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# (CYCLIC) SUBGROUP SEPARABILITY OF HNN AND SPLIT EXTENSIONS

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ABSTRACT. This work has been divided in two parts. In the first part we give a sufficient conditions on an HNN extension of a free group to be cyclic subgroup seperable. In the second part we define just subgroup separability on a split extension of special groups which is actually on holomorph.

## 1. Introduction

A group $G$ is said to be *cyclic subgroup separable* ($\pi_c$) if, for each cyclic subgroup $\langle x\rangle$ of $G$, and for each element $g \in G \setminus \langle x\rangle$, there exists $N \lhd_f G$ such that $g \notin N\langle x\rangle$. In the rest of the paper the notation $N \lhd_f G$ will be used to denote that $N$ is a normal subgroup of finite index in $G$.

Let $H$ be a subgroup of a group $G$. Then $G$ is said to be $H$*-separable* if, for each $x \in G\setminus H$, there exists $N \lhd G$ such that $x \notin NH$. If $G$ is $\langle 1\rangle$-separable, then $G$ is said to be *residually finite* ($\mathcal{RF}$). Moreover a group $G$ is said to be *subgroup separable* if $G$ is $H$-separable for all finitely generated subgroups $H$ of $G$.

In [3], it has been proved a residual finiteness condition for HNN extensions by Baumslag and Tretkoff. Actually this result has been used extensively in the study of the residual finiteness of HNN extensions. For example, in [10], Kim defined cyclic subgroups separability of HNN extensions on a finite base group. Similarly, in [17], Wong investigated the (cyclic) subgroup separability of certain HNN extensions of finitely generated abelian groups. Moreover the properties of an HNN extension find an important place in the study of one-relator groups (see, for instance, [1], [2]). Therefore, in [10], it has been

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proved that certain one-relator groups are $\pi_c$. Some other studies about (cyclic) subgroup separability of HNN extensions can be found in [13], [15], [18].

Unfortunately we could not find any references in the literature about the (cyclic) subgroup separability of split extensions. But we claim to show that, by using the definition, this subject can be hold under some conditions for split extensions as well (Section 3). To do that we use the equivalence of split extensions with the semi-direct product [5].

## 2. The HNN extension case

Throughout this section $\overline{G}$ denotes the homomorphic image of $G$ and so $\overline{g}$ is a homomorphic image of $g \in G$ in $\overline{G}$.

Let us recall some basic materials for HNN extensions which may be found, for instance, in [11]. Let

$$G = \langle A, t:\ t^{-1}ht = \varphi(h)\,,\ \ h \in H\rangle$$

denotes an HNN extension of a base group $A$ with stable letter $t$ and associated subgroups $H$ and $K$, where $\varphi\colon H \to K$ is an isomorphism. Each element $g \in G$ can be written in a reduced form such as

$$g = a_0 t^{\varepsilon_1} a_1^{\varepsilon_2} \cdots a_{n-1} t^{\varepsilon_n} a_n\,, \tag{1}$$

where $a_i \in A$, $\varepsilon_i = \pm 1$, and no subwords $t^{-1}ht$ $(h \in H)$ or $tkt^{-1}$ $(k \in K)$ occur.

Let $g$ be a reduced form as in (1). Then the sum of the positive (or negative) exponents of $t$ in the word $g$ is defined by $\exp_{t^+}(g)$ (or $\exp_{t^-}(g)$). Also an element $g = a_0 t^{\varepsilon_1} a_1^{\varepsilon_2} \cdots a_{n-1} t^{\varepsilon_n}$ is said to be *cyclically reduced* if all cyclic permutations of $g$ are reduced. Therefore every element of $G$ is conjugate to a cyclically reduced form.

In this paper we give the sufficient conditions to an HNN extension of a free group be $\pi_c$. So let us recover some results of this subject on free groups. Since a free group is subgroup separable ([7]), a finite extension of a free group is subgroup separable. Moreover free groups are strongly subgroup separable ([6]). In [13], N i b l o gave his attention to the HNN extensions of a free group $F$ given by a presentation

$$G = \langle F, t:\ tat^{-1} = a^{\pm 1}\rangle \tag{2}$$

where $a \in F$, and then proved the following result.

