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DARBOUX PROPERTY FOR FUNCTIONS OF SEVERAL VARIABLES

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In [1] there are investigated local properties of Darboux ($\mathcal{O}$) functions mapping E_n into R, where $\mathcal{O}$ is a basis of E_n with the following properties:

(1) Every element of $\mathcal{O}$ is a connected set;
(2) Any translation of an element of $\mathcal{O}$ is in $\mathcal{O}$;
(3) For $x \in E_n$ and $U \in \mathcal{O}$, $x \in \bar{U}$, there exists $V \in \mathcal{O}$ such, that $x \in \bar{V}$ and $\bar{V} - \{x\} < U$.($\bar{U}$denotes the closure of U.)

If $\mathcal{O}$ is the class of open intervals in E_n, $n > 1$, the condition (3) is not satisfied.

Troughout this paper $\mathcal{X}$ will be the set R^2 with the usual topology. Let ϱ be a mertic in $\mathcal{X}$ inducing the usual topology. $\mathcal{O}$ will denote the class of all open spheres in the metric space $(\mathcal{X}, \varrho)$. It is obvious that there is a metric ϱ such that $\mathcal{O}$ does not satisfy (3).

Let $\mathcal{N}$ be a class of subsets of real numbers such that $\emptyset \in \mathcal{N}$. We define a class $\mathcal{N}\mathcal{D}(\mathcal{O})$ of functions as follows [4].

A function f: $\mathcal{X} \to R$ belongs to the class $\mathcal{N}\mathcal{D}(\mathcal{O})$ if for every $G \in \mathcal{O}$ we have $(\inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x)) - f(G) \in \mathcal{N}$.

If $\mathcal{N} = \{\emptyset\}$, then $\mathcal{N}\mathcal{D}(\mathcal{O}) = \mathcal{D}(\mathcal{O})$ which is the class of all functions with the Darboux property as it is defined in [6]. If $\mathcal{N}$ is the class of all sets complements of which are dense in their convex hulls, then f belongs to the class $\mathcal{U}_0(\mathcal{O})$, which is the same class as the class $\mathcal{D}_0(\mathcal{O})$ defined in [7]. For the case of functions of real variable this is the same as the class $\mathcal{U}_0$ defined in [2] and the class of functions studied by Radaković in [9].

Denote by $\mathcal{U}(\mathcal{O})$ the class of all functions f: $\mathcal{X} \to R$ such that for every $G \in \mathcal{O}$ and every set $M \subset \mathcal{X}$ of cardinality less than c we have $\overline{f(G - M)} \supset \langle \inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x) \rangle$.

(c denotes the cardinality of the continuum.)

Theorems proved in this paper are similar to Theorem 1 of [1] for functions of classes $\mathcal{N}\mathcal{D}(\mathcal{O})$, $\mathcal{U}_0(\mathcal{O})$, $\mathcal{U}(\mathcal{O})$. For the case of functions of real variable this is proved in [3] for $\mathcal{D}$ and in [4] for $\mathcal{N}\mathcal{D}$. It is also shown that $\mathcal{U}(\mathcal{O})$ is the closure of the class $\mathcal{D}(\mathcal{O})$ with respect to uniform limits.

If x is an element of a metric space $(\mathscr{X}, \varrho)$ and r is a positive number, then the symbol $S(x, r)$ denotes the set $\{y; \varrho(x, y)<r\}$ and $\mathcal{O}$ denotes the class $\{S(x, r); x \in \mathscr{X}, r>0\}$.

Let α be a direction in the plane. Put $S_{\alpha xr}=S^0_{\alpha xr}\cup\{x\}$, where $S^0_{\alpha xr}$ is the open sphere with radius r whose centre c_x lies on the halfline with origin x and direction α and $\varrho(c_x, x)=r$.

Let us denote $f_\alpha(x)=\liminf_{r\to 0^+} \{f(y); y \in S_{\alpha xr}\}$,

$$f^\alpha(x)=\limsup_{r\to 0^+} \{f(y); y \in S_{\alpha xr}\} \text{ and}$$

$$I_\alpha(x)=(f_\alpha(x), f^\alpha(x)).$$

Theorem 1. *Let $\mathcal{N}$ be a hereditary σ-additive class of subsets of R such that if an open interval belongs to $\mathcal{N}$, then its closure also belongs to $\mathcal{N}$. Then $f \in \mathcal{N}\mathcal{D}(\mathcal{O})$ if and only if for every $x \in \mathscr{X}$, $\alpha \in \langle 0,2\pi)$ and $r>0$ we have $I_\alpha(x)-f(S_{\alpha xr}) \in \mathcal{N}$.*

To prove the theorem we need the following lemma of [4].

