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ON THE HAHN-BANACH THEOREM

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The Hahn—Banach theorem is a well-known proposition of functional analysis. It guarantees that for any given linear functional defined over a linear subspace of a linear space there exists at least one entire-space extension whose value is at no point greater than the corresponding value of a given subadditive and positively homogeneous real function, provided that this inequality holds for all points in the subspace. Not two whole decades ago, H. Nakano proved and formulated a generalization of this theorem in which he had replaced the subadditivity and positive homogeneity of the bounding function by its convexity and he did not require that this function assume finite values everywhere in the space. Soon after the appearance of Nakano's note, J. Musielak and W. Orlicz published another demonstration of the improved theorem, based on an extension theorem of S. Mazur and W. Orlicz. Still another argument, resting on several distinct assertions, is given for a slightly less general statement in the book of F. A. Valentine.

In [4] a different type of assumptions upon the bounding function is established to be satisfactory. Strangely enough, only a small class of mappings is dealt with simultaneously by both Nakano theorems.

This paper presents a short proof of the Nakano formulation of the Hahn—Banach theorem. Accidentally, this new argumentation permits a weakening of one of the hypotheses; the improvement is stated as Theorem 1. In the second theorem two assumptions of the modified Hahn—Banach theorem are replaced by a weaker hypothesis and the bounding function is not required to be defined over the whole space. Since these parallel results apply to essentially different classes of functions, an attempt is made to cover them by one theorem. With such an objective in mind, by combination of the two following proofs, the third theorem has been obtained. But for the requirements upon the domain of the bounding function this final theorem would contain both previous refinements as two extreme special cases.

Although in [5] mappings with values in a partially ordered linear space are considered, we choose to work here only with real-valued functions. However, the proofs as well as the notions employed, after the obvious change of terms, are suitable without any modification also for that more general setup just mentioned.

It can be easily seen that some additional condition upon the bounding function must be taken into consideration besides that of its being a convex mapping on a convex set. The condition used in the Nakano formulation can be weakened. It is sufficient to require convex domains that are suitably located relative to the given subspace so as to effect the non-emptiness of the collections of real numbers indicated below. The fulfilment of this for a particular choice of points determining the sequence of consecutive extensions would suffice. It should be noted, however, that our problem becomes easy to solve when this requirement is not satisfied.

Theorem 1. *If f is a convex real function on a convex set A in a linear space G, if L is a linear functional on a linear subspace H of the space G, if $L(a) \leqq f(a)$ for all $a \in H \cap A$, and if for any $a \in A - H$ there is a point $y \in H$, and a real number $\beta > 0$, such that $y - \beta a \in A$, then there is at least one extension $\bar{L}$ of L to the entire space G that is a linear functional such that $\bar{L}(a) \leqq f(a)$ for all $a \in A$.*

Proof. Let us suppose that $a \in G - H$. Let $\boldsymbol{R}$ denote the field of real numbers. Clearly, the set $H + \boldsymbol{R}a$ is a linear subspace of G, and we prove that there is an extension with the required properties to this subspace. Since over the space H the functional L is assumed to be defined, we are concerned with extending it to the points $x + \alpha a$, and points $y - \beta a$, where $x \in H$, $y \in H$, $\alpha > 0$, $\beta > 0$. In accordance with the assumptions, either none of these two collections or each of them contains an element of the set A. Let $x + \alpha a$, and $y - \beta a$, be two arbitrary members of the set A, each of which belongs just to one of the two considered collections of points. The chief idea of the proof is that for the real and positive number k such that $\frac{k}{\alpha} + \frac{k}{\beta} = 1$ it follows from the convexity of the function f that

$$\frac{k}{\alpha} f(x + \alpha a) + \frac{k}{\beta} f(y - \beta a) \geqq f\left[k\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\right].$$

The point $k\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)$ is necessarily an element of A since the set A is convex, and it is also a point of the subspace H. The rest of the proof is straightforward and should be a routine matter. Really, by our hypotheses

$$f\left[k\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\right] \geqq L\left[k\left(\frac{x}{\alpha} + \frac{y}{\beta}\right)\right],$$

and by the two preceding inequalities

$$\frac{k}{\alpha} f(x + \alpha a) + \frac{k}{\beta} f(y - \beta a) \geqq k \left[\frac{L(x)}{\alpha} + \frac{L(y)}{\beta} \right],$$

therefore,

$$\frac{1}{\alpha} f(x + \alpha a) - \frac{L(x)}{\alpha} \geqq \frac{L(y)}{\beta} - \frac{1}{\beta} f(y - \beta a).$$

The last inequality implies that the set

$$\left\{ \frac{1}{\alpha} f(x + \alpha a) - \frac{L(x)}{\alpha} : \alpha > 0, \; x \in H \right\}$$

is bounded from below, whence it has an infimum, say ω. Clearly,

$$\frac{1}{\alpha} f(x + \alpha a) - \frac{L(x)}{\alpha} \geqq \omega \geqq \frac{L(y)}{\beta} - \frac{1}{\beta} f(y - \beta a),$$

and it is easy tc see that the function $\bar{L}$: $x + \alpha a \to L(x) + \alpha\omega$ is a linear functional with the required properties.

