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ON THE COMPLETION OF CYCLICALLY ORDERED GROUPS

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ABSTRACT. In this paper there is presented a new construction of the completion $M(G)$ of the cyclically oredered group G. The results concerning the completion of a lenearly ordered group by the Dedekind cuts are applied.

The notion of a cyclically ordered set was introduced by E. Čech [1]. V. Novák [5] defined and studied the completion $C(G)$ of a cyclically ordered set G. Another approach to this notion was established by V. Novák and M. Noovotný in [6].

L. Rieger [7] (cf. also L. Fuchs [3], Chap. IV, §6) defined the notion of a cyclically orddered group. Swierzckowski [8] derived a representation theorem for cyclically ordered groups. Each linearly ordered group can be considered a cyclically ordered group.

In [4] there is investigated the completion G^* of a cyclically ordered group G. It is defined to be a certain subset of $C(G)$ which satisfies a maximality condition (cf. Section 2 below).

In this paper there is presented a new construction of the completion of the cyclically ordered group G. It seems to be simpler than that given in [4]. The results concerning the completion of a linearly ordered group by Dedekind cuts are applied. The completion $M(G)$ obtained in this way concides with G^*.

1. Preliminaries

Let G be a linearly ordered set. Let us denote by $X^u(X^l)$ the set of all upper (lower) bounds of a subset $X \subseteq G$. The system of all subsets of G of the form $(X^u)^l$, where X is a nonempty and upper bounded subset of G will be denoted by $D(G)$. Each element of $D(G)$ is called the Dedekind cut on G. If the system $D(G)$is partially ordered by inclusion, then $D(G)$ is a conditionally complete chain. The mapping $\varphi(g) = (\{g\}^u)^l$ is an isomorphism of G into $D(G)$ and φ preserves all intersections and joins existing in G. The elements g and $\varphi(g)$ will be identified. Then G is a sublattice of $D(G)$ and the following conditions are satisfied:

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(c_1) *For each element* $h \in D(G)$ *there exists a nonempty upper bounded subset* X *of* G *such that* $h = \sup X$ *in* $D(G)$.

(c_2) *For each nonempty upper bounded subset* X *of* G *there exists an element* $h \in D(G)$ *such that* $h = \sup X$ *in* $D(G)$.

If $G_1 \subseteq G$, then $D(G_1)$ can be embedded into $D(G)$ in the natural way.

Let G be a nonempty set and $[x, y, z]$ a ternary relation defined on G with the following properties:

I. *If* $[x, y, z]$, *then* x, y, z *are distinct*; *if* x, y, z *are distinct*, *then either* $[x, y, z]$ *or* $[z, y, x]$.

II. *If* $[x, y, z]$, *then* $[y, z, x]$.

III. *If* $[x, y, z]$ *and* $[y, u, z]$, *then* $[x, u, z]$.

Then the ternary relation $[x, y, z]$ is called a cyclic order on G (cf. E. Čech [1]). The set G is said to be a cyclically ordered set.

Let G be a linearly ordered set. Define a cyclic order on G by

$$[x, y, z] \equiv x < y < z \text{ or } y < z < x \text{ or } z < x < y .$$

We shall say that this cyclic order is generated by the linear order on G.

Remark. In the whole paper the cyclic order on the given linearly ordered set S will be assumed to be generated by the linear order on S.

Let G be a cyclically ordered set and let $(G, +)$ be a group. Assume that the following condition is fulfilled for all $x, y, z, a, b \in G$:

(iv) *If* $[x, y, z]$, *then* $[a + x + b, a + y + b, a + z + b]$.

Then $(G, +)$ is said to be a cyclically ordered group. We shall write G instead of $(G, +)$.

Let K be the set of all reals a with $0 \leqq a < 1$; the set K is linearly ordered in the natural way. If the operation $+$ is defined as addition mod 1 and if the cyclic order is generated by the linear order of K, then K is a cyclically ordered group.