Lemma 1. *Let $\mathcal{N}$ satisfy the hypothesis of Theorem 1. If $E \notin \mathcal{N}$, then there exists $y_0 \in E$ such that for every open interval $I\subset R$ containing y_0 we have $I\cap E \notin \mathcal{N}$ and for every open interval $J\subset R$ which has y_0 as its end point we have $J \notin \mathcal{N}$.*

Proof of Theorem 1. The necessity of the condition is obvious. The sufficiency will be proved indirectly. Let $f \notin \mathcal{N}\mathcal{D}(\mathcal{O})$. There exists $G \in \mathcal{O}$ such that $E=(\inf_{x\in G} f(x), \sup_{x\in G} f(x))-f(G) \notin \mathcal{N}$. According to Lemma 1, there exists a point $y_0 \in E$ such that for every open interval I containing y_0 we have $E\cap I \notin \mathcal{N}$ and for every open interval J for which y_0 is one of the end points we have $J \notin \mathcal{N}$. There exist points x_1 and x_2 in G such that $f(x_1)<y_0<f(x_2)$. If for every $x \in G$, $f(x)$ is less than y , there would be a point $x' \in \bar{G}$ such that $f(x')>y_0$. For α equal to the direction of the line connecting the point x' with the centre of G we have $f^\alpha(x')>y_0$ and $f_\alpha(x')\leq y_0$, which is in contradiction with the assumption of the theorem. So there exists $G_0 \in \mathcal{O}$ such that $x_1, x_2 \in G_0$ and $\bar{G}_0\subset G$.

Let us denote $A=\{x; x \in \bar{G}_0, f(x)>y_0\}$, and $B=\{x; x \in \bar{G}_0, f(x)<y_0\}$. We show that for $x \in A$ we have $f_\alpha(x)\geq y_0$ for all $\alpha \in \langle 0, 2\pi)$. Let $f_{\alpha_0}(x)<y_0$ for some α_0. Since $x \in A$, $f^{\alpha_0}(x)>y_0$ and so $y_0 \in I_{\alpha_0}(x)$. We obtain $E\cap I_{\alpha_0}(x)-f(S_{\alpha_0xr}) \in \mathcal{N}$, for $S_{\alpha_0xr}\subset G_0$, which contradicts a property of y_0. It follows that for $x \in A$ and any natural number m there exists an open set $H_A(x) \in \mathcal{O}$, $x \in H_A(x)$ such that for $z \in \bar{H}_A(x)$, $f(z)>y_0-1/m$. Similarly for $x \in B$ and any natural number m, $f^\alpha(x)\leq y_0$ and there exists an open sphere $H_B(x) \in \mathcal{O}$, $x \in H_B(x)$ such that for $z \in \bar{H}_B(x)$ we have $f(z)<y_0+1/m$. $\bar{G}_0\subset \bigcup_{x\in B} H_B(x)\cup\bigcup_{x\in A} H_A(x)$ and so $\bar{G}_0=\bigcup_{x\in A} H_A(x)\cap\bar{G}_0\cup\bigcup_{x\in B} H_B(x)\cap\bar{G}_0$. Since $\bar{G}_0$ is connected,

$\bigcup_{x \in A} H_A(x) \cap \bigcup_{x \in B} H_B(x) \neq \emptyset$. Hence there exist points $z_1 \in A$ and $z_2 \in B$ such that $H_A(z_1) \cap H_B(z_2) \neq \emptyset$. Let $\lambda_0 = \sup\{\lambda;\ f((1-\mu)z_1 + \mu z_2) > y_0,\ 0 \leq \mu \leq \lambda\}$ and $x_0 = (1-\lambda_0)z_1 + \lambda_0 z_2$. Then we show that $f(x_0) = y_0$. Let $f(x_0) > y_0$. Let α be the direction of the line $x_0 z_2$. Then $f_\alpha(x_0) \leq y_0$ and $f^\alpha(x_0) > y_0$ but this is contradictory to the property of y_0. Similarly we can show that $f(x_0)$ is not less than y_0. So $f(x_0) = y_0$ which contradicts the assumption that $y_0 \in E$.

