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Dedicated to Professor Tibor Šalát
on the occasion of his 70th birthday

$*$-MEDIAN

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(*Communicated by Tibor Katriňák*)

ABSTRACT. The investigation of completely normal topological spaces gives a motive for the definition of the $*$-median operation on distributive p-algebras. Basic properties of this operation are described in the following paper.

Several authors described the role of the median operation in distributive lattices. Let us remember G. Birkhoff [1], M. Sholander [5] and M. Kolibiar [4]. The *median operation* on a distributive lattice L is defined (see [1]) in the following way:

$$(a,b,c) = (a \vee b) \wedge (a \vee c) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \qquad \text{for} \quad a,b,c \in L\,.$$

We can investigate a lot of properties of topological spaces with the help of open sets only and transform these properties into locales. Recall, that a *locale* L is a complete lattice in which the infinite distributive law

$$a \wedge VS = V\{a \wedge s : \ s \in S\}$$

holds for all $a \in L$, $S \subseteq L$.

For example, the normality of a topological space T is possible to define in the locale $O(T)$ of all open sets in T in the following way:

$$a,b \in O(T)\,, \ a \vee b = 1 \implies \exists\, \ell {\in} O(T) \quad a \vee \ell^* = 1 = b \vee \ell\,,$$

where $*$ denotes pseudocomplements in L.

If we transform this condition into locales, then we have the category of normal locales (see [3]). This category has not many natural properties because subspaces, factor spaces and products of topological spaces need not be normal.

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For these reasons, we have some modifications of topological spaces, namely, so called completely normal spaces (see [2]). The corresponding category of locales called completely normal locales is introduced in [6] in the following way:

A locale L is *completely normal* when for any $a, b \in L$ there exists $\ell \in L$ such that $a \leq b \vee \ell$, $b \leq a \vee \ell^*$.

The properties of completely normal locales are studied in [6]. Let us introduce the following proposition.

PROPOSITION 1. *Let L be a locale. Then the following assertions are equivalent:*

1. *L is a completely normal locale.*
2. *Sublocales of L are normal.*
3. *For any $a, b \in L$ there exists $\ell \in L$ such that*

$$a \vee b = (a \wedge b) \vee (a \wedge \ell) \vee (b \wedge \ell^*) .$$

Proof. See [6; Proposition 2]. □

Assertion 3 from Proposition 1 motivates us to investigate a ternary operation analogously to the median operation on a distributive p-algebra $(L, \vee, \wedge, O, 1, *)$, i.e., a distributive lattice $(L, \vee, \wedge)$ with 0, 1, and pseudocomplements denoted by $*$.

PROPOSITION 2.

1. *In every distributive lattice, the identity $(*)$ holds:*

$$(a \vee b) \wedge (a \vee d) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge d) . \qquad (*)$$

If L is a lattice and for arbitrary elements $a, b, c \in L$ there exists $d \in L$ such that $(*)$ holds, then L is distributive.

2. *A p-algebra $(L, \vee, \wedge, 0, 1, *)$ is distributive if and only if $(a \vee b) \wedge (a \vee c^*) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c^*)$ holds for any $a, b, c \in L$.*

Proof.

1. $\Longrightarrow$: $(a \vee b) \wedge (a \vee d) \wedge (b \vee c) = (a \vee b) \wedge \big[(a \wedge b) \vee (a \wedge c) \vee (d \wedge b) \vee (c \wedge d)\big] = (a \wedge b) \vee (a \wedge c) \vee (d \wedge b)$.

$\Longleftarrow$: First, let us prove that L is a modular lattice. If $a, b, c \in L$, $a \geq b$ and $d \in L$ is such that $(*)$ is satisfied, then $a \wedge (b \vee c) = (a \vee b) \wedge (a \vee d) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge d) = b \vee (a \wedge c)$. Now, $a \wedge (b \vee c) = a \wedge \big[(a \vee b) \wedge (a \vee d) \wedge (b \vee c)\big] = a \wedge \big\{[(a \wedge b) \vee (a \wedge c)] \vee (b \wedge d)\big\} = \big[(a \wedge b) \vee (a \wedge c)\big] \vee \big[a \wedge (b \wedge d)\big] = (a \wedge b) \vee (a \wedge c)$ holds for any $a, b, c \in L$, and thus L is distributive.

2. This is a direct consequence of 1. □

DEFINITION 3. Let L be a distributive p-algebra. Then the ternary operation on L defined by

$$[a,b,c] = (a \vee b) \wedge (a \vee c^*) \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c^*)$$

for $a, b, c \in L$ is called the *$*$-median on L*.

THEOREM 4.

1. *Let L be a set with 0, 1, and a ternary operation $[\cdot, \cdot, \cdot]$ with the properties:*

1° $\big[a, 0, [c,d,e]\big] = \big[c, [a,0,d], [a,0,e]\big]$,
2° $[a,a,b] = a$,
3° $[a,b,1] = a$,
4° $[0,1,0] = 1$.