**Proposition 2.1.** *Let $G$ be an HNN extension as in (2). If $F$ is strongly subgroup separable then $G$ is subgroup separable.*

Actually Proposition 2.1 is a considerable generalisation of the following result given by Baumslag and Solitar [4].

**Proposition 2.2.** *Let $G$ be an HNN extension as in* (2). *Then $G$ is $\mathcal{RF}$.*

The following remark plays an important role in the proof of our main result (see Theorem 2.4 below).

**Remark 2.3.** Let $F$ be a free group with finite rank and let $H = \langle x_1, x_2, \dots, x_n \rangle$, $K = \langle x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1} \rangle$ be subgroups of $F$. Then, by defining an isomorphism

$$\phi \colon H \to K\,, \qquad x_i \mapsto x_i^{\pm 1}\,,$$

it is easy to see that the HNN extension $\langle F, t : \ t^{-1}Ht = K \rangle$ defines the same HNN extension, for $a \in H$, in (2).

By taking $F$, $H$ and $K$ as in the above remark, we can give one of the main results of this paper as follows.

**Theorem 2.4.** *Let $G = \langle F, t : \ tHt^{-1} = K \rangle$ be an HNN extension and let*

$$\Delta = \{ P \lhd_f F : \ \varphi(P \cap H) = P \cap K \}\,.$$

*Assume that*

(a) $\bigcap_{P \in \Delta} HP = H$ *and* $\bigcap_{P \in \Delta} KP = K$,

(b) $\bigcap_{P \in \Delta} P\langle x \rangle = \langle x \rangle$, *for all* $x \in F$.

*Then $G$ is $\pi_c$.*

### 2.1. The proof of Theorem 2.4

Before we proceed our proof (in which there will be used a similar method as in [10, Theorem 2.2]), we also need the following lemma which is a considerable generalisation of the result by Kim and Shirvani. So we refer ([10]), ([15]) for the proof.

**Lemma 2.5.** *Let $G$ and $\Delta$ be as in* Theorem 2.4. *Then, for each $S \in \Delta$, we have a homomorphism*

$$\phi_S \colon G \to \langle F/S, t_S; \ t_S^{-1}\overline{h}t_S = \overline{h}\overline{\varphi}, \ \overline{h} \in \overline{H} \rangle\,,$$

*where $\overline{F} = F/S$, $t\phi_S = t_S$ and $\overline{\varphi} \colon HS/S \to KS/S$ is an isomorphism induced by $\varphi$.*

Proof. Let $g$, $x$ be reduced forms in $G$ such that $g \in G \setminus \langle x \rangle$. Since every element in $G$ is conjugate to a cyclically reduced form, we may assume that $x$ is cyclically reduced. Moreover, since $G$ is $\mathcal{RF}$ by Proposition 2.2, we may assume that $x \neq 1$.

**(Case 1):** Suppose that $g \notin \langle x \rangle$. Then we have the following subcases.

*(Subcase 1):* $\exp_{t^+}(x) = -\exp_{t^-}(x)$ and $\exp_{t^+}(g) \neq -\exp_{t^-}(g)$,

*(Subcase 2):* $\exp_{t^+}(g) = -\exp_{t^-}(g)$ and $\exp_{t^+}(x) \neq -\exp_{t^-}(x)$,

*(Subcase 3):* $\exp_{t^+}(g) \neq -\exp_{t^-}(g)$, $\exp_{t^+}(x) \neq -\exp_{t^-}(x)$ and $\exp_{t^+}(x) - \exp_{t^-}(x)$ does not divide $\exp_{t^+}(g) - \exp_{t^-}(g)$.