The proof can be completed by using Zorn's lemma in a conventional manner.

We now turn to our second problem. The two are certainly alike but it looks as if subadditivity and convexity had also something in common in themselves. It may seem that many different extension theorems of the present type can be constructed. Nevertheless, we have no reason here to attempt more than to remove the requirement of positive homogeneity from the Hahn—Banach theorem. To start with, we can easily see that the bounding function must be required not to attain too negative values at points not far from the centre of the space. But, to our disappointment, even when subjected to such a law, a subadditive mapping may be discontinuous to a high degree and the extension may fail to exist. However, we have our next result.

Theorem 2. *If f is a subadditive real function on a semigroup A contained in a linear space G,*

if for any $a \in A$ there is an interval (γ, δ), and a positive number K, such that $f(\nu a) \leqq K$ for all $\nu \in (\gamma, \delta)$,

if L is a linear functional on a linear subspace H of the space G,

if $L(a) \leqq f(a)$ for all $a \in H \cap A$,

and if for any $a \in A - H$ there is a point $y \in H$, and a real number $\beta > 0$, such that $y - \beta a \in A$,

then there is at least one extension $\bar{L}$ of L to the entire space G that is a linear functional such that $\bar{L}(a) \leqq f(a)$ for all $a \in A$.

Proof. We have first to demonstrate that for each $a \in A$ the hypotheses ensure that

$$-\infty < \inf\left\{\frac{f(\nu a)}{\nu} : \nu a \in A,\ \nu > 0\right\}.$$

Suppose that this statement is false. Then there is a point $a \in A$, and a sequence of positive reals ν_k, for which

$$\frac{f(\nu_k a)}{\nu_k} \leqq -k .$$

From the inequality

$$\frac{f(\nu_k a + \nu_k a)}{2\nu_k} \leqq \frac{f(\nu_k a)}{\nu_k}$$

we infer that there is an increasing, and converging to infinity, sequence of positive reals satisfying the inequality above.

By hypotheses there is $y \in H$, and $\beta > 0$, such that $y - \beta a \in A$. The line Rb, where $b = y - \beta a$, contains, by hypotheses, at least one segment on which the function f is defined and bounded from above. Let $f(\nu b) \leqq K$ for all $\nu \in (\gamma, \delta)$. Since the set A is closed with respect to addition, the function f is defined over at least one of the halflines determined by the intervals $(n\gamma, +\infty)$, $(-\infty, m\delta)$, where n, m, are the smallest natural numbers not less than $\frac{\gamma}{\delta - \gamma}$, $\frac{\delta}{\gamma - \delta}$, respectively. In the latter case the function f is defined over all points of the line Rb. We deduce that for any $\nu_k \geqq n\gamma\beta$ the point $\mu_k b$, where $\mu_k = \frac{\nu_k}{\beta}$, belongs to the domain of f, and that $\nu_k a + \mu_k b = \mu_k y$. It follows that

$$L(\mu_k y) \leqq f(\mu_k y) \leqq f(\nu_k a) + f(\mu_k b) ,$$

whence

$$f(\mu_k b) \geqq \mu_k L(y) - f(\nu_k a) ,$$

and in view of our supposition

$$f(\mu_k b) \geqq \mu_k L(y) + \nu_k k ,$$

therefore,

$$f(\mu_k b) \geqq \mu_k [L(y) + \beta k] .$$

If $\delta > 0$, then the subadditivity implies that

$$f(\nu b) \leqq \left(\frac{\nu}{\delta} + 1\right)K$$

for all $\nu \geqq n\gamma$, $\nu \geqq 0$. This shows that there is an integer k for which the previous inequality cannot hold. If $\delta < 0$, and if our supposition were true, then it follows that the map $f(.b)$ assumes arbitrarily great values at positive reals arbitrarily close to 0. This leads to a contradiction because the subadditivity ot the function f implies that on the interval $\langle 0, \delta - \gamma)$ the map $f(.b)$ is bounded from above by the number $K + f(-\gamma b)$. Hence the desired property is ensured.