Let $\mathscr{U}_0(\mathcal{O})$ denote the class of all functions f: $\mathscr{X} \to R$ such that every $G \in \mathcal{O}$ we have $\overline{f(G)} \supset \langle \inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x) \rangle$.

As it was said, $\mathscr{U}_0(\mathcal{O}) = \mathscr{N}\mathscr{D}(\mathcal{O})$ if $\mathscr{N}$ is the class of all sets complements of which are dense in their convex hulls. But this class of sets is not additive, so Theorem 1 is not valid for functions of $\mathscr{U}_0(\mathcal{O})$. A similar theorem for $\mathscr{U}_0(\mathcal{O})$ can be proved.

Theorem 2. *A function f belongs to $\mathscr{U}_0(\mathcal{O})$ if and only if for every $x \in \mathscr{X}$, $\alpha \in \langle 0, 2\pi)$ and $r > 0$ we have $I_\alpha(x) \subset \overline{f(S_{\alpha x r})}$.*

To prove Theorem 2 we need the following generalization of Lemma of [3].

Lemma 2. *Let $g: M \to R$, where M is a connected set in $\mathscr{X}$. Let $g(x) \neq 0$ for $x \in M$ and let there be points $x_1, x_2 \in M$ such that $g(x_1) \cdot g(x_2) < 0$. Then there exists a point $x_0 \in M$ such that $g(x)$ takes both positive and negative values in every neighbourhood of x_0.*

Proof. As $M = \{x;\ x \in M,\ g(x) > 0\} \cup \{x;\ x \in M,\ g(x) < 0\}$ and M is connected, it follows that

$$\overline{\{x;\ x \in M,\ g(x) > 0\}} \cup \overline{\{x;\ x \in M,\ g(x) < 0\}} \cap M \neq \emptyset .$$

Proof of Theorem 2. It is obvious that the condition is necessary.

To prove the sufficiency of the condition, we argue by contradiction. Let $y_0 \in \langle \inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x) \rangle - \overline{f(G)}$ for some G. Then there are points $x_1, x_2 \in G$ such that $f(x_1) < y_0 < f(x_2)$. Let there be $f(x) < y_0$ for all $x \in G$, then there exists $x_0 \in \bar{G}$ such that $f(x_0) > y_0$. Let α be a direction of the line $x_0 x_1$, where x_1 is the centre of G, then $f_\alpha(x_0) < y_0$ and $f^\alpha(x_0) > y_0$ and $f(S^0_{\alpha x_0 r}) \subset \overline{f(G)}$ for $S^0_{\alpha x_0 r} \subset G$, $\overline{f(S_{\alpha x_0 r})} = \overline{f(S^0_{\alpha x_0 r})} \cup \{f(x_0)\}$ which is in contradiction with the fact that $I_\alpha(x) \subset \overline{f(S_{\alpha x r})}$. Put $y_1 = \sup\{z;\ z \in \overline{f(G)},\ z < y_0\}$, $y_2 = \inf\{z;\ z \in \overline{f(G)},\ z > y_0\}$. Then $y_1, y_2 \in \overline{f(G)}$ and $(y_1, y_2) \cap \overline{f(G)} = \emptyset$. Define a function g by $g(x) = f(x) - (y_1 + y_2)/2$. The function g satisfies on G the conditions of Lemma 2. Hence there is a point x_0 such that in every neighbourhood of x_0, $f(x)$ takes the values both greater or equal y_2 and smaller or equal y_1.

Now we prove that there is α_0 such that $I_{\alpha_0}(x_0) \supset (y_1, y_2)$ and the proof of Theorem 2 will be finished.

Put $A_1 = \{\alpha; f^\alpha(x_0) \leqslant y_1\}$ and $A_2 = \{\alpha; f_\alpha(x_0) \geqslant y_2\}$. We consider two cases.