For these subcases, we can find $S \in \Delta$ such that $\overline{g} \neq 1$ is reduced, $\exp_{t^+}(\overline{g}) - \exp_{t^-}(\overline{g}) = \exp_{t^+}(g) - \exp_{t^-}(g)$ and $\overline{x} \neq 1$ is cyclically reduced. Also $\exp_{t^+}(\overline{x}) - \exp_{t^-}(\overline{x}) = \exp_{t^+}(x) - \exp_{t^-}(x)$, where, by Lemma 2.5,

$$\overline{G} = G_{\phi_S} = \left\langle F/S, t_S;\ t_S^{-1}\overline{H}t_S = \overline{K} \right\rangle.$$

It follows that $\overline{g} \notin \langle \overline{x} \rangle$. Since, by Proposition 2.1, $\overline{G}$ is $\pi_c$, there exist $\overline{N} \lhd \overline{G}$ such that $\overline{g} \notin \overline{N}\langle \overline{x} \rangle$. Let $N$ be the preimage of $\overline{N}$ in $G$. Then $g \notin N\langle x \rangle$ and $N \lhd G$, as required.

**(Case 2):** $\exp_{t^+}(g) = -\exp_{t^-}(g)$ and $\exp_{t^+}(x) = -\exp_{t^-}(x)$.

By assumption $(b)$, there exists $S \in \Delta$ such that $g \notin S\langle x \rangle$. By considering $\overline{G} = G\phi_S$ as in previously, we then have $\overline{g} \notin \langle \overline{x} \rangle$, and therefore one can find a normal subgroup $N$ of $G$ such that $g \notin N\langle x \rangle$, as required.

**(Case 3):** $\exp_{t^+}(g) \neq -\exp_{t^-}(g)$, $\exp_{t^+}(x) \neq -\exp_{t^-}(x)$ and $\exp_{t^+}(x) - \exp_{t^-}(x)$ divides $\exp_{t^+}(g) - \exp_{t^-}(g)$.

Since $x$ is cyclically reduced, we write $x = a_0 t^{\delta_1} a_1 t^{\delta_2} \cdots a_{n-1} t^{\delta_n}$ where $a_j \in F$, $n \geq 1$ and $\delta_{j+1} = \pm 1$. Let $\exp_{t^+}(g) - \exp_{t^-}(g) = m = nk$ $(k \in \mathbb{Z}^+)$ and let $g = b_0 t^{\varepsilon_1} b_1 t^{\varepsilon_2} \ldots b_{m-1} t^{\varepsilon_m} b_m$ be reduced, where $b_i \in F$ and $\varepsilon_i = \pm 1$ by the condition $(a)$, we can find $S_1 \in \Delta$ such that $a_i \notin HS_1$ if $a_i \notin H$ or $a_i \notin KS_1$ if $a_i \notin K$, for each $i = 0, \ldots, n-1$. Similarly we can find $S_2 \in \Delta$ such that $b_j \notin HS_2$ if $b_j \notin H$ or $b_j \notin KS_2$ if $b_j \notin K$, for each $j = 0, \ldots, m$.

Now, since $g^{-1}x^k \neq 1 \neq gx^k$ and since, by Proposition 2.2, $G$ is $\mathcal{RF}$, there exist $M \lhd G$ such that $g^{-1}x^k \notin M$ and $gx^k \notin M$. Then $M \cap F \in \Delta$ and $P = S_1 \cap S_2 \cap (M \cap F) \in \Delta$. Since $P \subset S_1 \cap S_2$, then $\overline{g}$ is reduced and $\overline{x}$ is cyclically reduced, where $\overline{G} = G_{\phi_P}$. Moreover we have

$$\begin{aligned} \exp_{t^+}(\overline{g}) - \exp_{t^-}(\overline{g}) &= \exp_{t^+}(g) - \exp_{t^-}(g) = m = nk \\ &= \exp_{t^+}(x^k) - \exp_{t^-}(x^k) = \exp_{t^+}\left(\overline{x}^k\right) - \exp_{t^-}\left(\overline{x}^k\right) \end{aligned}$$

and $\overline{g} \neq \overline{x}^{\pm k}$, where $\overline{G} = G_{\phi_P}$. It follows that $\overline{g} \notin \langle \overline{x} \rangle$. Then, as in Case 1, we can find $N \lhd G$ such that $g \notin N\langle x \rangle$. This completes the proof.

