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THE MULTIPLICITY CRITERIA FOR ZERO POINTS OF SECOND ORDER DIFFERENTIAL EQUATIONS

ONDŘEJ DOŠLÝ

ABSTRACT. Sufficient conditions on the function $p(x)$ are given which guarantee the existence of a nontrivial solution of the equation $y'' + p(x)y = 0$ having at least $(n+1)$ zeros, $n \ge 1$, on a given interval.

1. Introduction

The aim of the present paper is to investigate the oscillation behaviour of the second order differential equation

$$\big(r(x)y'\big)' + p(x)y = 0, \tag{1.1}$$

where $r(x) \in C^1(I)$, $r(x) > 0$ on I, $p(x) \in C(I)$, $x \in I = (a,b)$, $-\infty \le a < b \le \infty$. Particularly, we give conditions on the functions r, p which guarantee the existence of a nontrivial solution of (1.1) having at least $(n+1)$ zeros on I.

Recall briefly the history of the problem. Hawking and Penrose [7] showed that the equation

$$y'' + p(x)y = 0 \tag{1.2}$$

is conjugate on $\mathbb{R} = (-\infty, \infty)$ (i.e., there exists a nontrivial solution of (1.2) having at least 2 zeros – the so-called conjugate points – on $\mathbb{R}$) whenever $p(x) \ge 0$ and $p(x) \not\equiv 0$. This result was generalized by Tipler who proved that (1.2) is conjugate on $\mathbb{R}$ whenever

$$\liminf_{t_1 \downarrow -\infty,\ t_2 \uparrow \infty} \int_{t_1}^{t_2} p(x)\,\mathrm{d}x > 0\,. \tag{1.3}$$

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Concerning equation (1.1), M ü l l e r - P f e i f f e r [11] proved that this equation is conjugate on $I = (a, b)$ if

$$\int_a r^{-1}(x)\,\mathrm{d}x = \infty = \int^b r^{-1}(x)\,\mathrm{d}x\,, \tag{1.4}$$

$$\liminf_{t_1 \downarrow a,\ t_2 \uparrow b} \int_{t_1}^{t_2} p(x)\,\mathrm{d}x \geq 0 \tag{1.5}$$

and $p(x) \not\equiv 0$ on I. The results of [7] and [12] are based on the Riccati technique, the criterion of Müller-Pfeiffer is proved via the variational principle.

Recently the author proved (using a combination of the Riccati technique and the transformation method) that (1.2) is conjugate on I if there exist $\varepsilon_1, \varepsilon_2 > 0$, $c \in I$ such that

$$\varepsilon_1 \int_c^b \exp\left\{ \int_c^x \left[\int_c^t p(s)\,\mathrm{d}s - \varepsilon_1 \right] \mathrm{d}t \right\} \mathrm{d}x > \pi/2\,, \tag{1.6$_1$}$$

$$\varepsilon_2 \int_a^c \exp\left\{ \int_c^x \left[\int_c^t p(s)\,\mathrm{d}s + \varepsilon_2 \right] \mathrm{d}t \right\} \mathrm{d}x > \pi/2\,. \tag{1.6$_2$}$$

These conditions were proved to be less restrictive than (1.3).

Note that the transformation of the independent variable $t = t(x) = \int^x r^{-1}(s)\,\mathrm{d}s$ transforms (1.1) into an equation of the form (1.2). Hence, in the sequel, we consider only the second order equations in this form.

The paper is designed as follows. In the next section we introduce the criterion which guarantees the existence of a nontrivial solution of (1.2) having at least $(n+1)$ zeros on $I = \mathbb{R}$. We also discuss the situation when the interval I is bounded or one-side bounded. In Section 3 we present an improved version of the conjugacy criterion given by (1.6), which is shown to be more general than another conjugacy criterion of T i p l e r given in [12]. The last section is devoted to remarks concerning the extension of the results of Sections 2, 3 to partial differential equations and self-adjoint equations of higher orders.

2. The existence of $(n+1)$ zero points

Consider the equation (1.2) as a perturbation of the equation

$$y'' = 0\,. \tag{2.1}$$

If $a = -\infty$, $b = \infty$, the latter equation is 1-special according to the terminology introduced by B o r ů v k a [1], i.e., this equation is disconjugate on I and there exists a unique (up to a multiple by a nonzero real constant) solution, namely $y_0 = 1$, which does not vanish on I. An equivalent formulation of this fact is that the principal solutions of (2.1) at $-\infty$ and ∞ are identical (recall that a solution y_b of a second-order equation is said to be principal at a point b if $\lim_{x \to b} y_b(x)/y(x) = 0$ for any solution $y(x)$ linearly independent of $y_b(x)$ and $y_b(x) \neq 0$ in a neighbourhood of b).