Case 1. $\langle 0, 2\pi) - (A_1 \cup A_2) \neq \emptyset$. Then for $\alpha_0 \in \langle 0, 2\pi) - (A_1 \cup A_2)$ we have $f^{\alpha_0}(x_0) \geqslant y_2$ and $f_{\alpha_0}(x_0) \leqslant y_1$. Thus $I_{\alpha_0}(x_0) \supset (y_1, y_2)$.

Case 2. $A_1 \cup A_2 = \langle 0, 2\pi)$. Let $\alpha_0 \in b(A_1) \cap b(A_2)$, where $b(A)$ denotes the boundary of A with respect to the usual topology on the interval. There is a sequence $\{\alpha_n^1\}_{n=1}^\infty$ convergent to α_0, $\alpha_n^1 \in A_1$ for $n = 1, 2, 3, \ldots$ and we have $f_{\alpha_0}(x_0) \leqslant y_1$. Similarly there exists a sequence $\{\alpha_n^2\}_{n=1}^\infty$ convergent to α_0, $\alpha_n^2 \in A_2$ for $n = 1, 2, 3, \ldots$ and we have $f^{\alpha_0}(x_0) \geqslant y_2$ so that $I_{\alpha_0}(x_0) \supset (y_1, y_2)$.

The following theorem can be proved in a similar way.

Theorem 3. *A function f belongs to $\mathscr{U}(\mathcal{O})$ if and only if for every x, $\alpha \in \langle 0, 2\pi)$, $r > 0$ and for every set $M \subset \mathscr{X}$ of cardinality less then c we have $I_\alpha(x) \subset \overline{f(S_{\alpha x r} - M)}$.*

As it is shown in [2] for the case of real variable, the class $\mathscr{U}$ is the class of uniform limits of sequences of Darboux functions. A similar characterization for the class $\mathscr{U}(\mathcal{O})$ is given.

Let us denote $C_0(f, x)$ ($C(f, x)$) the set of all points y such that for every set $G \in \mathcal{O}$ such that $x \in G$ and for every neighbourhood N of y the set $f^{-1}(N) \cap G \neq \emptyset$ (has cardinality c).

Let us denote $C_0^\alpha(f, x, \mathcal{O})$ ($C^\alpha(f, x, \mathcal{O})$) the set of all points y such that for every neighbourhood N of y and for every $r > 0$ the set $f^{-1}(N) \cap S_{\alpha x r} \neq \emptyset$ (has cardinality c).

Theorem 4. *For a function f: $\mathscr{X} \to R$ the following conditions are equivalent:*

(a) *$f \in \mathscr{U}_0(\mathcal{O})$.*

(b) *$C_0(f, x)$ is a closed interval for every $x \in \mathscr{X}$.*

(c) *For every $G \in \mathcal{O}$ we have $\bigcup_{x \in \bar{G}} C_0(f, x) = \langle \inf_{x \in \bar{G}} f(x), \sup_{x \in G} f(x)\rangle$ (for $x \in \bar{G} - G$ instead of $C_0(f, x)$ we take $C_0^\alpha(f, x, \mathcal{O})$ for α equal to the direction of x_0x, where x_0 is the centre of G).*

Proof. (a) implies (b). Suppose that $C_0(f, x_0)$ is not an interval. Then the convex hull $\mathrm{co}(C_0(f, x_0))$ of $C_0(f, x_0)$ contains a point y which does not belong to $C_0(f, x_0)$. Since $f \in \mathscr{U}_0(\mathcal{O})$, we have $\overline{f(G)} \supset \langle \inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x)\rangle \supset \mathrm{co}(C_0(f, x_0))$ for G which contains x_0. Let N be neighbourhood of y. Then $f^{-1}(N) \cap G \neq \emptyset$ and consequently $y \in C_0(f, x_0)$.

(b) implies (c). Let $C_0(f, x)$ be a closed interval for every $x \in \mathscr{X}$. Let $G \in \mathcal{O}$ and $K = \bigcup_{x \in \bar{G}} C_0(f, x)$.

