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Dedicated to Professor Sylvia Pulmannová on the occasion of her 65th birthday

LORENZEN'S THEOREM FOR PSEUDO-EFFECT ALGEBRAS

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(*Communicated by Gejza Wimmer*)

ABSTRACT. We present a variation of the Lorenzen theorem for pseudo-effect algebras satisfying a kind of the Riesz decomposition property. We show that the representability of pseudo-effect algebras as a subdirect product of antilattice pseudo-effect algebras depends on the notion of the polar of a pseudo-effect algebra.

1. Introduction

The famous Lorenzen theorem ([Lor], [Gla]) says that an ℓ-group G is representable, i.e., it is a subdirect product of linearly ordered groups if and only if the polars of G^+ are ℓ-ideals.

Recently, new partial algebraic structures, called pseudo-effect algebras and pseudo MV-algebras (as total algebraic structures), were introduced in [DvVe1], [DvVe2] and [GeIo]. They are a non-commutative generalization of effect algebras and MV-algebras, respectively, which are studied in many branches of mathematics and its applications. For example, such structures serve as models of quantum structures ([DvPu]) as well as in mathematical logic. Under some natural conditions, supposing a kind of Riesz decomposition property, they are always intervals in unital po-groups, see [DvVe1], [DvVe2]. Moreover, every pseudo MV-algebra is an interval in a unital ℓ-group, see [Dvu1].

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A generalization of the Lorenzen theorem for directed interpolation groups was presented by Glass [Gla; Theorem 42]; however in its proof, there are some unclear points. The Lorenzen theorem for pseudo MV-algebras was proved in [GeIo].

Inspired by these results, we present a variation of the Lorenzen theorem for pseudo-effect algebras satisfying a kind of the Riesz decomposition property. For this aim we introduce the notion of a polar and of a C-polar. The paper is organized as follows. In Section 2, we introduce elements of pseudo-effect algebras and pseudo MV-algebras. In Section 3, the polars for pseudo-effect algebras are presented and some results are proved. C-polars, where C is an ideal, are studied in Section 4. C-carriers are investigated in Section 5. Section 6 defines representable pseudo-effect algebras. Finally, the main result is given in Section 7, showing when a pseudo-effect algebra is a subdirect product of antilattice pseudo-effect algebras.

2. Pseudo-effect algebras

A partial algebra $(E;+,0,1)$, where $+$ is a partial binary operation and 0 and 1 are constants, is called a *pseudo-effect algebra* ([DvVe1], [DvVe2]) if, for all $a,b,c \in E$, the following hold

(i) $a+b$ and $(a+b)+c$ exist if and only if $b+c$ and $a+(b+c)$ exist, and in this case $(a+b)+c = a+(b+c)$;

(ii) there is exactly one $d \in E$ and exactly one $e \in E$ such that $a+d = e+a = 1$;

(iii) if $a+b$ exists, there are elements $d,e \in E$ such that $a+b = d+a = b+e$;

(iv) if $1+a$ or $a+1$ exists, then $a=0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a+c=b$, then $\le$ is a partial ordering on E such that $0 \le a \le 1$ for any $a \in E$. It is possible to show that $a \le b$ if and only if $b = a+c = d+a$ for some $c,d \in E$. We write $c = a \,/\, b$ and $d = b \setminus a$. Then

$$(b \setminus a) + a = a + (a \,/\, b) = b\,,$$

and we write $a^- = 1 \setminus a$ and $a^\sim = a \,/\, 1$ for any $a \in E$.

For basic properties of pseudo-effect algebras see [DvVe1], [DvVe2]. We recall that if $+$ is commutative, E is said to be an *effect algebra*. For properties of effect algebras see [DvPu].

For example, if (G,u) is a unital (not necessarily Abelian) po-group with strong unit u (in fact it is sufficient to take a positive element u in G),[1] and

$$\Gamma(G,u) := \{g \in G:\ 0 \le g \le u\},$$

then $\big(\Gamma(G,u); +, 0, u\big)$ is a pseudo-effect algebra if we restrict the group addition $+$ to $\Gamma(G,u)$.

According to [DvVe1], we introduce for pseudo-effect algebras the following forms of the *Riesz decomposition properties*:

(a) For $a,b \in E$, we write $a \,\mathbf{com}\, b$ to mean that for all $a_1 \le a$ and $b_1 \le b$, a_1 and b_1 commute.

(b) We say that E fulfils the *Riesz interpolation property*, (RIP) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1,a_2 \le b_1,b_2$, there is a $c \in E$ such that $a_1,a_2 \le c \le b_1,b_2$.

(c) We say that E fulfils the *weak Riesz decomposition property*, (RDP_0) for short, if for any $a,b_1,b_2 \in E$ such that $a \le b_1 + b_2$, there are $d_1,d_2 \in E$ such that $d_1 \le b_1$, $d_2 \le b_2$ and $a = d_1 + d_2$.

(d) We say that E fulfils the *Riesz decomposition property*, (RDP) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1,d_2,d_3,d_4 \in E$ such that $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$.

(e) We say that E fulfils the *commutational Riesz decomposition property*, (RDP_1) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1+a_2 = b_1+b_2$, there are $d_1,d_2,d_3,d_4 \in E$ such that
   (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$,
   (ii) $d_2 \,\mathbf{com}\, d_3$.

(f) We say that E fulfils the *strong Riesz decomposition property*, (RDP_2) for short, if for any $a_1,a_2,b_1,b_2 \in E$ such that $a_1 + a_2 = b_1 + b_2$, there are $d_1,d_2,d_3,d_4 \in E$ such that
   (i) $d_1 + d_2 = a_1$, $d_3 + d_4 = a_2$, $d_1 + d_3 = b_1$, $d_2 + d_4 = b_2$,
   (ii) $d_2 \wedge d_3 = 0$.

We introduce analogical notions for po-groups. Let G be a po-group and for $a,b \in G^+$, we write $a \,\mathbf{com}\, b$ if and only if, for all $a_1,b_1 \in G^+$ such that $a_1 \le a$ and $b_1 \le b$, we have $a_1 + b_1 = b_1 + a_1$.

Let $(G;+,0,\le)$ be a directed po-group. According to [DvVe1], [DvVe2], we say that G fulfills (RIP), (RDP_0), (RDP), (RDP_1), and (RDP_2), respectively, if

[1] We say that a positive element u of a po-group G is a *strong unit* if, for any $g \in G$, there is an integer $n \ge 1$ such that $g \le nu$.

analogical properties as those for pseudo-effect algebras hold also for the positive cone G^+ of G.

A mapping $h\colon E \to F$, where E and F are pseudo-effect algebras, is said to be a *homomorphism* if

(i) $h(0) = 0$ and $h(1) = 1$,

(ii) $h(a+b) = h(a) + h(b)$ whenever $a+b$ is defined in E.

If h is injective and surjective such that also h^{-1} is a homomorphism, then h is said to be an *isomorphism*, and E and F are *isomorphic*. It is clear that a one-to-one homomorphism f from E onto F is an isomorphism if and only if $f(a) \le f(b)$ implies $a \le b$.

According to [GeIo], a *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2,1,1,0,0)$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^- \oplus y^-)^\sim$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;

(A2) $x \oplus 0 = 0 \oplus x = x$;

(A3) $x \oplus 1 = 1 \oplus x = 1$;

(A4) $1^\sim = 0$; $1^- = 0$;

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;

(A6) $x \oplus x^\sim \odot y = y \oplus y^\sim \odot x = x \odot y^- \oplus y = y \odot x^- \oplus x$;[2]

(A7) $x \odot (x^- \oplus y) = (x \oplus y^\sim) \odot y$;

(A8) $(x^-)^\sim = x$.

If we define $x \le y$ if and only if $x^- \oplus y = 1$, then $\le$ is a partial order such that M is a distributive lattice with $x \vee y = x \oplus (x^\sim \odot y)$ and $x \wedge y = x \odot (x^- \oplus y)$. For basic properties of pseudo MV-algebras see [GeIo] or [DvPu].

If we define a partial binary operation $+$ on M via: $x+y$ is defined if and only if $x \le y^-$, and in this case $x + y := x \oplus y$, then $(M; +, 0, 1)$ is a pseudo-effect algebra. Moreover, a pseudo-effect algebra E can be converted into a pseudo MV-algebra such that the $+$ derived from $\oplus$ and the original $+$ coincide if and only if E satisfies (RDP_2) ([DvVe2]).