Then L is a distributive p-algebra with regard to the operations $a \vee b = [1,a,b]$, $a \wedge b = [a,0,b]$.

2. *If L is a distributive p-algebra, then the $*$-median on L has properties $1^\circ - 4^\circ$.*

Proof.

1. We shall prove in the following parts:

a) Properties 1°, 2° and 3° imply

(i) $[a,0,1] = a$,
(ii) $[1,a,1] = 1$,
(iii) $[1,1,a] = 1$,
(iv) $[a,0,0] = \big[a, [0,0,1], [0,0,1]\big] = \big[0,0,[a,1,1]\big] = [0,0,a] = 0$,
(v) $[a,0,a] = \big[a,0,[a,1,1]\big] = \big[a,[a,0,1],[a,0,1]\big] = [a,a,a] = a$.

b) Now, we shall use only properties 1° (i)–(v) and prove that L is a distributive lattice:

We have $a \wedge (b \vee c) = \big[a,0,[1,b,c]\big] = \big[1,[a,0,b],[a,0,c]\big] = (a \wedge b) \vee (a \wedge c)$, $a \wedge b = [a,0,b] = \big[a,0,[b,0,1]\big] = \big[b,[a,0,0],[a,0,1]\big] = [b,0,a] = b \wedge a$.

Now, we shall prove the following formulas: $a \wedge (a \vee b) = \big[a,0,[1,a,b]\big] = \big[1,[a,0,a],[a,0,b]\big] = \big[1,a,[a,0,b]\big] = \big[1,[a,0,1],[a,0,b]\big] = \big[a,0,[1,1,b]\big] = [a,0,1] = a$, $a \wedge (b \vee a) = \big[a,0,[1,b,a]\big] = \big[1,[a,0,b],[a,0,a]\big] = \big[1,[a,0,b],a\big] = \big[1,[a,0,b],[a,0,1]\big] = \big[a,0,[1,b,1]\big] = [a,0,1] = a$ and together $a \vee b = \big\{a \wedge (b \vee a)\big\} \vee \big\{b \wedge (b \vee a)\big\} = \big\{(b \vee a) \wedge a\big\} \vee \big\{(b \vee a) \wedge b\big\} = (b \vee a) \wedge (a \vee b) = (a \vee b) \wedge (b \vee a) = \big\{(a \vee b) \wedge b\big\} \vee \big\{(a \vee b) \wedge a\big\} = \big\{b \wedge (a \vee b)\big\} \vee \big\{a \wedge (a \vee b)\big\} = b \vee a$. Finally, the introduced formula $a \wedge (a \vee b) = a$ together with $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (b \wedge a) \vee (c \wedge a) = (c \wedge a) \vee (b \wedge a)$ fulfils assumptions of the Sholander's theorem (see [1; p. 35, Theorem 10']) implying that $(L, \vee, \wedge)$ is a distributive lattice.

c) The fact that L is a p-algebra follows from 4°: Let us introduce $a^* = [0,1,a]$ for any $a \in L$ and prove that a^* is a pseudocomplement of a. Namely,

$a \wedge a^* = \bigl[a, 0, [0,1,a]\bigr] = \bigl[0, [a,0,1], [a,0,a]\bigr] = [0,a,a] = \bigl[0, [a,0,1], [a,0,1]\bigr] = \bigl[a, 0, [0,1,1]\bigr] = [a,0,0] = 0$. If $x \wedge a = 0$, i.e., $[x,0,a] = 0$, then $x \wedge a^* = \bigl[x, 0, [0,1,a]\bigr] = \bigl[0, [x,0,1], [x,0,a]\bigr] = [0,x,0] = \bigl[0, [x,0,1], [x,0,0]\bigr] = \bigl[x, 0, [0,1,0]\bigr] = [x,0,1] = x$.

2. We have $\bigl[a, 0, [c,d,e]\bigr] = a \wedge [c,d,e] = a \wedge (c \vee d) \wedge (c \vee e^*) \wedge (d \vee e)$, $\bigl[c, [a,0,d], [a,0,e]\bigr] = [c, a \wedge d, a \wedge e] = \bigl\{c \vee (a \wedge d)\bigr\} \wedge \bigl\{c \vee (a \wedge e)^*\bigr\} \wedge \bigl\{(a \wedge d) \vee (a \wedge e)\bigr\} = (c \vee a) \wedge (c \vee d) \wedge \bigl\{c \vee (a \wedge e)^*\bigr\} \wedge a \wedge (d \vee e) = a \wedge (c \vee d) \wedge (c \vee e^*) \wedge (d \vee e)$ since $a \wedge (c \vee e^*) = a \wedge \bigl\{c \vee (a \wedge e)^*\bigr\}$. Namely, $a \wedge e^* \leq a \wedge (a \wedge e)^*$ and $a \wedge (a \wedge e)^* \leq (a \wedge e)^* \implies 0 = \bigl\{a \wedge (a \wedge e)^*\bigr\} \wedge (a \wedge e) = \bigl\{a \wedge (a \wedge e)^*\bigr\} \wedge e \implies a \wedge (a \wedge e)^* \leq a \wedge e^*$. It means that $a \wedge e^* = a \wedge (a \wedge e)^*$, and thus $a \wedge (c \vee e^*) = (a \wedge c) \vee (a \wedge e^*) = (a \wedge c) \vee \bigl\{a \wedge (a \wedge e)^*\bigr\} = a \wedge \bigl\{c \vee (a \wedge e)^*\bigr\}$. We proved property 1°, and properties 2° – 4° follow from Definition 3, immediately. □

Remarks.