As condition (1.3) shows, equation (2.1) is in a certain sense oscillatory unstable. A small perturbation of this equation by a term $p(x)y$ with $p(x) \not\equiv 0$ and $p(x)$ having essentially a non-negative mean value on $\mathbb{R}$ makes equation (1.2) conjugate on $\mathbb{R}$. This fact is not too surprising if we observe that $y_0(x) = 1$ is the eigenfunction corresponding to the least eigenvalue $\lambda_0 = 0$ of the minimal closed symmetric operator l_0 generated by the differential expression

$$l(y) = (1 + x^2)^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} y, \qquad x \in \mathbb{R}, \tag{2.2}$$

(see [13] for necessary terminology). This idea also suggests the method which we use in looking for conditions on $p(x)$ implying the existence of a nontrivial solution of (1.2) with at least $(n + 1)$ zeros.

The main result of this section is based on the following variational lemma.

LEMMA 1. *Let $x_0, x_1 \in \mathbb{R}$, $x_0 < x_1$ and let $y_1, \dots, y_n \in W^{1,2}(x_0, x_1)$ be linearly independent functions (as the members of the Hilbert space $W^{1,2}(x_0, x_1)$) such that $y_k(x_0) = y_k(x_1) = 0$ and y_k has exactly $(k - 1)$ zeros on (x_0, x_1), $k = 1, \dots, n$. If*

$$I(y_k;\ x_0,\ x_1) = \int_{x_0}^{x_1} \left[{y_k'}^2(x) - p(x) y_k^2(x) \right] \mathrm{d}x < 0, \tag{2.3}$$

$k = 1, \dots, n$, then there exists a nontrivial solution of (1.2) having at least $(n + 1)$ zeros on $[x_0, x_1]$.

P r o o f. By the Courant variational principle (see, e.g., [2]) (2.3) implies that the boundary value problem $y'' + p(x)y = 0$, $y(x_0) = y(x_1) = 0$ has at least n negative eigenvalues. The eigenfunction corresponding to the eigenvalue $\lambda_n < 0$ (the eigenvalues are ordered by size, $\lambda_1 < \lambda_2 < \dots < \lambda_n$) has $(n + 1)$ zeros on $[x_0, x_1]$. Now, if the interval $[x_0, x_1]$ is shrinking, the eigenvalues increase and after some time $\lambda_n = 0$. The standard regularity argument implies that the corresponding eigenfunction y_n is of the class C^2, i.e., we have the solution of (1.2) having $(n + 1)$ zeros on $[x_0, x_1]$.

THEOREM 1. *Suppose* $p(x) - (k^2-1)(1+x^2)^{-2} \not\equiv 0$ *on* $\mathbb{R}$ *and*

$$\liminf_{t_1\downarrow-\infty,\ t_2\uparrow\infty} \int_{t_1}^{t_2} \left[p(x) - (k^2-1)(1+x^2)^{-2}\right](1+x^2)\sin^2 k\left(\operatorname{arctg} x + \frac{\pi}{2}\right) \mathrm{d}x \geq 0$$

$$k = 1,\dots,n. \tag{2.4}$$

Then (1.2) *possesses a nontrivial solution having at least* $(n+1)$ *zeros on* $\mathbb{R}$.

Proof. Denote $y_k = (1+x^2)^{1/2}\sin k\left(\operatorname{arctg} x + \frac{\pi}{2}\right)$, $f_k = (k^2-1)(1+x^2)^{-2}$, $p_k = p - f_k$. Since $p_k \not\equiv 0$ and (2.4) holds, there exists $t_k \in \mathbb{R}$, $d_k > 0$ and $\varrho_k > 0$ such that $p_k(x) > d_k$ for $x \in (t_k - \varrho_k, t_k + \varrho_k)$. According to (2.4) there exist T_k, S_k such that $\int_{t_1}^{t_2} p_k(x) y_k^2(x)\,\mathrm{d}x > -\varepsilon$, $\varepsilon > 0$ sufficiently small, whenever $t_1 < T_k$, $t_2 > S_k$. Choose $x_1 \leq \min_k\{T_k\}$, $x_2 \geq \max_k\{S_k\}$ such that all zero points of the functions y_k and all ϱ_k-neighbourhoods of t_k are in the interior of the interval (x_1, x_2). Moreover, the numbers t_k and ϱ_k can be chosen in such a way that the interval $[t_k - \varrho_k, t_k - \varrho_k]$ does not contain zero points of the function $y_k(x)$. Define the function $h_k(x)$ and the test function $u_k(x)$ for (2.3) as follows:

$$h_k(x) = \begin{cases} \delta_k(1 - \varrho_k^{-1}|x - t_k|) & \text{if } |x - t_k| \leq \varrho_k \\ 0 & \text{if } |x - t_k| > \varrho_k, \end{cases}$$

the precise value of the constant δ_k will be determined latter,

$$u_k(x) = \begin{cases} 0, & x \leq x_0, \\ y_k(x)\int_{x_0}^{x} y_k^{-2}(s)\,\mathrm{d}s\left(\int_{x_0}^{x_1} y_k^{-2}(s)\,\mathrm{d}s\right)^{-1}, & x \in [x_0, x_1], \\ y_k(x)(1 + h_k(x)\cdot\operatorname{sgn} y_k(x)), & x \in [x_1, x_2], \\ y_k(x)\int_{x}^{x_3} y_k^{-2}(s)\,\mathrm{d}s\left(\int_{x_2}^{x_3} y_k^{-2}(s)\,\mathrm{d}s\right)^{-1}, & x \in [x_2, x_3], \\ 0, & x \geq x_3, \end{cases}$$

the values x_0, x_3 will be also determined later. Obviously $u_k \in \mathring{W}^{1,2}[x_0, x_3]$ ($\mathring{W}^{1,2}[x_0, x_3] = \{u \in W^{1,2}(x_0, x_3),\ u(x_0) = 0 = u(x_3)\}$). To simplify formally the computations which follow, we write y, u, f, $\tilde{p}$, t, ϱ, h, d, δ instead of y_k, u_k, f_k, p_k, t_k, ϱ_k, h_k, d_k, δ_k respectively.

Using the fact that y and u are solutions of

$$y'' + f(x)y = 0 \tag{2.5}$$

on $(x_0, t-\varrho) \cup (t+\varrho, x_3)$ and $y \neq 0$ on $(x_0, x_1) \cup (x_2, x_3)$, we have

$$\int_{x_0}^{x_1} (u'^2 - fu^2)$$

$$= u'u\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} u(u'' + fu) = u'(x_1)u(x_1) = y'(x_1)y(x_1) + \left(\int_{x_0}^{x_1} y^{-2}(s)\,\mathrm{d}s\right)^{-1}.$$

Similarly,

$$\int_{x_2}^{x_3} (u'^2 - fu^2) = -y'(x_2)y(x_2) + \left(\int_{x_2}^{x_3} y^{-2}(s)\,\mathrm{d}s\right)^{-1}.$$

Since y is the principal solution of (2.5) both at ∞ and $-\infty$ (this may be verified by a direct computation), we have

$$\lim_{x_0 \downarrow -\infty} \left(\int_{x_0}^{x_1} y^{-2}(s)\,\mathrm{d}s\right)^{-1} = 0 = \lim_{x_3 \uparrow \infty} \left(\int_{x_2}^{x_3} y^{-2}(s)\,\mathrm{d}s\right)^{-1}.$$

Further,

$$\int_{x_1}^{x_2} (u'^2 - fu^2)$$

$$= y'y\Big|_{x_1}^{t-\varrho} - \int_{x_1}^{t-\varrho} y(y'' + fy) + y'y\Big|_{t-\varrho}^{t+\varrho} - \int_{t-\varrho}^{t+\varrho} u(u'' + fu) + y'y\Big|_{t+\varrho}^{x_2} - \int_{t+\varrho}^{x_2} y(y'' + fy)$$

$$= y'y\Big|_{x_1}^{x_2} + \int_{t-\varrho}^{t+\varrho} y^2\left[(1 + h \cdot \operatorname{sgn} y)'\right]^2 = y'y\Big|_{x_1}^{x_2} + \int_{t-\varrho}^{t+\varrho} y^2 h'^2.$$

Computing $\int_{x_0}^{x_1} \tilde{p}u^2$ and $\int_{x_2}^{x_3} \tilde{p}u^2$ observe that the function u/y is monotonic on (x_0, x_1) and (x_2, x_3). Hence, using the second mean value theorem of integral calculus, we have

$$\int_{x_0}^{x_1} \tilde{p}u^2 = \int_{x_0}^{x_1} \tilde{p}y^2(u/y)^2 = \int_{\xi_1}^{x_1} \tilde{p}y^2\,, \qquad \xi_1 \in (x_0, x_1)\,,$$

$$\int_{x_2}^{x_3} \tilde{p}u^2 = \int_{x_2}^{\xi_2} \tilde{p}y^2\,, \qquad \xi_2 \in (x_2, x_3)\,.$$