Next we claim that

$$\inf\left\{\frac{f(\nu a)}{\nu} : \nu a \in A,\ \nu > 0\right\} = \inf\left\{\frac{f(\nu a)}{\nu} : \nu a \in A,\ \nu \text{ is a positive integer},\ \nu > \alpha\right\}$$

for each $a \in A$, and each real number α. To establish this assertion, let $a \in A$, let $\alpha \in R$, and let $f(\nu a) \leqq K$ for all $\nu \in (\gamma, \delta)$. For any $\varepsilon > 0$ there is $\nu > 0$ such that

$$\frac{f(\nu a)}{\nu} < \frac{\varepsilon}{3} + \inf\{\ldots\text{positive real}\ldots\}\,.$$

Let n be a natural number larger than

$$\max\left\{\frac{3\max\{|\gamma|, |\delta|\}\,|f(\nu a)|}{\varepsilon\nu^2} - \frac{\gamma}{\nu},\ \frac{3K - \gamma\varepsilon}{\varepsilon\nu}\right\}.$$

According to a well-known theorem, there is a real number $l \varepsilon (\gamma, \delta)$ such that $n\nu + l$ is a rational number. We have

$$\frac{f[(n\nu + l)a]}{n\nu + l} \leqq \frac{f(n\nu a) + f(la)}{n\nu + l} < \frac{f(\nu a)}{\nu + \frac{l}{n}} +$$

$$+ \frac{K}{\frac{3K - \gamma\varepsilon}{\varepsilon\nu}\nu + l} < \left|\frac{f(\nu a)}{\nu + \frac{l}{n}} - \frac{f(\nu a)}{\nu}\right| + \frac{f(\nu a)}{\nu} + \frac{\varepsilon}{3} <$$

$$< \left|\frac{lf(\nu a)}{(n\nu + l)\nu}\right| + \frac{\varepsilon}{3} + \inf + \frac{\varepsilon}{3} < \varepsilon + \inf,$$

whence,

$$\inf\{\ldots\text{positive real}\ldots\} = \inf\{\ldots\text{positive rational}\ldots\}\,.$$

Now let ν be an arbitrary positive rational number. Obviously, there is a natural number n such that $n\nu$ is a positive integer greater than α. We have $f(n\nu a) \leqq nf(\nu a)$, and from this the needful identity can be readily obtained.

Perhaps it should be remarked that a partially ordered linear space must be assumed to contain a unit element and to satisfy the Archimedean property, if all foregoing steps are to be carried out.

We have now made preparations to define a function $\bar{f}$ by the expression

$$\bar{f}(x) = \inf_{\nu > 0} \frac{f(\nu x)}{\nu} .$$

This is a subadditive function since for points of the space G and for positive integers we get

$$\frac{f[\mu\nu(x + y)]}{\mu\nu} \leqq \frac{f(\mu\nu x)}{\mu\nu} + \frac{f(\mu\nu y)}{\mu\nu} \leqq \frac{f(\nu x)}{\nu} + \frac{f(\mu y)}{\mu} .$$

Furthermore, it is a positively homogeneous function since for $a \in G$, $\alpha \in R$, $\alpha > 0$, we obtain

$$\bar{f}(\alpha a) = \inf_{\nu > 0} \frac{f(\nu\alpha a)}{\nu} = \alpha \inf_{\nu > 0} \frac{f(\nu\alpha a)}{\nu\alpha} = \alpha\bar{f}(a) .$$

Clearly, $\bar{f}(x) \leqq f(x)$ for all $x \in A$. And finally, since

$$L(a) = \frac{L(\nu a)}{\nu} \leqq \frac{f(\nu a)}{\nu}$$

for any $a \in H$, and each ν such that $\nu a \in A$, we deduce that

$$L(a) \leqq \inf_{\nu>0} \frac{f(\nu a)}{\nu} = \bar{f}(a) .$$

To complete the proof, we can now repeat all the steps used in the demonstration of Theorem 1. The constant k must be, however, chosen sufficiently large this time.

By the preceding methods it is not difficult to verify the concluding theorem.

Theorem 3. *If f is the sum of a convex function and of a subadditive function defined over a semigroup A that is a convex subset of a linear space G,*

if for any $a \in A$ there is an interval (γ, δ), and a positive number K, such that $f(\nu a) \leqq K$ for all $\nu \in (\gamma, \delta)$,

if L is a linear functional on a linear subspace H of the space G,

if $L(a) \leqq f(a)$ for all $a \in H \cap A$, *and if for any* $a \in A - H$ *there is a point* $y \in H$, *and a real number* $\beta > 0$, *such that* $y - \beta a \in A$,

then there is at least one extension $\bar{L}$ *of* L *to the entire space* G *that is a linear functional such that* $\bar{L}(a) \leqq f(a)$ *for all* $a \in A$.

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