For example, if u is a strong unit of a (not necessarily Abelian) ℓ-group G,

$$\Gamma(G, u) := [0, u]$$

and

$$\begin{aligned} x \oplus y &:= (x+y) \wedge u, \\ x^- &:= u - x, \\ x^\sim &:= -x + u, \\ x \odot y &:= (x - u + y) \vee 0, \end{aligned}$$

[2] $\odot$ has a higher priority than $\oplus$.

then $\left(\Gamma(G,u);\oplus,^-,^\sim,0,u\right)$ is a pseudo MV-algebra ([GeIo]).

The basic representation theorem for pseudo-effect algebras is the following result [DvVe1], [DvVe2], and for pseudo MV-algebras see also [Dvu1].

THEOREM 2.1. *For a pseudo-effect algebra E fulfilling* (RDP_1), *there is a unique (up to isomorphism of unital po-groups) unital po-group (G,u) fulfilling* (RDP_1) *such that $E\cong\Gamma(G,u)$.*

If M is a pseudo MV-algebra, there is a unique (up to isomorphism of unital ℓ-groups) unital ℓ-group (G,u) such that $M\cong\Gamma(G,u)$.

A non-empty subset I of a pseudo-effect algebra E is said to be an *ideal* of E if

(i) $x+y\in I$ whenever $x,y\in I$ and if $x+y$ is defined in E,
(ii) if $x\le y$ for $x\in E$ and $y\in I$, then $x\in I$.

Then E as well as $\{0\}$ are ideals of E.

Let $\mathcal{I}(E)$ denote the set of all ideals of a pseudo-effect algebra E. According to [Dvu3] if E satisfies (RDP), then $\mathcal{I}(E)$ is a lattice with respect to the set-theoretical inclusion with meets and joins denoted simply by $\wedge$ and $\vee$.

An ideal I of E is

(i) *normal* if $a+I=I+a$ for all $a\in E$,[3]
(ii) *maximal* if I is a proper subset of E and it is not included in any proper ideal of E as a proper subset,
(iii) *prime* if $I_0(a)\cap I_0(b)\subseteq I$ implies $a\in I$ or $b\in I$ for all $a,b\in E$.[4]

We denote by $\mathcal{N}(E)$, $\mathcal{M}(E)$, and $\mathcal{P}(E)$ the set of all normal ideals, maximal ideals, and prime ideals, respectively, of E. Using the Zorn lemma, we see that $\mathcal{M}(E)$ is non-void. Under some conditions on E, [Dvu3], we can prove that $\mathcal{M}(E)\subseteq\mathcal{P}(E)$.

We recall that if E satisfies (RDP), then an ideal I is prime if and only if E/I is an antilattice, see [Dvu3; Proposition 4.6].

3. Polars and pseudo-effect algebras

For $\emptyset\ne A\subseteq E$, we set $A^\perp:=\{x\in E:\ x\wedge a=0 \text{ for all } a\in A\}$, and we refer to $A^\perp$ as the *polar* of A. We define $a^\perp:=\{a\}^\perp$ for $a\in E$. Then

$$a^\perp\cap a^{\perp\perp}=\{0\}, \qquad a\in E, \tag{3.1}$$

[3] If A is a non-empty subset of E, then $a+A:=\{a+x:\ x\in A \text{ and } a+x \text{ is defined in } E\}$. In a similar way we define $A+a$.

[4] By $I_0(a)$ and $N_0(a)$ we define any ideal and any normal ideal generated by $a\in E$.

and, for $\emptyset \neq A \subseteq E$,

$$A^{\perp} \cap A^{\perp\perp} = \{0\}, \qquad A \subseteq A^{\perp\perp}, \qquad A^{\perp} = A^{\perp\perp\perp}, \tag{3.2}$$

$A^{\perp} = \bigcap\{a^{\perp} : a \in A\}$, $B^{\perp} \subseteq A^{\perp}$ if $A \subseteq B \subseteq E$, and $b^{\perp} \subseteq a^{\perp}$ if $a \leq b$, $a, b \in E$.

We recall that if E satisfies (RDP_0) and $I_0(a)$ is the ideal of E generated by an element $a \in E$, and A is a non-void subset of E, then

$$a^{\perp} = I_0(a)^{\perp} \qquad \text{and} \qquad A^{\perp} = I_0(A)^{\perp},$$

where $I_0(A)$ is the ideal of E generated by A.

PROPOSITION 3.1. *Let E be a pseudo-effect algebra with (RDP_0). If $\emptyset \neq A \in E$, then $A^{\perp}$ is an ideal of E. In addition, if $a + b \in E$, then*

$$(a+b)^{\perp} = a^{\perp} \cap b^{\perp}.$$

Proof. $0 \in A^{\perp}$. If $x, y \in E$ and $x \leq y \in A^{\perp}$, then $x \in A^{\perp}$. Assume now $x, y \in A^{\perp}$ and let $x + y \in E$. Fix $a \in A$. If $z \leq x + y$ and $z \leq a$, then $z = x_1 + y_1$, where $x_1 \leq x$, $y_1 \leq y$, and $x_1, y_1 \in a^{\perp}$. While $x_1, y_1 \leq a$, we have $x_1 = x_1 \wedge a = 0 = y_1 \wedge a = y_1$, which proves $z = 0$.

In a similar way we prove the equation. □

PROPOSITION 3.2. *If A is an ideal of a pseudo-effect algebra E with (RDP_0), then $A \cap A^{\perp} = \{0\}$ and $A^{\perp}$ is the greatest ideal of E whose intersection with A is the null ideal.*

Proof. The first statement follows from (3.2). Assume that I is an ideal of E such that $I \cap A = \{0\}$. Let $x \in I$ and $a \in A$, then $x \wedge a = 0$, which yields $x \in A^{\perp}$. □

PROPOSITION 3.3. *Let E be a pseudo-effect algebra with (RDP_0). If A and B are ideals of E, then*

$$(A \cap B)^{\perp\perp} = A^{\perp\perp} \cap B^{\perp\perp}. \tag{3.3}$$

In particular, if $a, b \in E$, then

$$\big(I_0(a) \cap I_0(b)\big)^{\perp\perp} = a^{\perp\perp} \cap b^{\perp\perp}.$$

Proof. It is necessary to verify that $A^{\perp\perp} \cap B^{\perp\perp} \subseteq (A \cap B)^{\perp\perp}$. Choose $x \in A^{\perp\perp} \cap B^{\perp\perp}$, $y \in (A \cap B)^{\perp}$, and $a \in A$, $b \in B$. Assume $w \leq x, y, a, b$. Then $w \in A \cap B$, and since $w \leq w, y$, we have $w = 0$. So if $g \leq x, y, a$, then $g \in b^{\perp}$, therefore, $g \in B^{\perp}$. Since $x \in B^{\perp\perp}$ and $0 \leq g \leq g, x$, we have $g = 0$. Hence, if $v \leq x, y$ and $w \leq v, a$, then $w = 0$, i.e., $v \in a^{\perp}$ and $v \in A^{\perp}$. But $v \leq x \in A^{\perp\perp}$, which by (3.1) gives $v = 0$, consequently, $x \in (A \cap B)^{\perp\perp}$. □

PROPOSITION 3.4. *Let A and B be two ideals of a pseudo-effect algebra E with* (RDP_0). *Then*

$$(A \cap B)^{\perp} = (A^{\perp} \cup B^{\perp})^{\perp\perp}.$$

Proof. Since $A \cap B \subseteq A, B$, we have $A^{\perp} \cup B^{\perp} \subseteq (A \cap B)^{\perp}$. Hence, $(A \cap B)^{\perp\perp} \subseteq (A^{\perp} \cup B^{\perp})^{\perp}$. By Proposition 3.3, $A^{\perp\perp} \cap B^{\perp\perp} \subseteq (A^{\perp} \cup B^{\perp})^{\perp}$. Hence, if $x \in (A^{\perp} \cup B^{\perp})^{\perp}$ and $y \in A^{\perp} \cup B^{\perp}$, then $x \wedge y = 0$. If now $y \in A^{\perp}$, then $x \in A^{\perp\perp}$; if $y \in B^{\perp}$, then $x \in B^{\perp\perp}$, i.e., $x \in A^{\perp\perp} \cap B^{\perp\perp}$. □

4. C-polars in pseudo-effect algebras

According to [Gla], we generalize the notion of a polar as follows. Let C be an ideal of a pseudo-effect algebra E. The *C-polar* of a non-void subset A of E is the set $A^{\perp_C} := \{g \in E :\ (\forall a \in A)(c \le g, a \implies c \in C)\}$. We set $g^{\perp_C} := \{g\}^{\perp_C}$ if $g \in E$. We define $A^{\perp_C \perp_C} = \left(A^{\perp_C}\right)^{\perp_C}$. For example, if $C = \{0\}$, then $A^{\perp_{\{0\}}} = A^{\perp}$.