Let L be a linearly ordered group. We denote by $L \otimes K$ the direct product of the groups L and K with the ternary relation defined in the following way. For each three elements $u = (x, a)$, $v = (y, b)$, $w = (z, c)$ of $L \times K$ we put $[u, v, w]$ if some of the following conditions is satisfied:

(i) $[a, b, c]$;

(ii) $a = b \neq c$ *and* $x < y$;

(iii) $b = c \neq a$ *and* $y < z$;

(iv) $c = a \neq b$ *and* $z < x$;

(v) $a = b = c$ *and* $[x, y, z)$.

Then $L \otimes K$ is a cyclically ordered group.

1.1 Theorem. (Swierczkowski [8]) *Let* G *be a cyclically ordered group. Then there exists a linearly ordered group* L *such that* G *is isomorphic to a subgroup of* $L \otimes K$.

Let f be an isomorphism of G into $L \otimes K$. The elements g and $f(g)$ will be identified. Hence G is a subgroup of $L \otimes K$. Let us form the sets

$$L_1 = \{x \in L\colon \text{ there exist } a \in K \text{ and } g \in G \text{ with } g = (x, a)\},$$
$$K_1 = \{a \in K\colon \text{ there exist } x \in L \text{ and } g \in G \text{ with } g = (x, a)\},$$
$$G_0 = \{g \in G\colon \text{ there exists } x \in L \text{ with } g = (x, 0)\}.$$

Then L_1 and K_1 are subgroups of L and K, respectively. G_0 is an invariant subgroup of G. Let $g \in G$; if we put $g > 0$ whenever $x > 0$, then G_0 is a linearly ordered group (cf. [4]).

Let A and B be linearly ordered sets. Define the relation $\leqq$ on the set $C = \{(a, b)\colon a \in A, b \in B\}$ by $(a_1, b_1) \leqq (a_2, b_2)$ if $b_1 < b_2$ or $b_1 = b_2$ and $a_1 \leqq a_2$ for each $(a_1, b_1), (a_2, b_2) \in C$. Then C is a linearly ordered set which is called the lexicographic product of A and B. We shall write $C = A \circ B$.

2. Completion of a cyclically ordered set and of a cyclically ordered group

Let G be a cyclically ordered set. V. Novák [5] constructed a completion of G in the following way. Assume that g is a fixed element of G. For each $x, y \in G$ put $x <_g y$ if either $[g, x, y]$ or $g = x \neq y$. Then $<_g$ is a linear order on G with the least element g. A linear order $<$ on G is called a cut on G if the cyclic order on G generated by $<$ concides with the original cyclic order on G. A cut $<$ is said to be regular if some of the following conditions is valid:

(i) $(G, <)$ *has neither the least nor the greatest element.*

(ii) $(G, <)$ *has the least element.*

Let $C(G)$ be the set of all regular cuts on G and let $h_i <_i$ $(i = 1, 2, 3)$ be elements of $C(G)$. Define a cyclic order on $C(G)$ by putting $[h_1, h_2, h_3]$ if there exist elements $x, y, z \in G$ such that

$$x <_1 y <_1 z,\ y <_2 z <_2 x,\ z <_3 x <_3 y.$$

Let $\varphi(g) = <_g$ for each $g \in G$. Then φ is an isomrphism of the cyclically ordered set G into $C(G)$. The elements g and $\varphi(g)$ will be identified. In this sense G is a subset of $C(G)$. The cyclically ordered set $C(G)$ is called the completion of G.

Let G be a cyclically ordered group. In [4] there is defined a completion of G. By the completion of G is meant a group $(G^*, +^*)$ fulfilling the following conditions:

(a) $G \subseteq G^* \subseteq C(G)$

(b) $(G^*, +^*)$ *is a cyclically ordered group under the cyclic order induced by* $C(G)$.

(c) $(G, +)$ *is a subgroup of* $(G^*, +^*)$

(d) *If* $(G_1, +_1)$ *is a group satisfying* (a)—(c) (*with* $(G_1, +_1)$ *instead of* $(G^*, +^*)$), *then* $(G_1, +_1)$ *is a subgroup of* $(G^*, +^*)$.

We shall write G^* instead of $(G^*, +^*)$. In [4] it is proved the completion G^* does exist; the definition of G^* implies that it is uniquely determined.