Hence the result. □

We remark that M o s t o w s k i [12] has shown that the word problem is solvable for finitely presented, residually finite groups. In the same way one can think that the power problem is solvable for finitely presented $\pi_c$ groups. Therefore let us take the HNN extension as depicted in Theorem 2.4. We then get:

**Corollary 2.6.** *The group $G$ has solvable power problem.*

## 3. The split extension case

Our work in this section is based on the fact that the split extension is semi-direct product (see [5] for the proof). Therefore, as in the previous section, let us recover some basic facts about semi-direct product of two groups.

Let $A$, $K$ be groups, and let $\theta$ be a homomorphism defined by $\theta\colon A \longrightarrow \operatorname{Aut}(K)$, $a \longmapsto \theta_a$ for all $a \in A$. Then the semi-direct product $G = K \rtimes_\theta A$ of $K$ by $A$ is defined as follows.

The elements of $G$ are all ordered pairs $(a,k)$ $(a \in A,\ k \in K)$ and the multiplication is given by

$$(a,k)(a',k') = \big(aa', (k\theta_{a'})k'\big)\,.$$

Similiar definitions of a semi-direct product can be found in [14] or [16].

Suppose that $\mathcal{P}_K = \langle \mathbf{y};\ \mathbf{s}\rangle$ and $\mathcal{P}_A = \langle \mathbf{x};\ \mathbf{r}\rangle$ are presentations for the groups $K$ and $A$, respectively. In [9, Proposition 10.1, Corollary 10.1], Johnson showed that the semi-direct product $G = K \rtimes_\theta A$ has the presentation

$$\mathcal{P} = \langle \mathbf{y}, \mathbf{x};\ \mathbf{s}, \mathbf{r}, \mathbf{t}\rangle$$

where $\mathbf{t} = \big\{yx\lambda_{yx}^{-1}x^{-1} :\ y \in \mathbf{y}\,,\ \ x \in \mathbf{x}\big\}$, and $\lambda_{yx}$ is a word on $\mathbf{y}$ representing the element $(k_y)\theta_{a_x}$ of $K$ $(a \in A,\ k \in K,\ x \in \mathbf{x},\ y \in \mathbf{y})$.

As we said previously, the subgroup separability will be investigated on a special semi-direct product, actually on holomorph in this paper. The *holomorph* of a group is the semi-direct product of the group with the automorphism group, with respect to the obvious action. We recall that the automorphism group of a non-trivial finite cyclic group of order $r$ is well known to be cyclic if and only if the number $r$ is of the kind $r = 4$, $r = p^t$, $r = 2p^t$ where $p$ is an odd prime; in these cases, the holomorph is thus a split metacyclic group.

Let $t \geq 2$ and let $N$ be the cyclic group of order $r = 2^t$. As usual, we will identify the automorphism group of $N$ with the group $\mathbb{Z}_{2^t}^*$ of units of $\mathbb{Z}_{2^t}$, viewed as a commutative ring. Now let us consider the holomorph

$$G = \mathbb{Z}_{2^t} \rtimes \mathbb{Z}_{2^t}^* \tag{3}$$

of $N$. For the case $t = 2$, the group $G$ comes down to the dihedral group, the group $\mathbb{Z}_4^*$ being cyclic of order 2, generated by the class of $-1$. We note that $t = 2$ case will not be considered in the subgroup separability in this paper.

Henceforth we suppose that $t \geq 3$. Now the group $\mathbb{Z}_{2^t}^*$ decomposes as a direct product of a copy of $\mathbb{Z}_2$, generated by the class of $-1$, and a copy of $\mathbb{Z}_{2^{t-2}}$, generated by the class of 5. Let us write $s = 2^{t-1}$. By [8], the cyclic groups being written multiplicatively, the semi-direct product $G$, given in (3), has thus the presentation

$$\mathcal{P}_G = \langle x, y, z;\ y^r,\ x^s,\ xyx^{-1} = y^5,\ zyz^{-1} = y^{-1},\ [x, z] \rangle, \tag{4}$$

where the normal cyclic subgroup $N$ is generated by $y$ and the cyclic subgroups of order $s$ and 2 are generated by $x$ and $z$, respectively.