Many analogical properties as those for polars hold also for C-polars. We recall that C-polars for interpolation groups were studied in [Gla].

PROPOSITION 4.1. *Let E be a pseudo-effect algebra, $\emptyset \neq A \subseteq E$, and $C \in \mathcal{I}(E)$.*

(o) $A^{\perp_C} = \bigcap\{a^{\perp_C} :\ a \in A\}$.
(i) $C \subseteq A^{\perp_C}$.
(ii) $B^{\perp_C} \subseteq A^{\perp_C}$ *if* $A \subseteq B \subseteq E$.
(iii) $A^{\perp_C \perp_C \perp_C} = A^{\perp_C}$.
(iv) $A \subseteq A^{\perp_C \perp_C}$.
(v) $A^{\perp_C} \cap A^{\perp_C \perp_C} = C$.

Let E satisfy (RDP_0).

(vi) $A^{\perp_C} \in \mathcal{I}(E)$.
(vii) $\left(I_0(A)\right)^{\perp_C} = A^{\perp_C}$.
(viii) *If $x + y \in E$, then* $(x+y)^{\perp_C} = x^{\perp_C} \cap y^{\perp_C}$.
(ix) *If $C \subseteq A \in \mathcal{I}(E)$, then $A \cap A^{\perp_C} = C$, and $A^{\perp_C}$ is the largest ideal of E whose intersection with A is C.*

Proof. It follows he same ideas as those for polars. □

PROPOSITION 4.2. *If A is a non-void subset of a pseudo-effect algebra E, the following statements are equivalent.*

(i) $A \subseteq C$.
(ii) $A^{\perp_C} = E$.
(iii) $A \subseteq A^{\perp_C}$.

P r o o f . The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are evident. Assume now (iii). Then $A \subseteq A^{\perp_C}$ and, for any $a \in A$, we have $a \in A^{\perp_C} \subseteq a^{\perp_C}$. Therefore, if $c \leq a$, then $c \in C$, i.e., $a \in C$. □

As a consequence, we have $g^{\perp_C} = E$ if and only if $g \in C$. The following statement is direct.

PROPOSITION 4.3. *Let E be a pseudo-effect algebra and A a non-void subset of E.*

(i) *If $C_1, C_2 \in \mathcal{I}(E)$, $C_1 \subseteq C_2$, then $A^{\perp_{C_1}} \subseteq A^{\perp_{C_2}}$.*
(ii) *If $C_1, C_2 \in \mathcal{I}(E)$, then $A^{\perp_{C_1}} \cap A^{\perp_{C_2}} = A^{\perp_{(C_1 \cap C_2)}}$.*
(iii) *If $A, C \in \mathcal{I}(E)$, then $A^{\perp_C} = A^{\perp_{(A \cap C)}}$.*

PROPOSITION 4.4. *If $A, B, C \in \mathcal{I}(E)$, where E is a pseudo-effect algebra with* (RDP_0)*, then*

$$(A \cap B)^{\perp_C \perp_C} = A^{\perp_C \perp_C} \cap B^{\perp_C \perp_C},$$
$$(A \cap B)^{\perp_C} = (A^{\perp_C} \cup B^{\perp_C})^{\perp_C \perp_C}.$$

P r o o f . It follows the proof of (3.3), where we change $w = 0$ and $v = 0$ to $w \in C$ and $v \in C$, respectively. □

PROPOSITION 4.5. *Let $\{A_t\}_t$ be a non-void system of ideals of a pseudo-effect algebra E satisfying* (RDP_0)*. If $A = \bigcup_t A_t$, then $A^{\perp_C} = \bigcap_t A_t^{\perp_C}$.*

P r o o f . Since $A \supseteq A_t$ for any t, we have $A^{\perp_C} \subseteq A_t^{\perp_C}$, i.e., $A^{\perp_C} \subseteq \bigcap_t A_t^{\perp_C}$. Choose now $x \in \bigcap_t A^{\perp_C}$ and $a \in A$, and assume $w \leq x, a$. Then $w \in A_t^{\perp_C}$ for any t and simultaneously $w \in A_{t_0}$ for some t_0. Hence, $w \in C$ proving $x \in A^{\perp_C}$. □

Let C be an ideal of E. We denote by

$$\mathrm{Pol}_C(E) := \{A \subseteq E : \ A = A^{\perp_C \perp_C}\}.$$

By (i) of Proposition 4.1, we have $C \subseteq A \subseteq E$ for any $A \in \mathrm{Pol}_C(E)$.

THEOREM 4.6. *Let E be a pseudo-effect algebra with* (RDP)*. Then $(\mathrm{Pol}_C(E); \subseteq, ^{\perp_C}, C, E)$ is a complete Boolean algebra such that for the corresponding meets and joins we have $\bigwedge_t^{\mathrm{C}} A_t = \bigcap_t A_t$, $\bigvee_t^{\mathrm{C}} A_t = \left(\bigcup_t A_t\right)^{\perp_C \perp_C}$, and $A \wedge^C \left(\bigvee_t^{\mathrm{C}} A_t\right) = \bigvee_t^{\mathrm{C}} (A \wedge^C A_t)$.*

In addition, the mapping $\pi_C : \mathcal{I}(E) \to \mathrm{Pol}_C(E)$ *given by* $\pi_C(A) := A^{\perp_C \perp_C}$, $A \in \mathcal{I}(E)$, *is a lattice homomorphism of* $\mathcal{I}(E)$ *onto* $\mathrm{Pol}_C(E)$, *and* C *is the largest element of the set* $\{A \in \mathcal{I}(E) : \pi_C(A) = C\}$. *If* ϕ *is a lattice homomorphism of* $\mathcal{I}(E)$ *into a lattice* $\mathcal{X}$ *with* 0 *such that* C *is the largest element in the set* $\{A \in \mathcal{I}(E) : \phi(A) = 0\}$, *then* $\phi(I_1) = \phi(I_2)$ *implies* $\pi_C(I_1) = \pi_C(I_2)$.

Proof. According to Proposition 4.4, $\mathrm{Pol}_C(E)$ is a de Morgan lattice with $A \wedge^C B = A \cap B$ and $A \vee^C B = (A \cup B)^{\perp_C \perp_C}$, and $A \wedge^C A^{\perp_C} = C$ and $A \vee^C A^{\perp_C} = E$. In view of Proposition 4.5, $\bigvee_t^C A_t = \left(\bigcup_t A_t\right)^{\perp_C \perp_C} \in \mathrm{Pol}_C(E)$ and $\bigcap_t A_t = \bigcap_t (A_t^{\perp_C})^{\perp_C} \in \mathrm{Pol}_C(E)$. Hence, $\bigwedge_t^C A_t = \bigcap_t A_t$.

Further, $A \wedge^C \left(\bigvee_t^C A_t\right) = A \cap \left(\bigcup_t A_t\right)^{\perp_C \perp_C} = A^{\perp_C \perp_C} \cap \left(I_0\left(\bigcup_t A_t\right)\right)^{\perp_C \perp_C} = \left(A \cap \left(\bigvee_t A_t\right)\right)^{\perp_C \perp_C} = \left(\bigvee_t (A \cap A_t)\right)^{\perp_C \perp_C} = \left(I_0\left(\bigcup_t (A \cap A_t)\right)\right)^{\perp_C \perp_C} = \left(\bigcup_t (A \cap A_t)\right)^{\perp_C \perp_C} = \bigvee_t^C (A \wedge^C A_t)$, where we have used distributivity in the lattice $\mathcal{I}(E)$, see [Dvu3; Proposition 3.2].