Using the previous computations we get

$$\int_{x_0}^{x_3} (u'^2 - pu^2) = \int_{x_0}^{x_3} (u'^2 - fu^2) - \int_{x_0}^{x_3} \tilde{p}u^2$$

$$= \Big(\int_{x_0}^{x_1} y^{-2}\Big)^{-1} + \Big(\int_{x_2}^{x_3} y^{-2}\Big)^{-1} - \int_{\xi_1}^{\xi_2} \tilde{p}y^2 + \int_{t-\varrho}^{t+\varrho} y^2h'^2 - 2\int_{t-\varrho}^{t+\varrho} \tilde{p}yh\cdot\operatorname{sgn} y - \int_{t-\varrho}^{t+\varrho} \tilde{p}y^2h^2\,.$$

The last term is negative since $\tilde{p} > 0$ and if x_0 and x_3 are sufficiently close to $-\infty$ and ∞, the sum of first two terms is less than ε. Moreover, since $\xi_1 < \min_k\{T_k\}$, $\xi_2 > \max_k\{S_k\}$, we have $-\int_{\xi_1}^{\xi_2} \tilde{p}y^2 \leq \varepsilon$. Consequently,

$$\int_{x_0}^{x_3} (u'^2 - pu^2) \leq 2\varepsilon + \int_{t-\varrho}^{t+\varrho} y^2h'^2 - 2\int_{t-\varrho}^{t+\varrho} \tilde{p}yh\cdot\operatorname{sgn} y\,.$$

Denote $y_0 = \max\{|y(x)|,\ x \in [t-\varrho, t+\varrho]\}$, $y_1 = \min\{|y(x)|,\ x \in [t-\varrho, t+\varrho]\}$. Then $\int_{t-\varrho}^{t+\varrho} y^2h'^2 - 2\int_{t-\varrho}^{t+\varrho} \tilde{p}yh\cdot\operatorname{sgn} y \leq y_0\delta^2\varrho^{-2} - 2dy_1\varrho\delta$ since by our assumption y does not change its sign on $[t-\varrho, t+\varrho]$ and $yh\cdot\operatorname{sgn} y \geq 0$ on this interval. By setting $\delta = \varepsilon^{1/2}$ we get $\int_{x_0}^{x_3} (u'^2 - pu^2) \leq \varepsilon^{1/2}\big((2 + y_0^2\varrho^{-2})\varepsilon^{1/2} - 2dy_1\varrho\big)$, hence $I(u\,;\ x_0, x_3) < 0$ if ε is sufficiently small. It is clear that the same values x_0, x_3 can be chosen for each $k \in \{1,\ldots,n\}$, hence by Lemma 1 the equation (1.2) possesses a nontrivial solution having at least $(n+1)$ zeros on $[x_0, x_3] \subset \mathbb{R}$. The proof is complete.

In the theorem we have just proved we suppose that $I = (a,b) = (-\infty,\infty)$. If both a and b are finite, to find a condition which guarantees the existence of a nontrivial solution of (1.2) with at least $n+1$ zeros is simple since in this case one can directly compute the eigenvalues and eigenfunctions of the operator $-y''$ on $W^{1,2}[a,b]$. Particularly, $\lambda_k = (k\pi)^2/(b-a)^2$, $y_k = \sin \lambda_k^{1/2}(x-a)$. If one of a, b, is infinite, say b, we need to compute the eigenfunctions and eigenvalues of the minimal closed symmetric extension of (2.2) on (a,∞). To this end we use the transformation $y(x) = (1+x^2)^{1/2}u(t)$, $t = \operatorname{arctg} x$, which transforms (2.2), considered on $[a,\infty)$, into the operator $\tilde{l}(u) = \ddot{u} + u$ ($\dot{}$ $= \mathrm{d}/\mathrm{d}t$), considered on $[\operatorname{arctg} a,\ \pi/2)$. Now, it suffices to compute the eigenvalues and eigenfunctions of $\tilde{l}(u)$ on $W^{1,2}[\operatorname{arctg} a,\ \pi/2]$ and to transform them "back" into (2.2) on $[a,\infty)$. As a result we get the following reformulation of Theorem 1.

