \mathrm{D})\varphi_n - f\|_{p'} \to 0$ with $n \to \infty$.

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