Finally assume that $\mathcal{X}$ is a lattice with 0 and that $\phi : \mathcal{I}(E) \to \mathcal{X}$ is a lattice homomorphism with C the largest element of the set $\{A \in \mathcal{I}(E) : \phi(A) = 0\}$. Let I be an ideal of E and define $\hat{I} = \{M \in \mathcal{I}(E) : \phi(M) \wedge_{\mathcal{X}} \phi(I) = \phi(C)\}$. If $M \in \hat{I}$, then $M \cap I \subseteq C$, which yields $M \subseteq I^{\perp_{(C \cap I)}} = I^{\perp_C}$ by (iii) of Proposition 4.3. In addition, $\phi(I^{\perp_C} \cap I) = \phi(I^{\perp_{(C \cap I)}} \cap I) = \phi(I \cap C) = \phi(C)$. Hence, $I^{\perp_C} \in \hat{I}$, and so is the largest element of $\hat{I}$. Consequently, if $\phi(I_1) = \phi(I_2)$, $I_1^{\perp_C} = I_2^{\perp_C}$ yielding $\pi_C(I_1) = \pi_C(I_2)$. □

In the rest of the present section, we show the relation among prime ideals and C-polars.

We say that an ideal C of a pseudo-effect algebra E is *prime* in an ideal A of E if

(i) $C \subseteq A$,
(ii) for $a, b \in A$, $I_0(a) \cap I_0(b) \subseteq C$ implies $a \in C$ or $b \in C$.

Using ideas from [Dvu3], we have that an ideal C of a pseudo-effect algebra E with (RDP) is prime in A ($C \subseteq A$) if and only if $I \cap J \subseteq C$ for $I, J \subseteq A$, $I, J \in \mathcal{I}(E)$, implies $I \subseteq C$ or $J \subseteq C$ or if and only if $I \cap J = C$ for $I, J \subseteq A$, $I, J \in \mathcal{I}(E)$, implies $I = C$ or $J = C$.

THEOREM 4.7. *Let C and A, $C \subseteq A$, be ideals of a pseudo-effect algebra E with* (RDP). *The following statements are equivalent.*

(i) *C is prime in $A^{\perp c \perp c}$.*
(ii) *C is prime in A.*
(iii) *$A^{\perp c}$ is a prime ideal of E.*
(iv) *$A^{\perp c} = a^{\perp c}$ for all $a \in A \setminus C$.*
(v) *$A^{\perp c}$ is a maximal C-polar of an ideal containing C.*
(vi) *$A^{\perp c \perp c}$ is a minimal C-polar of an ideal containing C.*
(vii) *$A^{\perp c \perp c}$ is an ideal maximal with respect to the property of being C prime in it.*

Proof.

(i) $\Longrightarrow$ (ii). Since $C \subseteq A \subseteq A^{\perp c \perp c}$, the implication is evident.

(ii) $\Longrightarrow$ (iii). Let $I, J \in \mathcal{I}(E)$ be such that $I \cap J = A^{\perp c}$. Then $(A \cap I) \cap (A \cap J) = C$. Therefore, $A \cap I = C$ or $A \cap J = C$. Hence, $I \subseteq A^{\perp c}$ or $J \subseteq A^{\perp c}$ (by (ix) of Proposition 4.1), which proves $A^{\perp c}$ is a prime ideal of E.

(iii) $\Longrightarrow$ (ii). Let $A^{\perp c}$ be a prime ideal of E and let $I, J \in \mathcal{I}(E)$ be subsets of A such that $I \cap J = C$. Then $(I \vee A^{\perp c}) \cap (J \vee A^{\perp c}) = A^{\perp c}$, where $\vee$ denotes the join in the lattice $\mathcal{I}(E)$, which yields $I \vee A^{\perp c} \subseteq A^{\perp c}$ or $J \vee A^{\perp c} \subseteq A^{\perp c}$. Hence, $I \subseteq A^{\perp c}$ and in view of hypothesis $I \subseteq A$, we have $I \subseteq A^{\perp c} \cap A = C$. In a similar way we proceed in the second case.

(ii) $\Longrightarrow$ (iv). Assume that C is a prime ideal of A. Then, for all $a \in A$, $A^{\perp c} \subseteq a^{\perp c}$. If there exists $a \in A \setminus C$ such that $A^{\perp c} \neq a^{\perp c}$, then we can choose an element $x \in a^{\perp c} \setminus A^{\perp c}$. Since $A^{\perp c} = \bigcap\{a^{\perp c} : a \in A\}$, there exists $a_0 \in A$ such that $x \notin a_0^{\perp c}$. Consequently, there exists $y \in E \setminus C$ such that $y \leq a_0, x$. Then $y \in a^{\perp c} \cap A$. But C is prime in A, so we have by (v) of Proposition 4.1 $C = a^{\perp c} \cap a^{\perp c \perp c} = (a^{\perp c} \cap A) \cap (a^{\perp c \perp c} \cap A)$, so that $C = a^{\perp c} \cap A$ or $C = a^{\perp c \perp c} \cap A$. However, $y \in (a^{\perp c} \cap A) \setminus C$ and $a \in (a^{\perp c \perp c} \cap A) \setminus C$, which is absurd.

(iv) $\Longrightarrow$ (ii). Suppose now that $A^{\perp c} = a^{\perp c}$ for all $a \in A \setminus C$, and let $x, y \in A \setminus C$ satisfy $I_0(x) \cap I_0(y) \subseteq C$. Then $y \in y^{\perp c \perp c}$ and $y \in x^{\perp c} = A^{\perp c} = y^{\perp c}$, which yields $y \in y^{\perp c} \cap y^{\perp c \perp c} = C$, a contradiction. Hence, C is prime in A.

(iv) $\Longrightarrow$ (v). Suppose $C \subset D \in \mathcal{I}(E)$ and let $A^{\perp c} \subseteq D^{\perp c}$. We claim $A^{\perp c} = D^{\perp c}$. We have $D \nsubseteq A^{\perp c}$, otherwise $D = D \cap A^{\perp c} \subseteq D \subseteq D^{\perp c} = C$, a contradiction. Hence, there exists $d \in D \setminus A^{\perp c}$ and by (o) of Proposition 4.1, there exists an element $u \in E \setminus C$ such that $u \leq a, d$. Consequently, $u \in (D \cap A) \setminus C$. By (iv), $D^{\perp c} \subseteq u^{\perp c} = A^{\perp c} \subseteq D^{\perp c}$.

(v) $\Longrightarrow$ (vi) and (vii) $\Longrightarrow$ (i). They are evident.

(vi) $\Longrightarrow$ (vii). First, we prove C is prime in $A^{\perp_C \perp_C}$. If not, there are two ideals I and J of E such that $C \subset I, J \subseteq A^{\perp_C \perp_C}$ and $C = I \subseteq J$. There exist two elements $a \in I \setminus C$ and $b \in J \setminus C$, and define $D = C \vee I_0(a)$. Then $A^{\perp_C \perp_C} \subset D$ and $C \subset D$ while $a \in D^{\perp_C \perp_C} = A^{\perp_C \perp_C}$, i.e., $D^{\perp_C} = A^{\perp_C}$. Let $x \in D$, and as $b \in A^{\perp_C} \cap J \subseteq A^{\perp_C} \cap A^{\perp_C \perp_C} = C$, we have a contradiction. Hence, C is prime in $A^{\perp_C \perp_C}$.

Second, assume there exists an ideal B of E such that $B \supseteq A^{\perp_C \perp_C}$ and C is prime in B. Therefore, for C and B the statement (vi) holds, i.e., $B^{\perp_C} = A^{\perp_C}$, and, consequently, $B \subseteq B^{\perp_C \perp_C} = A^{\perp_C \perp_C} \subseteq B$, which gives $B = A^{\perp_C \perp_C}$. □

THEOREM 4.8. *Let P be an ideal of a pseudo-effect algebra with* (RDP). *The following statements are equivalent.*

(i) *P is prime.*
(ii) *$P = a^{\perp_P}$ for all $a \in E \setminus P$.*
(iii) $\mathrm{Pol}_P(E) = \{P, E\}$.

Proof.
(i) $\iff$ (ii). It follows from Proposition 4.7 while $E^{\perp_P} = P$.

(i) $\Longrightarrow$ (iii). Let $I \in \mathrm{Pol}_P(E)$ and P be prime. Since $P = I^{\perp_P} \cap I^{\perp_P \perp_P}$, we have $P = I^{\perp_P}$ or $P = I$, i.e., $I = E$ or $I = P$.