THEOREM 1'. *Suppose that* $p(x) - \lambda_k(1+x^2)^{-1} \not\equiv 0$ *on* $I = (a,\infty)$ *and*

$$\liminf_{t_2 \uparrow \infty} \int_a^{t_2} (p(x) - \lambda_k(1+x^2)^{-1}) y_k(x)\,\mathrm{d}x \geq 0, \qquad k = 1,\dots,n,$$

where $\lambda_k = 4\pi^2k^2(\pi - 2\operatorname{arctg} a)^{-2} - 1$, $y_k = (1+x^2)^{1/2} \sin \dfrac{2k\pi}{\pi - 2\operatorname{arctg} a} \cdot (\operatorname{arctg} x - \operatorname{arctg} a)$. *Then* (1.2) *has a nontrivial solution with at least* $(n+1)$ *zeros on* (a,∞).

3. The existence of two zero points

In this section we present an improved version of the conjugacy criterion given by (1.6). In the form given here it offers a unified approach to the investigation of the conjugacy of (1.2) on an arbitrary interval.

THEOREM 2. *Suppose that there exist* $\varepsilon_1, \varepsilon_2 > 0$ *and* $c \in (a,b)$ *such that*

$$\begin{aligned} \varepsilon_1 \int_c^b \exp\left\{ 2\int_c^x \left[\int_c^t p(s)\,\mathrm{d}s - \varepsilon_1 \right] \mathrm{d}t \right\} \mathrm{d}x &> A \\ \varepsilon_2 \int_a^c \exp\left\{ 2\int_c^x \left[\int_c^t p(s)\,\mathrm{d}s + \varepsilon_2 \right] \mathrm{d}t \right\} \mathrm{d}x &> B. \end{aligned} \tag{3.1$_{1,2}$}$$

If

$$\varepsilon_1 + \varepsilon_2 - \pi(\varepsilon_1 B + \varepsilon_2 A)/2AB \geq 0, \tag{3.2}$$

then (1.2) *is conjugate on* (a,b).

P r o o f. Denote $c_R = \varepsilon_1(1 - \pi/2A)$, $c_L = \varepsilon_2(\pi/2B - 1)$. Let y_1, z_1 be the solutions of (1.2) given by the initial condition $y_1(c) = 1 = z_1(c)$, $y_1'(c) = c_R$,

$z_1'(c) = \varepsilon_1$. Suppose that the function $y_1(x)$ does not vanish on (c,b). Then the function $\alpha(x) = \operatorname{arctg}(z_1/y_1)$ is well defined on (c,b), $\alpha(c) = \pi/4$, $\alpha(b-) \le \pi/2$ and $\alpha' = \varepsilon_1\pi/2A(y_1^2 + z_1^2)$. For $x \in (c,b)$ we have $z_1(x) > y_1(x) > 0$ and the function $w = z_1'/z_1$ is a solution of the Riccati equation $w' + w^2 + p(x) = 0$, i.e., $w(x) = w(c) - \int_c^x p(s)\,\mathrm{d}s - \int_c^x w^2(s)\,\mathrm{d}s$ and hence

$$z_1(x) = \exp\left\{\int_c^x\left(\varepsilon_1 - \int_c^t (p+w^2)\right)\mathrm{d}t\right\} \le \exp\left\{\int_c^x\left(\varepsilon_1 - \int_c^t p\right)\mathrm{d}t\right\} \qquad \text{for } x > c.$$

It follows

$$\begin{aligned}\alpha(b-) &= \alpha(c) + \int_c^b \alpha' \\ &= \pi/4 + \varepsilon_1(\pi/2A)\int_c^b (y_1^2 + z_1^2)^{-1}\,\mathrm{d}x \ge \pi/4 + \varepsilon_1(\pi/2A)\int_c^b (2z_1^2)^{-1}\,\mathrm{d}x \\ &> \pi/4 + \varepsilon_1(\pi/4A)\int_c^b \exp\left\{2\int_c^x\left[\int_c^t p - \varepsilon_1\right]\mathrm{d}t\right\}\mathrm{d}x \ge \pi/4 + \pi/4 = \pi/2,\end{aligned}$$

a contradiction. Let y_2, z_2 be the solutions of (1.2) given by the initial condition $y_2(c) = 1 = z_2(c)$, $y_2'(c) = c_L$, $z_2'(c) = -\varepsilon_2$. Similarly as above we get a contradiction supposing that $y_2(x) > 0$ on (a,c). Now, let $y(x)$ be a solution of (1.2) given by the initial condition $y(c) = 1$, $y'(c) = \gamma \in [c_L, c_R]$. According to the Sturm separation theorem for zeros of linearly independent solutions of (1.2), $y(x)$ has zero points in both intervals (a,c) and (c,b). This completes the proof.