Let $G_0 \neq \{0\}$. We say that a cut $h \in C(G)$ is of the type of G_0 if the following conditions are fulfilled:

(i) *There exist* $g_1, g_2 \in G$ *such that* $g_2 - g_1 \in G_0$, $g_2 - g_1 > 0$,

(ii) $[g_1, h, g_2]$ *in* $C(G)$.

The set of all cuts from $C(G)$ of the type of G_0 will be denoted by $D_1(G)$.

Let $g \in G$; $t \in G_0$. The mapping $\psi(t) = g + t$ is a one-to-one mapping from the set G_0 onto $g + G_0$. Assume that $g_1, g_2 \in g + G_0$. Then $\psi(t_1) = g_1$, $\psi(t_2) = g_2$ for some $t_1, t_2 \in G_0$. If we put $g_1 \leq g_2$ in $g + G_0$ whenever $t_1 \leq t_2$ in G_0, then $g + G_0$ is a linearly ordered set.

Since $D_1(G)$ and $\bigcup_{g \in G} D(g + G_0)$ are isomorphic, in the following $D_1(G)$ and $\bigcup_{\in \, g \, G} D(g + g_0)$ will be identified. Let $h \in D_1(G)$. Then there exists $g \in G$ such that $h \in D(g + G_0)$. Put

$$l(h) = \{x \in g + G_0 \colon x \leqq h\}.$$

Then $h = \sup l(h)$ in $D(g + G_0)$.

Let $h_1, h_2 \in D_1(G)$. There exist $g_1, g_2 \in G$ such that $h_1 \in D(g_1 + G_0)$, $h_2 \in$ $\in D(g_2 + G_0)$. Define the operation $+^\wedge$ on $D_1(G)$ as follows

$$h_1 +^\wedge h_2 = \sup \{l(h_1) + l(h_2)\} \text{ in } D((g_1) + g_2) + G_0).$$

It is evident, that the operation $+^\wedge$ is independent of the choice of the elements g_1, g_2. The set $G^\wedge$ of all elements of $D_1(G)$ having inverses is a cyclically ordered group.

The following results were established in [4]:

2.2. Theorem. *Let G be a cyclically ordered group. Then*

(i) $G^* = G$ *if* $G_0 \neq \{0\}$.

(ii) $G^* = G$ *if G is finite.*

(iii) G^* *is isomorphic to K if G is infinite and if* $G_0 = \{0\}$.

3. The cyclically orderedd group $M(G)$

In this section we shall construct an extension $M(G)$ of a cyclically ordered group G. Then it will be shown that G^* and $M(G)$ coincide.

Let G be a cyclically ordered group and let $L, K, L_1, K_1, L_1 \otimes K_1$ be as above. Let us form the lexicographic product $L_1 \circ K_1$ of the linearly ordered sets L_1 and

K_1. The linear order on $L_1 \circ K_1$ will be denoted by $<_1$. Then G is a subset of $L_1 \circ K_1$ and $G \subseteq D(G) \subseteq D(L_1 \circ K_1)$.

If the system $\bar{D}(G) = D(G) \cup \{G\}$ is partially ordered by inclusion, then $\bar{D}(G)$ is a conditionally complete lattice with the greatst element.

Suppose that $T \subseteq G$, $h \in \bar{D}(G)$. Dentote

$T(L_1) = \{x \in L_1$: there exist $g \in G$ and $a \in K_1$ with $g = (x, a)\}$,
$T(K_1) = \{a \in K_1$: there exist $g \in G$ and $x \in L_1$ with $g = (x, a)\}$,
$U(h) = \{g \in G: g \geqq h\}$, $V(h) = \{g \in G: g \leqq h\}$.

Then (c_1) implies

$$h = \sup V(h) \text{ in } \bar{D}(G). \tag{1}$$

Let $h_1, h_2 \in \bar{D}(G)$. Then according to (1) we have

$$h_i = \sup V(h_i) \, (i = 1, 2).$$

The usual operation on the group of reals will be denoted by $+_r$.