(iii) $\Longrightarrow$ (i). Assume that $a \in E \setminus P$ and $P \subset a^{\perp_P}$. Since $a^{\perp_P} \in \mathrm{Pol}_P(E)$, we have $a^{\perp_P} = E$, i.e., $a \in a^{\perp_P \perp_P} = E^{\perp_P} = P$, a contradiction. □

5. C-Carriers of pseudo-effect algebras and C-regularity

Let a be an element of a pseudo-effect algebra E and let C be an ideal of E. The *C-carrier* of a, $a^{\wedge(C)}$, is the set

$$a^{\wedge(C)} = \{b \in E : b^{\perp_C} = a^{\perp_C}\}.$$

In particular, if $C = \{0\}$, we call $a^{\wedge} := a^{\wedge(\{0\})}$ the *carrier* of a.

The following basic properties of C-carriers can be easily proved.

PROPOSITION 5.1. *Let E be a pseudo-effect algebra and let $a \in E$ and $C \in \mathcal{I}(E)$. Then*

(i) *$a^{\wedge(C)} = C$ for any $a \in C$. In particular, $0^{\wedge} = \{0\}$.*
(ii) *$a \in a^{\wedge(C)} \subseteq a^{\perp_C \perp_C}$, $a^{\perp_C} = \left(a^{\wedge(C)}\right)^{\perp_C}$.*

Let E satisfy (RDP_0).

(iii) *If $b_1, b_2 \in a^{\wedge(C)}$ and $b_1 + b_2 \in E$, then $b_1 + b_2 \in a^{\wedge(C)}$.*
(iv) *If $a \in E \setminus C$, then $C \cap a^{\wedge(C)} = \emptyset$.*

We say that a pseudo-effect algebra E is C*-regular* if C is a normal ideal of E, and $a^{\perp_C}$ is normal for any $a \in E$.

PROPOSITION 5.2. *Let E be a pseudo-effect algebra with* (RDP_0) *and let C be an ideal of E. Then E is C-regular if and only if $a+x \in E$ and $y+a \in E$ imply $a^{\wedge(C)} = \big(x \,/\, (a+x)\big)^{\wedge(C)} = \big((y+a) \setminus y\big)^{\wedge(C)}$.*

Proof. Let E be C regular, and let $z \in a^{\perp_C}$. Then $a \in z^{\perp_C}$ and the normality of $z^{\perp_C}$ yields $x \,/\, (a+x)$, $(y+a) \setminus y \in z^{\perp_C}$, i.e., $z \in \big(x \,/\, (a+x)\big)^{\perp_C}$ and $z \in \big((y+a) \setminus y\big)^{\perp_C}$. Conversely, if $z \in \big((y+a) \setminus y\big)^{\perp_C}$, then $z \in \big(x \,/\, (a+x)\big)^{\perp_C}$, i.e., $a \in z^{\perp_C}$, $z \in a^{\perp_C}$, and similarly $z \in \big((y+a) \setminus a\big)^{\perp_C}$ implies $z \in a^{\perp_C}$.

Assume now $a^{\wedge(C)} = \big(x \,/\, (a+x)\big)^{\wedge(C)} = \big((y+a) \setminus y\big)^{\wedge(C)}$. Let $x_0 \in a^{\perp_C}$ and let $y_0 \,/\, (x_0+y_0) \in E$. Then $a \in x^{\perp_C} = \big(y_0 \,/\, (x_0+y_0)\big)^{\perp_C}$. Hence, $y_0 \,/\, (x_0+y_0) \in a^{\perp_C}$, and similarly we can prove $(y_0' + x_0) \setminus y_0' \in a^{\perp_C}$ for some $y_0' \in E$ for which $y_0' + x_0$ is defined in E. □

Let C be an ideal of a pseudo-effect algebra E. Let us set

$$\mathcal{K}_C(E) := \{a^{\wedge(C)} : \ a \in E\},$$

and define a partial order $\le$ on $\mathcal{K}_C(E)$ as follows: $a^{\wedge(C)} \le b^{\wedge(C)}$ if and only if $b^{\perp_C} \subseteq a^{\perp_C}$. Then, for all $a, b \in E$ such that $a \le b$, we have

$$0^{\wedge(C)} \le a^{\wedge(C)} \le b^{\wedge(C)} \le 1^{\wedge(C)}.$$

THEOREM 5.3. *Let E be a pseudo-effect algebra with* (RDP).

(i) *If $c = a+b$, then $c^{\wedge(C)}$ is the join of $a^{\wedge(C)}$ and $b^{\wedge(C)}$ in the space $\mathcal{K}_C(E)$.*

(ii) *$a^{\wedge(C)} \vee b^{\wedge(C)}$ is defined in $\mathcal{K}_C(E)$ for all $a, b \in E$. Moreover, there exists an element $d \in E$ such that $d \ge a, b$ and $d^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)}$. For an element $e \in E$, we have $e^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)}$ if and only if $e^{\perp_C} = a^{\perp_C} \cap b^{\perp_C}$.*

(iii) *If $a \vee b$ is defined in E, then $(a \vee b)^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)}$. If $a \wedge b$ is defined in E, then $(a \wedge b)^{\wedge(C)} = a^{\wedge(C)} \wedge b^{\wedge(C)}$.*

(iv) *If $d^{\perp_C} = (a^{\perp_C} \cup b^{\perp_C})^{\perp_C \perp_C}$, then $d^{\wedge(C)} = a^{\wedge(C)} \wedge b^{\wedge(C)}$.*

(v) *Let $a^{\wedge(C)} \le b^{\wedge(C)}$. Then, for any $a_1 \in a^{\wedge(C)}$ there exists $b_1 \in b^{\wedge(C)}$ such that $a_1 \le b_1$.*

(vi) *If $a^{\wedge(C)} \wedge b^{\wedge(C)}$ is defined in $\mathcal{K}_C(E)$, then so is $\big(a^{\wedge(C)} \vee c^{\wedge(C)}\big) \wedge \big(b^{\wedge(C)} \vee c^{\wedge(C)}\big)$, and it is equal to $\big(a^{\wedge(C)} \wedge b^{\wedge(C)}\big) \vee c^{\wedge(C)}$, and if also $a^{\wedge(C)} \wedge d^{\wedge(C)}$ exists in $\mathcal{K}_C(E)$, then so does $a^{\wedge(C)} \wedge \big(b^{\wedge(C)} \vee d^{\wedge(C)}\big)$ and it is equal to $\big(a^{\wedge(C)} \wedge b^{\wedge(C)}\big) \vee \big(a^{\wedge(C)} \wedge d^{\wedge(C)}\big)$.*

(vii) *If $\mathcal{K}_C(E)$ is finite, then it is a Boolean algebra.*

Proof.

(i) Let $c = a + b$. According to (viii) of Proposition 4.1, we have $c^{\perp_C} = a^{\perp_C} \cap b^{\perp_C}$, which proves easily $c^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)}$.

(ii) Let a and b be arbitrary elements of E. (RDP) implies that there are three elements $a_1, b_1, c \in E$ such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c = b_1 + a_1 + c \in E$. Let $d := a_1 + b = b_1 + a$. Then $d^{\perp_C} = a_1^{\perp_C} \cap b^{\perp_C} = b_1^{\perp_C} \cap a^{\perp_C}$, i.e., $d^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)} = a^{\perp_C} \cap b^{\perp_C}$. Assume $y^{\wedge(C)} \ge a^{\wedge(C)}, b^{\wedge(C)}$. Hence, $y^{\perp_C} \subseteq a^{\perp_C} \cap b^{\perp_C} = d^{\wedge(C)}$, i.e., $d^{\wedge(C)} \le y^{\wedge(C)}$.

The rest is evident.

(iii) Assume $a \vee b \in E$. Then $a, b \le a \vee b \le d$, where d is the element from (ii). This gives $a^{\wedge(C)}, b^{\wedge(C)} \le (a \vee b)^{\wedge(C)} \le d^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)}$.

Assume now $a \wedge b \in E$. Hence, $(a \wedge b)^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)}$. Suppose $x^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)}$. Since $I_0(a \wedge b) = I_0(a) \cap I_0(b)$, according to Proposition 4.4, we have $(a \wedge b)^{\perp_C} = (a^{\perp_C} \cup b^{\perp_C})^{\perp_C \perp_C} \subseteq x^{\perp_C}$. This gives $(a \wedge b)^{\wedge(C)} \ge x^{\wedge(C)}$.

(iv) Suppose $d^{\perp_C} = (a^{\perp_C} \cup b^{\perp_C})^{\perp_C \perp_C}$. Then $d^{\perp_C} \supseteq a^{\perp_C}, b^{\perp_C}$, i.e., $d^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)}$. Assume $x^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)}$. Then $x^{\perp_C} \supseteq a^{\perp_C} \cup b^{\perp_C}$, i.e., $x^{\perp_C} \supseteq (a^{\perp_C} \cup b^{\perp_C})^{\perp_C \perp_C} = d^{\perp_C}$, which gives $x^{\wedge(C)} \le d^{\wedge(C)}$, and $d^{\wedge(C)} = a^{\wedge(C)} \wedge b^{\wedge(C)}$.