We have a subgroup, say $G_1$, of $G$ (by [8]) generated by $x$ and $y$ is metacyclic with presentation

$$\mathcal{P}_{G_1} = \langle x, y;\ y^r,\ x^s,\ xyx^{-1} = y^5 \rangle\,. \tag{5}$$

Again by [8], we have another subgroup, say $G_2$, of $G$ generated by $z$ and $y$ is metacyclic of the form $G_2 = N \rtimes \langle z;\ z^2 \rangle$ with the presentation

$$\mathcal{P}_{G_2} = \langle y, z;\ y^r,\ zyz^{-1} = y^{-1} \rangle\,. \tag{6}$$

Thus we have the following other main result of this paper.

**Theorem 3.1.** *Let $G$ be a semi-direct as in* (3) *with the presentation* (4). *Then, for the subgroups $G_1$ and $G_2$ of $G$, $G$ is $G_1$ and $G_2$-subgroup separable, respectively.*

Proof. In the proof we will just follow the definition of subgroup separability. Now a simple calculation shows that we have total $2rs$ elements in the group $G$, $rs$ elements in $G_1$ and $2r$ elements in $G_2$. The total $rs$ elements of $G \setminus G_1$ can be given in the set

$$\begin{aligned}\{&z, zy, zy^2, \dots, zy^{r-1}, xz, x^2z, x^3z,\\ &\dots, x^{s-1}z, yxz, y^2xz, y^3xz, \dots, y^{r-1}xz, yx^2z, y^2x^2z, y^3x^2z,\\ &\dots, y^{r-1}x^2z, \dots, yx^{s-1}z, y^2x^{s-1}z, y^3x^{s-1}z, \dots, y^{r-1}x^{s-1}z\}\,.\end{aligned}$$

Also the total $2r(s-1)$ elements of $G \setminus G_2$ can be given in the set

$$\begin{aligned}\{&x, x^2, x^3, \dots, x^{s-1}, yx, yx^2, yx^3, \dots, yx^{s-1}, y^2x, y^2x^2, y^2x^3,\\ &\dots, y^2x^{s-1}, y^{r-1}x, y^{r-1}x^2, y^{r-1}x^3, \dots, y^{r-1}x^{s-1}, xz, x^2z, x^3z,\\ &\dots, x^{s-1}z, yxz, y^2xz, y^3xz, \dots, y^{r-1}xz, yx^2z, y^2x^2z, y^3x^2z,\\ &\dots, y^{r-1}x^2z, \dots, yx^{s-1}z, y^2x^{s-1}z, y^3x^{s-1}z, \dots, y^{r-1}x^{s-1}z\}\,.\end{aligned}$$

Let us take the normal subgroup $N = \langle y;\ y^r \rangle$. It is easy to see that $\forall g_1 \in G \setminus G_1$ and $\forall g_2 \in G \setminus G_2$, $g_1 \notin NG_1$ and $g_2 \notin NG_2$. Therefore $G$ is $G_1$ and $G_2$-subgroup separable. This gives the result. □

Let us consider the subgroup $G_1$ with the presentation (5) of $G$ again. By choosing $z$ as a stable letter and consider the mappings

$$y \mapsto y^{-1}, \qquad x \mapsto x,$$

the group $G$ with the presentation (4) becomes a finite HNN extension of $G_1$. Thus we have the following easy consequence of Theorem 3.1.

**Corollary 3.2.** *The HNN extension $G$ of $G_1$ as above is just subgroup separable.*

However we cannot get a similar result as in the above corrollary for the subgroup $G_2$. But a simple calculation as in the proof of Theorem 3.1 shows that the subgroup $G_2 = N \rtimes \langle z;\ z^2 \rangle$, is metacyclic by [8], is not subgroup separable. So one can generalize this to

**Corollary 3.3.** *Not all metacyclic groups are subgroup separable.*

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