Suppose that for all elements $v_1 \in V(h_1)$, $v_2 \in V(h_2)$, $v_1 = (x_1, a_1)$, $v_2 = (x_2, a_2)$ the relation $a_1 +_r a_2 < 1$ is valid. In such a case denote

$$V(h_1, h_2) = \{v_1 + v_2 : v_1 \in V(h_1), v_2 \in V(h_2)\}.$$

It is clear that $a_1 +_r a_2 = a_1 + a_2$. If no ambiguity is likely to arise we shal often write V instead of $V(h_1, h_2)$. The set V is nonempty because the sets $V(h_1)$ and $V(h_2)$ are nonempty subsets of G.

Observe that it can happen that $V = G$ for some $h_1, h_2 \in \bar{D}(G)$, $h_1, h_2 \neq \{G\}$.

3.1. Lemma. *Let $g \in G$, $v \in V$, $g < v$. Then $g \in V$.*

Proof. Let $g \in G$, $g = (x, a)$, $v \in V$. There exist $v_1 \in V(h_1)$, $v_2 \in V(h_2)$, $v_1 = (x_1, a_1)$, $v_2 = (x_2, a_2)$ such that $v = v_1 + v_2$. Then $v = (x_1 + x_2, a_1 + a_2)$. If we denote $g' = v - g = v_1 + v_2 - g$, $g' = (x', a')$, then $x' = x_1 + x_2 - x$, $a' = a_1 + a_2 - a$ and $g = (-g' + v_1) + v_2$. From $g < v$ we infer that $g' > 0$. Hence $a' \geqq 0$.

Suppose that $a' = 0$. Hence $x' > 0$ and $a = a_1 + a_2$. Therefore $-x' + x_1 < x_1$ and so $-g' + v_1 < v_1$. We conclude that $-g' + v_1 \in V(h_1)$. Thus $g \in V$.

Now assume that $a' > 0$. Hence $a < a_1 + a_2$. Let $a_2 < a_1$ be valid. We distinguish two cases. First suppose that $a' < a_1$. Then $-a' + a_1 < a_1$ and so $-g' + v_1 < v_1$. Thus $-g' + v_1 \in V(h_1)$. Therefore $g \in V$. Let $a' \geqq a_1$. As for $a' \leqq a_1 + a_2$, we get $g = -a' + a_1 + a_2 \leqq a_2 < a_1$. Therefore $g < v_1$. From this it follows that $g \in V(h_1) \subseteq V$. The case $a_1 \leqq a_2$ is analogous.

Let $V \neq G$. Then the set V is upper bounded. However, assume that V is not upper bounded and that $g \in G$. There exists $v \in V$ such that $g < v$. Then 3.1 implies that $g \in V$, From this it follows that $G = V$, a contradiction.

Therefore there exists sup V in $\bar{D}(G)$ whenever $V \neq G$. Evidently, if $V = G$, then sup $V = \{G\}$ in $\bar{D}(G)$.

Now, assume that there are elements $v_1 \in V(h_1)$, $v_2 \in V(h_2)$, $v_1 = (x_1, a_1)$, $v_2 = (x_2, a_2)$ with $a_1 +_r a_2 \geqq 1$. Then

$$\bar{V}(h_1, h_2) = \{v_1 + v_2 : v_1 \in V(h_1), v_2 \in V(h_2), a_1 +_r a_2 \geqq 1\}$$

is a nonempty subset of G. Then symbol $\bar{V}(h_1, h_2)$ wil be often replaced by $\bar{V}$.

The following lemma can be proved in a similar way as 3.1.

3.2. Lemma. *Let $g \in G$, $v \in \bar{V}$, $g < v$. Then $g \in \bar{V}$.*

Analogously to the above we obtain tha $\bar{V}$ is upper bounded whenever $\bar{V} \neq G$. Therefore there exist sup $\bar{V}$ in $\bar{D}(G)$. If $\bar{V} = G$, then sup $\bar{V} = \{G\}$.

Define the operation $+$ on $\bar{D}(G)$ by putting

$$h_1 + h_2 = \begin{cases} \sup V(h_1, h_2), & \text{if } \bar{V}(h_1, h_2) = \emptyset, \\ \sup \bar{V}(h_1, h_2), & \text{if } \bar{V}(h_1, h_2) \neq \emptyset. \end{cases}$$

The following lemma is easy to verify.