Let us relate the result of Theorem 2 to some recent ones. First, if $A = (\pi/2) = B$, it is easy to see that (3.1) is more general than (1.6). Tipler proved in [12] that equation (1.2) is conjugate on an interval $[x_0,\infty)$ provided $p(x) \ge 0$ on $[x_0,\infty)$ and there exist $x_1, x_2 \in (x_0,\infty)$, $x_1 < x_2$ such that $\int_{x_1}^{x_2} p(x)\,\mathrm{d}x > (x_1 - x_0)^{-1}$. Next we show that this criterion can be obtained as a corollary of Theorem 2. Let $a = x_0$, $b = \infty$ and $c > x_2$ be arbitrary for a moment. Since $p(x) \ge 0$ on $[x_0,\infty)$, by the Sturm comparison theorem the statement will be proved to be a corollary of Theorem 2 if we prove it for a function $\tilde{p}(x)$ such that $\tilde{p}(x) = p(x)$ for $x \in [x_0, x_2]$ and $\tilde{p}(x) \equiv 0$ for $x \geq x_2$. Let $\varepsilon_1 > 0$ be arbitrary, we have

$$\varepsilon_1 \int_c^\infty \exp\left\{2\int_c^x \left[\int_c^t \tilde{p} - \varepsilon_1\right] \mathrm{d}t\right\} \mathrm{d}x \geq \varepsilon_1 \int_c^\infty \exp\{-2\varepsilon_1(x-c)\} \,\mathrm{d}x = 1/2\,.$$

For any $\varepsilon_2 > 0$ we have

$$\varepsilon_2 \int_{x_0}^c \exp\left\{2\int_c^x \left[\int_c^t \tilde{p} + \varepsilon_2\right] \mathrm{d}t\right\} \mathrm{d}x \geq \varepsilon_2 \int_{x_0}^c \exp\left\{2(x-c)\left[\int_{x_1}^{x_2} p - \varepsilon_2\right]\right\} \mathrm{d}x$$

$$= \varepsilon_2 \left(\int_{x_1}^{x_2} p - \varepsilon_2\right)^{-1} \left[\exp\left\{2\left[\int_{x_1}^{x_2} p - \varepsilon_2\right](c - x_0)\right\} - 1\right].$$

The inequality (3.2) can be written in the form $\varepsilon_1(1-\pi/2A)+\varepsilon_2(1-\pi/2B) \geq 0$. Since $A \geq 1/2$ independently of ε_1, letting $\varepsilon_1 \to 0$ it suffices to show that the term $\varepsilon_2(1-\pi/2B)$ is positive for some $\varepsilon_2 > 0$. But this is the case if $\varepsilon_2 = (x_1 - x_0)^{-1}$ and $c \to \infty$, since then $B \to \infty$ and thus $(1 - \pi/2B)$ is positive.

Remarks

i) Consider the partial differential equation

$$\Delta_n u + q(x)u = 0\,, \tag{4.1}$$

where $\Delta_n = \sum_{i=1}^n \partial^2/\partial x_i^2$ is the n-dimensional Laplace operator, $x = (x_1, \ldots, x_n)$ and $q(x)$ is a locally Lipschitz continuous real-valued function defined on $\mathbb{R}^n$. A bounded domain $G \subset \mathbb{R}^n$ is said to be a *nodal domain* of (4.1) if there exists a nontrivial solution u of (4.1) for which $u \in \overset{\circ}{W}{}^{1,2}(G) \cap W^{2,2}(G)$ (the Sobolev spaces $W^{1,2}(G)$, $W^{2,2}(G)$ are defined similarly as in the one-dimensional case). It is known (see, e.g, [10]) that a ball $B_R = \left\{x \in \mathbb{R}^n,\ |x| = \left(\sum x_i^2\right)^{1/2} \leq R\right\}$ contains a nodal domain of (4.1) if there exists a nontrivial function $v \in \overset{\circ}{W}{}^{1,2}(B_R)$ such that

$$\int_{B_R} \left(|\nabla v(x)|^2 - q(x)v^2(x)\right) \mathrm{d}x \leq 0\,. \tag{4.2}$$

Denote $\tilde{q}(r) = \int\limits_{|x|=r} q(x)\,\mathrm{d}\omega$, where ω is the surface of the sphere $S_r^{n-1} = \{x \in \mathbb{R}^n,\ |x| = r\}$ in $\mathbb{R}^n$. If the function $v = v(r)$ is radially symmetric, we have $\int\limits_{B_R} \left(|\nabla v|^2 + qv^2\right) \mathrm{d}x = \int\limits_0^R \left(r^{n-1}\left(\frac{\mathrm{d}}{\mathrm{d}r}v\right)^2 + \tilde{q}(r)v^2\right) \mathrm{d}r$. Using this substitution one can apply the results concerning the zero points of solutions of ordinary differential equations while investigating the problem of the existence of a nodal domain and negative eigenvalues of the Schrödinger partial differential operators. We hope to follow this idea in more detail elsewhere.