We will show that K is dense in its convex hull $\mathrm{co}(K)$. Let there be an interval

$(a, b) \subset \mathrm{co}(K) - K$. $C_0(f, x)$ is a closed interval hence either $C_0(f, x) \subset (-\infty, a\rangle$ or $C_0(f, x) \subset \langle b, \infty)$ for each $x \in \bar{G}$. Let $C_0(f, x_0) \subset (-\infty, a\rangle$. Let there be α_0 such that, for every $r > 0$, $S_{\alpha_0 x_0 r}$ contains a point x_r with the property $C_0(f, x_r) \subset \langle b, \infty)$. Then there exists $y \in C_0(f, x_0)$ such that $y \geq b$ and this is a contradiction. For all α let δ_0^α denote the supremum of all δ such that $y \in C_0(f, x)$ implies $y \leq a$ for every $x \in S_{\alpha x_0 \delta}$. Then $\delta_0^\alpha > 0$. Let $\delta_0^\alpha \neq \infty$ and z_1 be a point of the boundary of $S_{\alpha x_0 \delta_0^\alpha}$. Then every set $H \in \mathcal{O}$ containing the point z_1 contains also a point z_2 such that $\sup C_0(f, z_2) \leq a$ and it follows $\sup C_0(f, z_1) \leq a$. We can apply to the point z_1 the same consideration as for x_0. Thus we get that for all $x \in \mathfrak{X}$ we have $\sup C_0(f, x) \leq a$.

Let $c \in \mathrm{co}(K)$. According to the preceding for every n there is an $x_n \in \bar{G}$ such that $C_0(f, x_n) \cap (c - 1/n, c + 1/n) \neq \emptyset$. We can suppose that x_n converges to $x_0 \in G$. Then there exist points $z_n \in S(x_n, 1/n) \cap \bar{G}$ such that $f(z_n) \in (c - 1/n, c + 1/n)$. Hence $c \in C_0(f, x_0) \subset K$.

Since there exist points x_1, x_2 such that $\inf_{x \in \bar{G}} f(x) \in C_0(f, x_1)$ and $\sup_{x \in G} f(x) \in C_0(f, x_2)$ the condition (c) follows.

(c) implies (a). The proof is very similar to that for the case of a real variable [2, Theorem 3.1].

Let $A, B \subset \mathcal{X}$. The set A will be called c-dense in B if for every $G \in \mathcal{O}$ for which $G \cap B \neq \emptyset$ the set $G \cap A$ has cardinality c.

Theorem 5. *The following conditions are equivalent:*

(a) $f \in \mathcal{U}(\mathcal{O})$.

(b) *For every $G \in \mathcal{O}$ we have $\bigcup_{x \in \bar{G}} C(f, x) = \inf_{x \in \bar{G}} f(x), \sup_{x \in \bar{G}} f(x)\rangle$ (for $x \in \bar{G} - G$ instead of $C(f, x)$ we take $C^\alpha(f, x, \mathcal{O})$, for α equal to the direction of the line $x_0 x$, where x_0 is the centre of G).*

(c) *$f \in \mathcal{U}_0(\mathcal{O})$ and, for every open interval I, $f^{-1}(I)$ is empty or c-dense in itself.*

(d) *$f \in \mathcal{U}_0(\mathcal{O})$ and the graph of f is c-dense in itself.*

Proof will be omited because it is very similar to that for the case of one variable [2, Theorem 3.2].

It is easy to see that Lemma 4.1 [2] holds also in $\mathcal{X}$.

Lemma 3. *Any $A \subset \mathcal{X}$ c-dense in itself is a union of countably many disjoint, non-empty subsets each of which is c-dense in A.*

Theorem 6. *Let $f \in \mathcal{U}(\mathcal{O})$ and $\varepsilon > 0$. Then there exists $g \in \mathcal{U}(\mathcal{O})$ such that g is not constant on any sphere, the range of g is countable and $\|f - g\| < \varepsilon$.*

Proof. Let $f(x) = a$, for all $x \in \mathcal{X}$. Let $\{r_i\}_{i=1}^\infty$ be the rational numbers lying in the interval $(a-\varepsilon/2,\ a+\varepsilon/2)$. Let $\{A_i\}_{i=1}^{\infty}$ be a decomposition of $\mathscr{X}$ into subsets c-dense in $\mathscr{X}$ (as it is given by Lemma 3). Define $g(x)=r_i$, for $x \in A_i$.