3.3. Lemma. *$(\bar{D}(G), +)$ is a semigroup and $0 \in G$ is a neutral element of $(\bar{D}(G), +)$.*

Let $M(G)$ be the set of all elements of $\bar{D}(G)$ having an inverse in $\bar{D}(G)$. Then $(M(G)$ is a group.

3.4. Lemma. *The cyclically ordered set $\bar{D}(G)$ is isomorphic to $C(G)$.*

Proof. Let $h \in \bar{D}(G)$ and let $V'(h) = G \setminus U(h)$. Assume that $h \neq \{G\}$. Let us form the ordinal sum $W = U(h) \oplus V'(h)$ of the linearly ordered sets $U(h)$ and $V'(h)$. The linear order w on W is a regular cut on G. If we put $\psi(h) = w$ for each $h \neq \{G\}$ and $\psi(h) = <_1$ whenever $h = \{G\}$, then ψ is an isomorphism of $\bar{D}(G)$ onto $C(G)$.

We may identify $\bar{D}(G)$ and $C(G)$.

A) The case $G_0 \neq \{0\}$

Now assume that $G_0 \neq \{0\}$. Let $h \in \bar{D}(G)$, $a \in K_1$. Denote

$U_a(h) = \{u \in U(h)$: there exists $x \in L_1$ with $u = (x, a)\}$,

$V_a(h) = \{v \in V(h)$: there exists $x \in L_1$ with $v = (x, a)\}$.

Then one of the following cases must occur:

(α) $V(h)(K_1)$ has the greatest element $a \in K_1$ and $V_a(h) \subset v + G_0$ for each $v \in V_a(h)$.

(β) $V(h)(K_1)$ has the greatest element $a \in K_1$ and $V_a(h) = v + G_0$ for each $v \in V_a(h)$.

(γ) $V(h)(K_1)$ has no greatest element.

In the case of (α) we say that h is of type (α).

Remark 1. If h is of type (α), then $U(h)(K_1) \neq \emptyset$ and $h \neq \{G\}$. The greatest element of $V(h)(K_1)$ is at the same time the least element of $U(h)(K_1)$.

The verification of the following lemma is a routine.

3.5. Lemma. *Let h_1, h_2, h be elements of $\bar{D}(G)$ of type (α), $V(h_i) \subseteq G_0$ $(i = 1, 2)$. If $h_1 \leqq h_2$, then $h_1 + h \leqq h_2 + h$ and $h + h_1 \leqq h + h_2$.*

Remark 2. If the hypothesis $V(h_i) \subseteq G_0$ $(i = 1, 2)$ is omitted, the assertion does not in general hold.

Let $h \in \bar{D}(G)$. In the next we want to establish a necessary and sufficient condition for $h \in M(G)$ to be valid.

Let h_1, $h_2 \in \bar{D}(G)$ be of type (α) and let $a_1(a_2)$ be the greatest element of $V(h_1)(K_1)$ $(V(h_2)(K_1))$. The definition of the operation $+$ on $\bar{D}(G)$ implies that

$$h_1 + h_2 = \sup \{v_1 + v_2 \colon v_1 \in V_{a_1}(h_1),\ v_2 \in V_{a_2}(h_2)\} \text{ in } \bar{D}(G). \tag{2}$$

Let $h \in \bar{D}(G)$, $h \neq \{G\}$. Denote

$$W_1 = \{u - v \colon u \in U(h),\ v \in V(h)\},\quad W_2 = \{-v + u \colon u \in U(h),\ v \in V(h)\},$$

$$W_{i0} = \{w \in W_i \colon \text{there exists } x \in L_1 \text{ with } w = (x, 0)\}\ (i = 1, 2).$$

3.6. Lemma. *Let $h \in \bar{D}(G)$, $h \neq \{G\}$ and let* $\inf W_1 = 0$ *in G. Then*

(i) *h is of type (α).*

(ii) *h has a right inverse in $\bar{D}(G)$.*

Proof (i) $\inf W_1 = 0$ in G implies that $0 \in W_1(K_1)$. In fact, if $0 \notin W_1(K_1)$, then either the inf W_1 does not exist or the inf $W_1 > 0$ in G. Therefore there exist $a \in K_1$, $x_1, x_2 \in L_1$, $u \in U(h)$, $v \in V(h)$ with $u = (x_1, a)$, $v = (x_2, a)$, $x_2 < x_1$ and a is the greatest (least) element of $V(h)(K_1)$ $(U(h)(K_1))$. We obtain $V_a(h) \subset v + G_0$ for all $v \in V_a(h)$. We conclude that h is of type (α).