ii) Let us try to extend the method introduced in Section 2 to the higher order self-adjoint linear differential equation

$$(-1)^n \left(r(x)y^{(n)}\right)^{(n)} + p(x)y = 0\,, \tag{4.3}$$

where $r \in C^n$, $r > 0$, $p \in C$, $x \in I = (a, b)$, $-\infty \le a < b \le \infty$. For the sake of simplicity consider, as a model, the fourth order equation

$$y^{(iv)} + p(x)y = 0\,. \tag{4.4}$$

It was proved in [9] that this equation possesses a nontrivial solution having at least two conjugate points on $\mathbb{R}$ (i.e., the points $x_1, x_2 \in \mathbb{R}$ such that $y^{(i)}(x_1) = 0 = y^{(i)}(x_2)$, $i = 0, 1$, for some nontrivial solution y of (4.4)), whenever there exist c_1, $c_2 \in \mathbb{R}$ such that

$$\limsup_{t_1 \downarrow -\infty,\ t_2 \uparrow \infty} \int_{t_1}^{t_2} p(x)(c_1 x + c_2)^2\,\mathrm{d}x < 0\,. \tag{4.5}$$

Now, find a condition on the function $p(x)$ which would guarantee the existence of a nontrivial solution of (4.4) with 3 or more conjugate points on $\mathbb{R}$. In the case of the second-order equation we proceeded as follows: The transformation $y = (1+x^2)^{1/2}u$ transformed (1.2) into the equation $\left((1+x^2)u'\right)' + (1+x^2)^{-1}u = 0$. Then we took the equation $\left(k^{-1}(1+x^2)u'\right)' + k(1+x^2)^{-1}u = 0$, $k = 2, 3, \dots, n$, and transforming it "back" we got the equation $y'' + (k^2-1)(1+x^2)^{-2}y = 0$ which we used in Theorem 1.

Let us try to modify this method to be applicable to the equation (4.3). This equation can be written in the form of the linear Hamiltonian system

$$\boldsymbol{u}' = \mathbf{A}\boldsymbol{u} + \mathbf{B}(x)\boldsymbol{v}, \qquad \boldsymbol{v}' = \mathbf{C}(x)\boldsymbol{u} - \mathbf{A}^{\top}\boldsymbol{v}\,, \tag{4.6}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ are $n \times n$ matrices with entries $\mathbf{A} = \mathbf{A}_{i,j} = 1$ if $j = i+1$, $i = 1, \dots, n-1$, all the other entries of $\mathbf{A}$ equal 0, $\mathbf{B}(x) = \operatorname{diag}\{0, \dots, 0, r^{-1}(x)\}$, $\mathbf{C}(x) = \operatorname{diag}\{p(x), 0, \dots, 0\}$, $\boldsymbol{u} = \operatorname{col}(y, \dots, y^{(n-1)})$, $\boldsymbol{v} = \operatorname{col}((-1)^{n-1} \cdot (ry^{(n)})^{(n-1)}, (-1)^{n-2}(ry^{(n)})^{(n-2)}, \dots, ry^{(n)})$.

It was shown in [4] that there exist $n \times n$ matrices $\mathbf{H}(x)$, $\mathbf{K}(x)$, $\mathbf{H}(x)$ being nonsingular, such that the transformation

$$\boldsymbol{u} = \mathbf{H}\boldsymbol{s}, \qquad \boldsymbol{v} = \mathbf{K}\boldsymbol{s} + (\mathbf{H}^{\mathsf{T}})^{-1}\mathbf{c} \tag{4.7}$$

transforms (4.6) into the so-called trigonometric system

$$\boldsymbol{s}' = \mathbf{Q}(x)\boldsymbol{c}, \qquad \boldsymbol{c}' = -\mathbf{Q}(x)\boldsymbol{s}, \tag{4.8}$$

where $\mathbf{Q} = \mathbf{H}^{-1}\mathbf{B}(\mathbf{H}^{\mathsf{T}})^{-1}$, $\mathbf{H}\mathbf{H}^{\mathsf{T}} = \mathbf{U}_1\mathbf{U}_1^{\mathsf{T}}, + \mathbf{U}_2\mathbf{U}_2^{\mathsf{T}}$, $(\mathbf{U}_1, \mathbf{V}_1)$, $(\mathbf{U}_2, \mathbf{V}_2)$ are the $2n \times n$ matrix solutions of (4.6) for which $\mathbf{U}_i^{\mathsf{T}}\mathbf{V}_i - \mathbf{V}_i^{\mathsf{T}}\mathbf{U}_i = \mathbf{O}$, $i = 1, 2$ and $\mathbf{U}_1^{\mathsf{T}}\mathbf{V}_2 - \mathbf{V}_1^{\mathsf{T}}\mathbf{U}_2 = \mathbf{I}$ – the identity matrix.