Let f be not constant. Then we can assume that $f(\mathscr{X})=R$, $R=\bigcup_{n=1}^{\infty} I_n$, $|I_n|<\varepsilon$, $I_j \cap I_k=\emptyset$, for $j\neq k$, where $I_n=(a_{n-1},\ a_n\rangle$ are half-open intervals having irrational end points. Put $A_n=f^{-1}(I_n^0)$, where I_n^0 denotes the interior of I_n. Let $\{r_{n,k}\}_{k=1}^{\infty}$ be a sequence of rational numbers belonging to I_n. Since $f \in \mathscr{U}(\mathscr{O})$, A_n is c-dense in itself. If A_n were not c-dense in itself, then there would exist $G_n \in \mathscr{O}$ such that card $(G_n \cap A_n)<c$ and $f(G_n-(G_n\cap A_n))=f(G_n-A_n)=f(G_n)-I_n^0$ and this is in contradiction with the fact that $(\inf_{x\in \bar{G}} f(x),\ \sup_{x\in \bar{G}} f(x)) \subset \overline{f(G_n-(G_n\cap A_n))}=\overline{f(G_n)-I_n^0}$.

Put

$$g(x)=\begin{cases} r_{n,k} \text{ for } x \in B_{n,k}, \\ f(x) \text{ for } x \notin \bigcup_{n=1}^{\infty} A_n, \end{cases}$$

where $A_n=\bigcup_{k=1}^{\infty} B_{n,k}$ is the decomposition of the set A_n given in Lemma 3.

It is obvious that $\|f-g\|<\varepsilon$ and the range of g is countable, and since $B_{n,k}$ are c-dense in A_n, g cannot be constant on any sphere.

Let x, α be given. Denote $\mathscr{I}^{\alpha}(x)=\{I_n\,;\ I_n\cap I_{g(x)}^{\alpha}\neq\emptyset\}$, where $I_{g(x)}^{\alpha}=(g_{\alpha}(x),\ g^{\alpha}(x))$ (similarly $I_{f(x)}^{\alpha}$). Then for $I \in \mathscr{I}^{\alpha}(x)$ we have $I_{f(x)}^{\alpha}\cap I\neq\emptyset$. Indeed let $I_n\cap I_{f(x)}^{\alpha}=\emptyset$ for some $I_n \in \mathscr{I}^{\alpha}(x)$. Since $f^{-1}(I_n^0)=A_n$ and A_n is c-dense in itself, there exists r_0 such that for $r<r_0$ it follows that $S_{\alpha xr}\cap A_n=\emptyset$. Hence, for such r, $g(S_{\alpha xr})\cap I_n^0=\emptyset$ and since $g^{-1}(a_n)\cap S_{\alpha xr}=f^{-1}(a_n)\cap S_{\alpha xr}$ we have $g(S_{\alpha xr})\cap I_n=\emptyset$, a contradiction.

Let $g^{\alpha}(x)=\infty$. Then, since $f \in \mathscr{U}(\mathscr{O})$, we have $f(S_{\alpha xr}-C)\cap I_n^0\neq\emptyset$ for every $I_n \in \mathscr{I}^{\alpha}(x)$, $r>0$ and for every C with cardinality less then c. Then the set $g(S_{\alpha xr}-C)$ contains all rational numbers of the intervals of $\mathscr{I}^{\alpha}(x)$. Let $z \in S_{\alpha xr}-C$ be such that $f(z) \in I_n^0$ for $I_n \in \mathscr{I}^{\alpha}(x)$. Obviously, $z \in A_n$. Then $(B_{n,i}-C)\cap S_{\alpha xr}\neq\emptyset$ for every i. Therefore $g(S_{\alpha xr}-C)$ contains all rational numbers of $\cup\{I_n\,;\ I_n \in \mathscr{I}^{\alpha}(x)\}$.

Let $g^{\alpha}(x)<\infty$. Then there exists I_{n_0} such that $g^{\alpha}(x) \in I_{n_0}$. Let $f^{\alpha}(x) \in I_{n_0}^0$ or $f^{\alpha}(x)=a_{n_0}$. Since $f \in \mathscr{U}(\mathscr{O})$, according to Theorem 3 we have $f(S_{\alpha xr}-C)\cap I_{n_0}^0\neq\emptyset$ for every $r>0$. Therefore $g(S_{\alpha xr}-C)$ contains all rational numbers of the interval I_{n_0}.