(ii) The proof is similar to that in [2] (Theorem 6). We have $0 = \inf W_1 = \inf W_{10} = \inf \{u - v \colon u \in U_a(h),\ v \in V_a(h)\} = -\sup \{v - u \colon u \in U_a(h),\ v \in V_a(h)\}$ in G. Whence $\sup \{v - u \colon u \in U_a(h), v \in V_a(h)\} = 0$ is valid in G. Then $\sup \{v - u \colon u \in U_a(h), v \in V_a(h)\} = 0$ in $\bar{D}(G)$, too. it is clear that the set $-U(h)$ is nonempty and upper bounded in G and $-a$ is the greatest element in $-U(h)(K_1)$. There exist $h' \in \bar{D}(G)$, $h' \neq \{G\}$, $h' = \sup(-U(h))$. Obviously that $-U(h) = V(h')$, $-U_a(h) = V_{-a}(h')$. In view of (2) we obtain $h + h' = \sup \{v + u \colon v \in V_a(h),\ u \in V_{-a}(h')\} = \sup \{v + u \colon v \in V_a(h),\ u \in -U_a(h)\} = \sup \{v - u \colon v \in V_a(h), u \in U_a(h)\} = 0$ in $\bar{D}(G)$. Thus h' is a right inverse of h.

In an analogical way we prove

3.7. Lemma. *Let $h \in \bar{D}(G)$, $h \neq \{G\}$ and let* $\inf W_2 = 0$ *in G. Then*

(i) *h is of type (α).*

(ii) *h has a left inverse in $\bar{D}(G)$.*

The element $h' = \sup(-U(h))$ is a left inverse of h.

3.8. Lemma. *Let $h \in M(G)$. Then*

(i) *$h \neq \{G\}$.*

(ii) $\inf W_i = 0$ $(i = 1, 2)$ *in* G.

Proof. Let h' be an inverse of h in $\bar{D}(G)$.

Assume that $V(h)(K_1) \neq \{0\}$. Then there exists $a \in V(h)(K_1)$, $a > 0$. Therefore $\bar{V}(h, h') \neq \emptyset$. In fact, if $\bar{V}(h, h') = \emptyset$, then $0 = h + h' = \sup V(h, h')$ in $\bar{D}(G)$ and $0 < a + a' < 1$ for each $a' \in V(h')(K_1)$, a contradiction. Thus $0 = h + h' = \sup \bar{V}(h, h') = \sup \{v + v'\colon v \in V(h), v' \in V(h'), v = (x, a), v' = (x', a'), a + a' = 0\}$. Hence a is the greatest element of $v(h)(K_1)$ and $a' = -a$ is the greatest element of $V(h')(K_1)$. According to (2) we get $0 = h + h' = \sup \{v + v'\colon v \in V_a(h), v' \in V_{a}(h')\}$ in $\bar{D}(G)$. Hence $\{v + v'\colon v \in V_a(h), v' \in V_{-a}(h')\} \subset G_0$. Therefore $V_a(h) \subset v + G_0$, $V_{a}(h') \subset v' + G_0$ for all $v \in V_a(h)$, $v' \in V_{a}(h')$.

Now assume that $v(h)(K_1) = \{0\}$. Then $\bar{V}(h, h') = \emptyset$ and thus $0 = h + h' = \bar{=} \sup V(h, h') = \sup \{v + v'\colon v \in V(h), v' \in V(h')\}$ in $D[G)$. From this it follows that $V(h')(K_1) = \{0\}$ In a similar way as above we prove that $V_0(h) \subset \subset v + G_0$, $V_0(h') \subset v' + G_0$ for all $v \in V_0(h)$, $v' \in V_0(h')$.