(v) By (ii), there exists $b_1 \ge a, b$ such that $b_1^{\wedge(C)} = a_1^{\wedge(C)} \vee b^{\wedge(C)} = a^{\wedge(C)} \vee b^{\wedge(C)} = b^{\wedge(C)}$, which gives $b_1 \in b^{\wedge(C)}$.

(vi) Put $x^{\wedge(C)} = a^{\wedge(C)} \wedge b^{\wedge(C)}$. Then obviously $x^{\wedge(C)} \vee c^{\wedge(C)} \le a^{\wedge(C)} \vee c^{\wedge(C)}$ and $x^{\wedge(C)} \vee c^{\wedge(C)} \le b^{\wedge(C)} \vee c^{\wedge(C)}$. Assume that $u^{\wedge(C)} \le a^{\wedge(C)} \vee c^{\wedge(C)}$ and $u^{\wedge(C)} \le b^{\wedge(C)} \vee c^{\wedge(C)}$ but it is not less than $x^{\wedge(C)} \vee c^{\wedge(C)}$. By (v) and (ii), there is a $u^{\wedge(C)}$ such that

$$x^{\wedge(C)} \vee c^{\wedge(C)} < u^{\wedge(C)} \tag{$*$}$$

(we change $u^{\wedge(C)}$ to $u^{\wedge(C)} \vee x^{\wedge(C)} \vee c^{\wedge(C)}$ if necessary). As in the proof of (ii), we have $x_1 \le x$, $a_1 \le a$ and $b_1 \le b$ such that $(x_1 + c)^{\wedge(C)} = x^{\wedge(C)} \vee c^{\wedge(C)} = u^{\wedge(C)} \le (a_1 + c)^{\wedge(C)} = a^{\wedge(C)} \vee c^{\wedge(C)}$ and $u^{\wedge(C)} \le (b_1 + c)^{\wedge(C)} = b^{\wedge(C)} \vee c^{\wedge(C)}$. By (iv), we can assume that they satisfy also $x_1 + c < u < a_1 + c$, $u < b_1 + c$. Since $x_1^{\wedge(C)} \le (u \setminus c)^{\wedge(C)}$, we have $x_1^{\wedge(C)} < (u \setminus c)^{\wedge(C)}$, otherwise the equality $x_1^{\wedge(C)} = (u \setminus c)^{\wedge(C)}$ would imply, by (i), $(x_1 + c)^{\wedge(C)} = x^{\wedge(C)} \vee c^{\wedge(C)} = x_1^{\wedge(C)} \vee c^{\wedge(C)} = (u \setminus c)^{\wedge(C)} \vee c^{\wedge(C)} = u^{\wedge(C)}$ against $(*)$. Since $u \setminus c \le a_1, b_1$, i.e., $u \setminus c \le a, b$, we have $(u \setminus c)^{\wedge(C)} \le a^{\wedge(C)} \wedge b^{\wedge(C)}$, which contradicts the choice of $u^{\wedge(C)}$.

For the second equality. Let $a_1^{\wedge(C)} = a^{\wedge(C)} \wedge b^{\wedge(C)}$ and $a_2^{\wedge(C)} = a^{\wedge(C)} \wedge d^{\wedge(C)}$. Then $a_1^{\wedge(C)} \vee a_2^{\wedge(C)} \le a^{\wedge(C)}$ and $a_1^{\wedge(C)} \vee a_2^{\wedge(C)} \le b^{\wedge(C)} \vee d^{\wedge(C)}$. Assume $x^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)} \vee d^{\wedge(C)}$. Then $x^{\perp_C} \supseteq a^{\perp_C} \cup (b^{\perp_C} \cap d^{\perp_C})$, which gives by Theorem 4.6, $x^{\perp_C} \supseteq a^{\perp_C} \vee^C (b^{\perp_C} \wedge^C d^{\perp_C}) = (a^{\perp_C} \vee^C b^{\perp_C}) \wedge^C (a^{\perp_C} \vee^C d^{\perp_C}) = a_1^{\perp_C} \cap a_2^{\perp_C}$. Then $x^{\wedge(C)} \le a_1^{\wedge(C)} \vee a_2^{\wedge(C)}$.

(vii) Since $\mathcal{K}_C(E)$ is finite, for any two elements $a, b \in E$, there is only a finite number of elements $c^{\wedge(C)}$ of $\mathcal{K}_C(E)$ such that $c^{\wedge(C)} \le a^{\wedge(C)}, b^{\wedge(C)}$. Hence, the element $\bigvee c^{\wedge(C)}$ is the infimum of $a^{\wedge(C)}$ and $b^{\wedge(C)}$.

By (vi), $\mathcal{K}_C(E)$ is distributive.

Let $a_1^{\wedge(C)}, \dots, a_n^{\wedge(C)}$ be the atoms of $\mathcal{K}_C(E)$. Let $b^{\wedge(C)} \in \mathcal{K}_C(E)$ and let $a_1^{\wedge(C)}, \dots, a_k^{\wedge(C)}$ be the atoms which are less than $b^{\wedge(C)}$. Then $b^{\wedge(C)} = \bigvee_{i=1}^{k} a_i^{\wedge(C)}$, and the element $c^{\wedge(C)} := \bigvee_{i=k+1}^{n} a_i^{\wedge(C)}$ is the complement of $b^{\wedge(C)}$. Indeed, $b^{\wedge(C)} \wedge c^{\wedge(C)} = \bigvee_{i=k+1}^{n} \big(b^{\wedge(C)} \wedge a_i^{\wedge(C)}\big) = 0^{\wedge(C)}$, and $b^{\wedge(C)} \vee c^{\wedge(C)} = \bigvee_{i=1}^{n} a_i^{\wedge(C)}$ $= 1^{\wedge(C)}$. □

PROPOSITION 5.4. *Let E be a pseudo-effect algebra with* (RDP) *and let C be an ideal of E. The mapping $\phi\colon\ E \to \mathcal{K}_C(E)$ defined by $\phi(a) = a^{\wedge(C)}$, $a \in E$, is an order-preserving mapping of E onto $\mathcal{K}_C(E)$ preserving all existing finite suprema and infima which exist in E, and $\{a \in E:\ \phi(a) = 0^{\wedge(C)}\} = C$.*

Proof. It follows from Theorem 5.3. □

6. Representable pseudo-effect algebras

Let $\{E_i\}_{i \in I}$ be an indexed system of pseudo-effect algebras. The Cartesian product $\prod_{i \in I} E_i$ can be organized into a pseudo-effect algebra with the partial addition defined by coordinates. Each E_i has the property (RDP) ((RDP$_1$), (RDP$_2$)) if and only if $\prod_{i \in I} E_i$ has this property.

We say that a pseudo-effect algebra E is a *subdirect product* of pseudo-effect algebras $\{E_i\}_{i \in I}$ if there is an injective homomorphism of pseudo-effect algebras $f\colon\ E \to \prod_{i \in I} E_i$ such that $f(a) \le f(b)$ if and only if $a \le b$ $(a, b \in E)$, and for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from E onto E_j, where π_j is the jth projection of $\prod_{i \in I} E_i$ onto E_j.

We say that a po-group G is a *subdirect product* of a system $\{G_i\}_{i \in I}$ of po-groups if there exists an injective group homomorphism $f\colon\ G \to \prod_{i \in I} G_i$ such that $f(a) \le f(b)$ if and only if $a \le b$ $(a, b \in G)$, and for every $j \in I$, $\pi_j \circ f$ is a surjective homomorphism from G onto G_j, where π_j is the jth projection of $\prod_{i \in I} G_i$ onto G_j.

We recall that a poset $(E; \le)$ is an *antilattice* if only comparable elements of E have an infimum or a supremum. If E is a pseudo-effect algebra, then

E is an antilattice if and only if $a \wedge b = 0$ implies $a = 0$ or $b = 0$, while $\big(a \setminus (a \wedge b)\big) \wedge \big(b \setminus (a \wedge b)\big) = 0$, see [Dvu3].

We say that a pseudo-effect algebra E is *representable* if E is a subdirect product of antilattice pseudo-effect algebras such that all finite suprema and infima which exist in E are preserved in the subdirect product.