Similarly as in the case of $g^{\alpha}(x)=\infty$ we can prove that $g(S_{\alpha xr}-C)$ contains all rational numbers of $\{I_n\,;\ I_n \in \mathscr{I}^{\alpha}(x)-\{I_{n_0}\}\}$. Hence, by Theorem 3, it follows that $g \in \mathscr{U}(\mathscr{O})$.

Theorem 7. *Let $f \in \mathscr{U}(\mathscr{O})$ be a function with a countable range and not constant on any sphere. Then f is a uniform limit of a sequence of Darboux $(\mathscr{O})$ functions.*

The proof of Theorem 7 is very similar to that for the case of real variable [2, Theorem 4.2].

Lemma 4. *If* $f \in \mathscr{B}_1$ *and for each* x, $f(x) \in \bigcap_\alpha (I_\alpha(x))_0$, $\alpha \in \langle 0, 2\pi)$, *then* $f \in \mathscr{D}(\mathcal{O})$, *where* $(I_\alpha(x))_0 = \langle \lim_{r\to 0^+} \inf \{f(z),\ z \in S^0_{\alpha x r}\},\ \lim_{r\to 0^+} \sup \{f(z),\ z \in S^0_{\alpha x r}\} \rangle$.

($\mathscr{B}_1$ denotes the functions of Baire class 1)

Proof. Let there exist $G \in \mathcal{O}$ and a real number d such that $G \cap \{x; f(x) = d\} = \emptyset$, but none of the sets $A = G \cap \{x; f(x) > d\}$, $B = G \cap \{x; f(x) < d\}$ are empty. The boundary $b(A)$ of A is non-empty, because G is connected. If $b(A) \cap G = \emptyset$, then $b(A) \subset \bar{G} - G$, then either $A = G$ or $B = G$, which contradicts the assumption.

Let $b(A)$ contain an isolated point z. Then there exists $G_z \in \mathcal{O}$ such that $G_z \cap b(A) = \{z\}$. Therefore $G_z - \{z\}$ is connected and $G_z - \{z\} \subset B$ or $G_z - \{z\} \subset A$, which is in contradiction with the property that $f(z) \subset \bigcap_\alpha (I_\alpha(z))_0$, $\alpha \in \langle 0, 2\pi)$.

Hence $b(A) \cap G$ is non-empty and dense in itself. We will prove that $A \cap b(A)$ is dense in $G \cap b(A)$. Let $A \cap b(A)$ be not dense in $G \cap b(A)$. Then there exists an open sphere $H \subset G$ with the centre $z_0 \in B \cap b(A)$ such that $H \cap A \cap b(A) = \emptyset$. Let z_1 be a point of $A \cap H$ such that $\varrho(z_1, H^c) > 2\varrho(z_1, z_0)$. ($H^c$ denotes the complement of H.) Let $H_1 \in \mathcal{O}$ such that $z_1 \in H_1 \subset A \cap H$ and $\varrho(H_1, H^c) > \varrho(H_1, z_0)$. Let $z_2 \in B$ such that $\varrho(z_2, z_1) = \varrho(z_1, B)$ then $z_2 \in H$. For α equal to the direction of the line $z_1 z_2$ there is r such that $S^0_{\alpha z_2 r} \subset A$, which is contradictory to our assumption that $f(z_2) \in \bigcap_\alpha (I_\alpha(z_2))_0$, $\alpha \in \langle 0, 2\pi)$. Similarly $B \cap b(A)$ is dense in $G \cap b(A)$.

The function f is not continuous in the points of $b(A) \cap G$, which is in contradiction with the fact that $f \in \mathscr{B}_1$.

Theorem 8. $\mathscr{D}(\mathcal{O})\mathscr{B}_1 = \mathscr{U}(\mathcal{O})\mathscr{B}_1 = \mathscr{U}_0(\mathcal{O})\mathscr{B}_1$.