In both cases we obtain that h and h' are of type (α). Remark 1 implies that $h \neq \{G\}$.

(ii) we want to show that $\inf W_1 = 0$ in G. It suffices to prove that $0 = = \inf W_{10} = \inf \{u - v\colon u \in U_a(h), v \in V_a(h)\}$ in G_0. We have $0 \leqq u - v$ for each $u \in U_a(h)$, $v \in V_a(h)$. Assume that htere exists $g \in G_0$ such that $0 < g \leqq u - v$ for every $u \in U_a(h)$, $v \in V_a(h)$. Therefore $g + v \leqq u$. In view of (1) we obtain $g + v \leqq h$. The elements $g + v$ and h are of type (α). By using 3.5 and (1) we infer that the relations $v \leqq -g + h$ and $h \leqq -g + h$ are valid. Since $h \in M(G)$, h has an inverse. Thus $0 \leqq -g$ and $g \leqq 0$, a contradiction.

The proof of (ii) is analogous.

From 3.6, 3.7 and 3.8 there immediately follows

3.9. Lemma. *Let* $h \in \bar{D}(G)$. *Then the following conditions are satisfied*:

(i) *If* $h = \{G\}$, *then* $h \notin M(G)$.

(ii) *If* $h \neq \{G\}$, *then* $h \in M(G)$ *if and only if* $\inf W_i = 0$ $(i = 1, 2)$ *in* G.

3.10. Theorem. *Let* G *be a cyclically ordered group. Assume that* $G_0 \neq \{0\}$. *Then* $M(G) = G^*$.

Proof. The cyclically ordered group $M(G)$ fulfils the conditions (a)—(c). Hence $M(G) \subseteq G^*$. According to 2.2 we have $G^* = G^\wedge$. Further the relation $G^\wedge \subseteq C(G) = \bar{D}(G)$ is valid. Let $h_1, h_2 \in G^\wedge$. Then there exist $g_1, g_2 \in G$, $g_1 = (x_1, a_1)$, $g_2 = (x_2, a_2)$ with $h_1 \in D(g_1 + G_0)$, $h_2 \in D(g_2 + G_0)$ and $h_1 +^\wedge h_2 = = \sup \{l(h_1) + l(h_2)\}$ (in $D((g_1 + g_2) + G_0)) = \sup \{v_1 + v_2\colon v \in V_{a_1}(h_1), v_2 \in V_{a_2}(h_2)\}$ (in $\bar{D}(G)) = h_1 + h_2$. Therefore $G^\wedge$ is a subghroup of $\bar{D}(G)$. Since $M(G)$ is the greatest element of the semigroup $\bar{D}(G)$, we obtain $G^\wedge \subseteq M(G)$. Hence $G^* = M(G)$ is valid.

B) The case $G_0 = \{0\}$

Assume that $G_0 = \{0\}$. Let $g \in G$, $g = (x, a)$. If $\psi(g) = a$, then ψ is an isomorphism of the cyclically ordered group G into K. In this sense G will be considered a subgroup of K.

If G is finite, then $M(G) = G$ and $G_0 = \{0\}$. According to 2.2 we get

3.11. Theorem. *Let G be a finite cyclically ordered group. Then $M(G) = G^*$.*

Now let G be an infinite cyclically ordered group and let $G_0 = \{0\}$. Assume that $h \in M(G)$. Then $h = \sup V(h)$ in $\bar{D}(G)$. There exists $h' \in K$, $h' = \sup V(h)$ in K. The mapping $\psi(h) = h'$ is an isomorphism of the cyclically ordered group $M(G)$ onto K.

With respect to 2.2 we get

3.12. Theorem. *Let G be an infinite cyclically ordered group. Assume that $G_0 = \{0\}$. Then $M(G)$ is isomorphic to G^*.*

From 3.10, 3.11 and 3.12 we infer that the following theorem is valid:

3.13. Theorem. *Let G be a cyclically ordered group. Then $M(G)$ is the completion of G.*

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