If system (4.6) corresponds to the equation $y^{(iv)} = 0$, i.e., $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \operatorname{diag}\{0, 1\}$, $\mathbf{C} = \mathbf{O}$, the matrices $\mathbf{H}$, $\mathbf{K}$, $\mathbf{Q}$ can be computed explicitly, see [8] and also [5] for a more detailed comment concerning the computation of these matrices. Particularly, we have

$$\mathbf{H}\mathbf{H}^{\mathsf{T}} = \begin{pmatrix} 1 + x^2 + x^4/4 + x^6/36 & x + x^3/2 + x^5/12 \\ x + x^3/2 + x^5/12 & 1 + x^2 + x^4/4 \end{pmatrix}.$$

One can verify directly that the transformation (4.7) with $k^{-1/2}\mathbf{H}$, $k^{-1/2}\mathbf{K}$, $k \in \mathbb{N}$, instead of $\mathbf{H}$, $\mathbf{K}$, transforms the trigonometric system (4.8), with $k\mathbf{Q}$ instead of $\mathbf{Q}$, into the Hamiltonian system

$$\boldsymbol{u}' = \mathbf{A}\boldsymbol{u} + \mathbf{B}\boldsymbol{v}, \qquad \boldsymbol{v}' = (1 - k^2)(\mathbf{H}\mathbf{H}^{\mathsf{T}})^{-1}\mathbf{B}(\mathbf{H}\mathbf{H}^{\mathsf{T}})^{-1}\boldsymbol{u} - \mathbf{A}^{\mathsf{T}}\boldsymbol{v}. \tag{4.9}$$

Denote $f(x) = (\mathbf{H}\mathbf{H}^{\mathsf{T}})_{12}$, $g(x) = (\mathbf{H}\mathbf{H}^{\mathsf{T}})_{22}$, then (4.9) is equivalent to the fourth order equation

$$y^{(iv)} + (k^2 - 1)\big[-(g^2y' + fgy) + fgy' + f^2y\big] = 0 \tag{4.10}$$

and the corresponding quadratic functional is of the form

$$\int_a^b \big(y''^2 + (k^2 - 1)(fy + gy')^2\big)\, dx.$$

The main difficulty which prevents us to extend directly the "second-order method" to fourth order equations is the fact that we do not know solutions of (4.10), i.e., we do not know the eigenfunctions corresponding to higher eigenvalues of the operator associated with the operator d^4/dx^4. For second order equations we computed the eigenfunctions of (2.2) corresponding to the eigenvalues $(k^2 - 1)$ via the transformation $y = k^{-1/2}(1 + x^2)^{1/2}u$ which transforms the equation $\left(k^{-1}(1 + x^2)u'\right)' + k(1 + x^2)^{-1}u = 0$ into the equation $l(y) = (k^2 - 1)y$ (the operator l is given by (2.2)). Here, even if we know the solution of the trigonometric system (4.8) corresponding to the equation $y^{(iv)} = 0$ (this trigonometric system is a higher order analogy of the equation $\left((1 + x^2)u'\right)' + (1 + x^2)^{-1}u = 0\right)$, we do not know the solution of this system with $k\mathbf{Q}$ instead of $\mathbf{Q}$, thus we cannot use this method in order to compute solutions of (4.10).

We may also formulate some further problems. For example, the linear Hamiltonian system corresponding to the equation $y^{(iv)} = 0$ is 0-general in the terminology introduced in [3] (i.e., the principal solutions of the corresponding linear Hamiltonian system at $-\infty$ and ∞ coincide). Since the transformation (4.7) transforms principal solutions into principal solutions, the associated trigonometric system (4.8) has the same property. Has the trigonometric system (4.8) with $k\mathbf{Q}$, $k = 2, 3, \dots$ also this property? It is not difficult to verify that in the second order case the answer is affirmative.

The solution of these problems would be very useful in extending the knowledge of the oscillation and the spectral properties of higher order self-adjoint differential equations.

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