Proof. According to Lemma 4, it is sufficient to prove that if $f \in \mathscr{U}_0(\mathcal{O})\mathscr{B}_1$, then $f(x) \in \bigcap_\alpha (I_\alpha(x))_0$, $\alpha \in \langle 0, 2\pi)$, for every $x \in \mathscr{X}$. Let there exist x_0 such that $f(x_0) \notin \bigcap_\alpha (I_\alpha(x))_0$, $\alpha \in \langle 0, 2\pi)$. It follows that there exist α_0 such that $f(x_0) \notin (I_{\alpha_0}(x_0))_0$, Then $I_{\alpha_0}(x_0) \subset \overline{f(S_{\alpha_0 x_0 r})}$ cannot hold.

A similar characterization as is given in [8] for $\mathscr{U}\mathscr{B}_\alpha$ can be proved for functions of $\mathscr{U}(\mathcal{O})\mathscr{B}_\alpha$. ($\mathscr{B}_\alpha$ denotes the functions of Baire class α.)

Lemma 5. *Let there be given* $f \in \mathscr{U}(\mathcal{O})\mathscr{B}_\alpha$, $g \in \mathscr{B}_\beta$ *and* $\varepsilon > 0$ *such that* $\|f - g\| \leq \varepsilon$, *then there exists* $h \in \mathscr{D}(\mathcal{O})\mathscr{B}_{\max(\beta, 2)}$ *such that* $\|f - h\| \leq 2\varepsilon$.

Proof. We apply the method of the proof of Lemma 7 of [8], where J_n are the open spheres with rational radii centres of which have rational coordinates. The existence of nowhere dense perfect subsets P_i^k follows from Alexandroff's-Hausdorff's theorem [5, p. 355]. For details see [8].

Theorem 9. *A function f belongs to $\mathcal{U}(\mathcal{O})$ if and only if f is a uniform limit of a sequence of $\mathcal{D}(\mathcal{O})$ functions. Moreover if f is in Baire class α then the approximating function can be taken from Baire class α.*

Proof. i) If f is an arbitrary function then the necessity is proved by applying Theorems 6 and 7. The proof of sufficiency is similar to that for the case of one variable [2, Theorem 4.3].

(ii) If $f \in \mathcal{B}_\alpha$ then for $\alpha = 0$ the assertion is trivial. For $\alpha = 1$ it is a consequence of Theorem 8. For $\alpha \geqslant 2$ the theorem is a consequence of Lemma 5.

REFERENCES

[1] BRUCKNER, A. M.—BRUCKNER, J. B.: Darboux transformations. Trans. Amer. math. Soc., 128, 1967, 103—111.

[2] BRUCKNER, A. M.—CEDER, J. G.—WEISS, M.: Uniform limits of Darboux functions. Coll. Math., 15, 1966, 65—77.

[3] CSÁSZÁR, A.: Sur la propriéte de Darboux. C. R. Premiér Congr. des Math. Hong. Budapest 1952, 551—560.

[4] HALPERIN, I.: On the Darboux property. Pacific J. Math., 5, 1955, 703—705.

[5] KURATOWSKI, K.: Topologie I. PWN, Warszawa 1958.

[6] MIŠÍK, L.: Über die Funktionen der ersten Baireschen Klasse mit Eigenschaft von Darboux, Mat.-fyz. Čas. 1964, 44-49.

[7] MIŠÍK, L.: Über die Eigenschaft von Darboux und einiger Klassen von Funktionen. Rev. Roum. Math. pures et appl., 11, 1966, 411—430.

[8] PREISS, D.: Limits of derivatives and Darboux-Baire functions, Rev. Roum. Math. pures et appl., 14, 1969, 1201—1206.

[9] RADAKOVIĆ, T.: Über Darbouxsche und stetige Funktionen, Monatshefte Math. Phys., 38, 1931, 117—122.

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СВОЙСТВО ДАРБУ ДЛЯ ФУНКЦИИ НЕСКОЛЬКИХ ПЕРЕМЕННЫХ

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Резюме

В статье определяются классы функции $\mathcal{N}\mathcal{D}(\mathcal{O})$, $\mathcal{U}(\mathcal{O})$ и $\mathcal{U}_0(\mathcal{O})$, которые являются обобщением классов функции $\mathcal{N}\mathcal{D}$ [4], $\mathcal{U}$ и $\mathcal{U}_0$ [2], для функции двух переменных. Исследуются их локальные свойства и обобщаются результаты работ [2] и [8] касающиеся равномерной сходимости функции Дарбу.