In the paper [Dvu], we have proved that the system of all representable pseudo-effect algebras forms a variety. Not all pseudo MV-algebras are representable, but every effect algebra with (RDP) is representable, as it was proved in [Rav] and [Dvu2].

THEOREM 6.1. *Every effect algebra E with* (RDP) *is a subdirect product of antilattice effect algebras with* (RDP), *and all existing meets and joins in E are preserved in the subdirect product.*

PROPOSITION 6.2. *Let a pseudo-effect algebra E with* (RDP_1) *be representable. Then every polar $A^{\perp}$ is a normal ideal.*

Proof. Let E be a subdirect product of a system $\{E_i\}_{i\in I}$ of antilattice pseudo-effect algebras. Assume $x \in A^{\perp}$ and let $x+y$ be defined in E. We show that $y \,/\, (x+y) \in A^{\perp}$. Let $z \le y \,/\, (x+y)$ and $z \le a$ for any $a \in A$. Write $z = (z_i)_{i\in I}$, $y = (y_i)_{i\in I}$, $x = (x_i)_{i\in I}$ and $a = (a_i)_{i\in I}$, where $z_i, y_i, x_i, a_i \in E_i$, $i \in I$. Then $z_i \le y_i \,/\, (x_i + y_i)$ and $z_i \le a_i$ for any $i \in I$. Since $a_i \wedge x_i = 0$ for each $i \in I$, if $a_i = 0$, then $z_i = 0$, if $a_i > 0$, then $x_i = 0$, which yields $z_i \le y_i \,/\, (0 + y_i) = 0$. Hence $z = 0$, which proves $\big(y \,/\, (x+y)\big) \wedge a = 0$ for any $a \in A$.

In a similar way, if $x \in A^{\perp}$ and $u + x \in E$, then $(u+x) \setminus u \in A^{\perp}$. □

We recall that every polar is normal in E if and only if $a^{\perp}$ is normal for every $a \in E$. In addition, in [GeIo], it is proved that a pseudo MV-algebra is representable if and only if every polar is normal, while $A^{\perp} = \Big(\bigcup_{a\in A} \{a\}\Big)^{\perp} = \bigcap_{a\in A} a^{\perp}$.

7. Regular pseudo-effect algebras and Lorenzen's theorem

We say that a pseudo-effect algebra E is *regular* if $a^{\perp}$ is a normal ideal for any $a \in E$. This is equivalent with the statement $A^{\perp}$ is a normal ideal for any $\emptyset \ne A \subseteq E$. We recall that if a regular E satisfies (RDP_0), then for any $a \in E$, we have $N_0(a)^{\perp} = a^{\perp} = I_0(a)^{\perp}$, where $N_0(a)$ is the normal ideal of E generated by a. Indeed, we have $I_0(a) \subseteq N_0(a) \subseteq a^{\perp\perp}$. Hence, $a^{\perp} \subseteq N_0(a)^{\perp} \subseteq a^{\perp}$.

We say that a pseudo-effect algebra E is *finitely irreducible* if, for any two ideals I and J of E with $I \cap J = \{0\}$, we have $I = \{0\}$ or $J = \{0\}$.

We recall that according to [DvVe1], if a and b are two elements of a pseudo-effect algebra E with (RDP_0), then $a \wedge b = 0$ implies $a+b$, $b+a$, $a \vee b$ are defined in E, and

$$a + b = a \vee b = b + a\,. \tag{7.1}$$

PROPOSITION 7.1. *Any antilattice pseudo-effect algebra with* (RDP_0) *is finitely irreducible and regular.*

Proof. If a pseudo-effect algebra E with (RDP_0) is not finitely irreducible, then there exist two non-zero ideals I and J such that $I \cap J = \{0\}$. Hence, if $a \in I$ and $b \in J$ are non-zero elements, then $a \wedge b = 0$, whence E cannot be an antilattice.

Assume $x \in a^\perp$ and let $x+y$ be defined in E. We show that $y \,/\, (x+y) \in a^\perp$. Let $z \le y \,/\, (x+y)$ and $z \le a$ for any $a \in A$. Since $a \wedge x = 0$, then if $a = 0$, then $z = 0$, if $a > 0$, then $x = 0$, which yields $z \le y \,/\, (0+y) = 0$. Hence $z = 0$, which proves $\big(y \,/\, (x+y)\big) \wedge a = 0$.

In a similar way, if $x \in a^\perp$ and $u + x \in E$, then $(u+x) \setminus u \in a^\perp$, which proves E is regular. □

PROPOSITION 7.2. *Any regular finitely irreducible pseudo-effect algebra E with* (RDP_0) *is an antilattice.*

Proof. Assume that there are $a, b \in E \setminus \{0\}$ with $a \wedge b = 0$. Then $a \in b^\perp$ and $b \in a^\perp$. In view of (7.1), $0 \ne a+b = a \vee b \in E$, so that $a^\perp \cap b^\perp = (a+b)^\perp$. While $(a+b)^\perp \cap (a+b)^{\perp\perp} = \{0\}$ and $a+b \in (a+b)^{\perp\perp}$, the irreducibility implies $(a+b)^\perp = \{0\}$, i.e., $a^\perp \cap b^\perp = \{0\}$, which gives $b \in a^\perp = \{0\}$ or $a \in b^\perp = \{0\}$, i.e., $b = 0$ or $a = 0$, a contradiction. □

PROPOSITION 7.3. *Let E be a pseudo-effect algebra with* (RDP) *and let P be a proper normal ideal of E.*

(i) *If I is an ideal of E, so is I/P in E/P. Moreover, if I is a proper ideal of E containing P, then I/P is a proper ideal of E/P.*

(ii) *If M is an ideal of E/P, then*

$$\kappa(M) := \{x \in E : \ x/P \in M\} \tag{7.2}$$

is an ideal of E, and $\kappa(M)/P = M$. If M is a proper ideal of E so is $\kappa(M)$ in E.

(iii)
$$\mathcal{N}(E/P) = \{N/P : \ N \in \mathcal{N}(E) \ \text{and} \ P \subseteq N\}\,.$$

(iv) *If P is an o-ideal of a directed po-group G with* (RDP_1) *and if M is an o-ideal of G/P, then $\kappa(M) := \{x \in G : \ x/P \in M\}$ is an o-ideal of G, and $\kappa(M)/P = M$. In addition, $\mathcal{O}(G/P) = \{N/P : \ N \in \mathcal{O}(G)$ and $P \subseteq N\}$.*

Proof.

(i) $0/P \in I/P$. Let $x/P \le y/P$, where $y \in I$. There exists $x_1 \in [x]_P$ such that $x_1 \le y$, which gives $x_1 \in I$, and $x_1/P = x/P \le y/P$. Assume $x/P + y/P$ is defined in E/P for some $x, y \in I$. There are $x_1 \in [x]_P$, $y_1 \in [y]_P$ and $e, f, u, v \in P$ such that $x_1 \setminus e = x \setminus f \in I$, $y_1 \setminus u = y \setminus v \in I$, $x_1 + y_1 \in E$. Then $x/P + y/P = x_1/P + y_1/P = (x_1 + y_1)/P = \big((x \setminus f) + e + (y \setminus v) + u\big)/P = \big((x \setminus f) + (y \setminus v)\big)/P$ and $(x \setminus f) + (y \setminus v) \in I$.

Let now $I \supseteq P$ and $1/P = x/P$, where $x \in I$. There are $e, f \in P$ such that $1 \setminus e = x \setminus f$, i.e., $x / 1 = f / e \in P \subseteq I$, which gives a contradiction.

(ii) We have $\kappa(M) \supseteq P$. If $x \le y \in \kappa(M)$, then $x/P \le y/P \in M$, so that $x \in \kappa(M)$. Let now $x, y \in \kappa(M)$ and $x + y \in E$. Then $(x + y)/P = x/P + y/P \in M$, i.e., $x + y \in \kappa(M)$.

Finally, assume M is a proper ideal of E/P. Then $1/P \notin M$, hence, $1 \notin \kappa(M)$.

(iii) It follows from (ii).

(iv) It follows the same steps as (iii). □

PROPOSITION 7.4.

(1) *Let I and J be two normal ideals of a pseudo-effect algebra E with* (RDP_1) *such that $I \cap J = \{0\}$. Then E is a subdirect product of E/I and E/J with the embedding $f\colon E \to E/I \times E/J$ defined $f(a) = (a/I, a/J)$, $a \in E$.*

(2) *Let I and J be two o-ideals of a directed po-group G with* (RDP_1) *such that $I \cap J = \{0\}$. Then G is a subdirect product of G/I and G/J with the embedding $f\colon G \to G/I \times G/J$ defined $f(a) = (a/I, a/J)$, $a \in G$.*

Proof.

(1) The mapping $f\colon E \to E/I \times E/J$ given by $f(a) = (a/I, a/J)$, $a \in E$, is a homomorphism of pseudo-effect algebras. If $f(a) = f(b)$, then there are $e, f_1 \in I$ and $u_1, v \in J$ such that $a \setminus e = b \setminus f_1$ and $a \setminus u_1 = b \setminus v$. If we now take the addition and subtraction in the corresponding unital interpolation group (G, u) such that $E = \Gamma(G, u)$, then $a - b = e - f_1 \in \phi(I)$ and $a - b = u_1 - f_1 \in \phi(J)$, i.e., $a - b = 0$, and f is an injective homomorphism.

Assume $f(x) \le f(y)$ for some $x, y \in E$, i.e., $x/I \le y/I$ and $x/J \le y/J$. There are two elements $a \in I$ and $b \in J$ with $a, b \le x$ such that $x \setminus a \le y$ and $x \setminus b \le y$. Since $a \wedge b = 0$, then $x = x \setminus (a \wedge b) = (x \setminus a) \vee (x \setminus b)$ (while all existing meets in E are preserved in the corresponding representation group (G, u)), which gives $x \le y$.

Hence, E is a subdirect product of E/I and E/J, as claimed.

(2) The second statement follows the same ideas as the first one. □

PROPOSITION 7.5. *Let E be a pseudo-effect algebra with* (RDP_1). *The following statements are equivalent:*

(i) *E is finitely irreducible.*

(ii) *If E is a subdirect product of E_1 and E_2, and if f is an injective homomorphism from E into $E_1 \times E_2$ such that $f(x) \le f(y)$ whenever $x \le y$, and $\pi_1 \circ f$ and $\pi_2 \circ f$ being surjective, then* $\mathrm{Ker}(\pi_1 \circ f) = \{0\}$ *or* $\mathrm{Ker}(\pi_2 \circ f) = \{0\}$.

Proof.

$\neg$(i) $\implies$ $\neg$(ii). Suppose E is not finitely irreducible, i.e., there are two normal non-zero ideals A and B of E such that $A \cap B = \{0\}$. By Proposition 7.4, E is a subdirect product of E/A and E/B with the embedding $f(a) = (a/A, a/B)$, $a \in E$. Hence, for the mappings $f_A\colon a \mapsto a/A$ and $f_B\colon a \mapsto a/B$, we have $\mathrm{Ker}(f_A) = A \ne \{0\}$ and $\mathrm{Ker}(f_B) = B \ne \{0\}$, so that E does not satisfy (ii).

$\neg$(ii) $\implies$ $\neg$(i). Suppose E is a subdirect product of E_1 and E_2 and let $f\colon E \to E_1 \times E_2$ be an injective homomorphism with $f(x) \le f(y)$ if and only if $x \le y$ such that, for every $A_i = \{a \in E : \pi_i \circ f(a) = 0\} \ne \{0\}$, $i = 1,2$. Then A_1 and A_2 are normal non-zero ideals of E. Assume $x \in A_1 \cap A_2$, then $f(x) = (0,0)$, and the injectivity of f gives $x = 0$, which proves $A_1 \cap A_2 = \{0\}$. Hence, E is not finitely irreducible. □

THEOREM 7.6. *Every pseudo-effect algebra E with* (RDP_1) *is a subdirect product of finitely irreducible pseudo-effect algebras with* (RDP_1) *preserving all finite joins and meets from E.*

Proof. Without loss of generality, we can assume that $E = \Gamma(G,u)$, where (G,u) is a unital po-group with (RDP_1). Let $g \in G$, $g \not\le 0$, and set $U(g) := \{h \in G : h \ge g\}$. We denote by $A(g)$ a proper normal ideal of E which is maximal among normal proper ideals A of E with respect to the property $U(g) \cap A = \emptyset$. Since $0 \notin U(g)$, $A(g)$ exists due to the Zorn lemma. Moreover, $\bigcap_g A(g) = \{0\}$.

We assert that E is a subdirect product of $\{E/A(g)\}_g$. Let $f(a) := \{a/A(g)\}_g \le \{b/A(g)\}_g =: f(b)$, $a,b \in E$. Then $(a-b)/\phi(A(g)) \le 0$ for any $g \not\le 0$. Set $g_0 = a - b$. If $g_0 \not\le 0$, there is an element $e \in A(g_0)$ such that $a - b \le e$, which implies $e \in U(g_0) \cap A(g_0)$, which is absurd.

Therefore, E is a subdirect product of $\{E/A(g)\}_g$, moreover, the embedding $a \mapsto f(a)$ $(a \in E)$ preserves all existing finite joins and meets from E.

To prove the finite irreducibility of $E/A(g)$, assume that I and J are normal ideals of $E/A(g)$ such that $I \cap J = \{0\}$. By Proposition 7.3, the sets $\kappa(I) = \{a \in E : a/A(g) \in I\}$ and $\kappa(J) = \{b \in E : b/A(g) \in J\}$ are normal ideals of E containing $A(g)$ such that $\kappa(I)/A(g) = I$ and $\kappa(J)/A(g) = J$. Since

$I = \{0\}$ if and only if $\kappa(I) = A(g)$, assume $\kappa(I) \supset A(g)$ and $\kappa(J) \supset A(g)$. The maximality of $A(g)$ implies there are $a \in \kappa(I) \cap U(g)$ and $b \in \kappa(J) \cap U(g)$. Hence, $0, g \leq a, b$. (RIP) holding in G entails there exists an element $c \in G$ such that $0, g \leq c \leq a, b$. Then $c \in E$, $c \in U(g)$, $c \notin A(g)$, and $c \in \kappa(I) \cap \kappa(J)$, i.e., $0 \neq c/A(g) \in I$ and $c/A(g) \in J$, which is a contradiction. Hence, $I = \{0\}$ or $J = \{0\}$. □

THEOREM 7.7. *Let E be a pseudo-effect algebra with* (RDP_1). *If E is representable, then E is regular.*

If E is C-regular for any normal ideal C of E, then E is representable.

If E is a pseudo-effect algebra with (RDP_2), *then E is representable if and only if E is regular.*

Proof. The first statement follows from Proposition 6.2.

Suppose now that $E = \Gamma(G, u)$ for some unital po-group (G, u) with (RDP_1). For any element $g \in G$, $g \not\leq 0$, let $A(g)$ be a normal ideal of E having the same sense as that in the proof of Theorem 7.6. If E is C-regular for any normal ideal C of E, then $A(g)$ is prime. Indeed, set $C = A(g)$, and let $A(g) = I \cap J$, where $I, J \in \mathcal{I}(E)$. Then $A(g) = A(g)^{\perp_C \perp_C} = I^{\perp_C \perp_C} \cap J^{\perp_C \perp_C}$ by Proposition 4.4. Since $I^{\perp_C \perp_C}$ and $J^{\perp_C \perp_C}$ are normal ideals of E, we have $A(g) = I^{\perp_C \perp_C} = I$ or $A(g) = J^{\perp_C \perp_C} = J$. Applying the proof of Theorem 7.6, we have that E is a subdirect product of $\{E/A(g)\}_g$, and the embedding $a \mapsto f(a)$ $(a \in E)$ preserves all existing finite joins and meets from E.

Finally, let E satisfy (RDP_2). Then E is a lattice. Assume $a/A(g) \wedge b/A(g) = 0$. Hence, if $a \wedge b = 0$, then $a \in b^{\perp} \subseteq A(g)$ or $b \in b^{\perp\perp} \subseteq A(g)$, i.e., $a/A(g) = 0$ or $b/A(g) = 0$. If $a \wedge b \in A(g)$, then $\big(a \setminus (a \wedge b)\big) \wedge \big(b \setminus (a \wedge b)\big) = 0$, which gives again $a/A(g) = 0$ or $b/A(g) = 0$. Consequently, $A(g)$ is prime, which yields that E is a subdirect product of $\{E/A(g)\}_g$. □

We note that we do not know whether the condition E is C-regular for any normal ideal C of E can be replaced by the condition E is regular in order to be E representable.

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