1. Property 1° from 4.1 can be reformulated to $a \wedge [c,d,e] = [c, a \wedge d, a \wedge e]$.
2. Let us mention that the $*$-median is no symmetric operation.

COROLLARY 5. *Let $(L, \leq)$ be a partially ordered set with 0, 1, and $[\cdot, \cdot, \cdot]$ be a ternary operation on L such that $b \geq a \iff [a,0,b] = a \iff [1,a,b] = b$ and $[1,1,a] = 1$.*

Then it holds:

1. *If L has property 1°, then L is a distributive lattice.*
2. *If L has properties 1°, 4° and $[0,1,1] = 0$, then L is a distributive p-algebra.*
3. *If L has properties 1°, 4°, and $\bigl[0,1,[0,1,a]\bigr] = a$ for $a \in L$, then L is a Boolean algebra.*

Proof.

1. We have $[a,0,1] = a$, $[a,0,a] = a$, $[1,a,1] = 1$, and $[a,0,0] = \bigl[a, [0,0,1], [0,0,1]\bigr] = \bigl[0, 0[a,1,1]\bigr] = 0$. Part b) from the proof of Theorem 4 implies that L is a distributive lattice.

2. Parts b) and c) from the proof of Theorem 4 imply that L is a distributive p-algebra.

3. Let us remark that $[0,1,1] = \bigl[0,1,[0,1,0]\bigr] = 0$. Then L is a distributive p-algebra, and $a^{**} = a$ holds for $a^* = [0,1,a]$ and $a \in L$, i.e., L is a Boolean algebra. □

COROLLARY 6. *Let L be a set with elements 0, 1, and $[\cdot, \cdot, \cdot]$ be a ternary operation on L. Then it holds:*

If L has properties 1°, 2°, 3°, and $\bigl[0,1,[0,1,a]\bigr] = a$ for all $a \in L$, then L is a Boolean algebra.

P r o o f. It holds $[0,1,0] = [0,1,[0,1,1]] = 1$. Then L is a distributive p-algebra, and $a^* = [0,1,a]$ is a pseudocomplement of a (see 4.1). The fact $a^{**} = [0,1,[0,1,a]] = a$ implies that L is a Boolean algebra. □

PROPOSITION 7. *Properties* $1° - 4°$ *from the Theorem* 4.1 *are independent.*

P r o o f. Let L be a Boolean algebra with $|L| > 5$. If we define $[a,b,c] = b$, then $1°$, $2°$, $4°$ hold, and $3°$ does not hold.

If we define $[a,b,c] = a \wedge (b \vee c)$, then $1°$, $2°$, $3°$ hold, and $4°$ does not hold.

If we define $[a,b,0] = b$, and $[a,b,c] = a$ for $c \neq 0$, then $2°$, $3°$, $4°$ hold, and $1°$ does not hold.

Let $L = \{0,1\}$ be a Boolean algebra. If we define $[1,0,1] = [1,1,1] = [0,1,0] = 1$ and $[0,1,1] = [0,0,1] = [0,0,0] = [1,1,0] = [1,0,0] = 0$, then $1°$, $3°$, $4°$ hold, and $2°$ does not hold. □

THEOREM 8. *Let* L *be a distributive* p*-algebra, and* $[\cdot,\cdot,\cdot]$ *be a ternary operation on* L *fulfilling* $1° - 4°$, *and* $a \vee b = [1,a,b]$, $a \wedge b = [a,0,b]$, *for* $a,b \in L$. *Then* $[\cdot,\cdot,\cdot]$ *is the* $*$*-median if and only if* $x \vee [a,b,c] = [x \vee a, x \vee b, c]$ *for* $a,b,c,x \in L$.

P r o o f.

$\Longrightarrow$: We have $x \vee [a,b,c] = x \vee \{(a \vee b) \wedge (b \vee c) \wedge (a \vee c^*)\} = (x \vee a \vee b) \wedge (x \vee b \vee c) \wedge (x \vee a \vee c^*) = [x \vee a, x \vee b, c]$.

$\Longleftarrow$: The proof has the following steps:

a) $[a,1,0] = [a \vee 0, a \vee 1, 0] = a \vee [0,1,0] = a \vee 1 = 1 \Longrightarrow [a,b,0] = [a, b \wedge 1, b \wedge 0] = b \wedge [a,1,0] = b \wedge 1 = b \Longrightarrow c^* \wedge [a,b,c] = [a, c^* \wedge b, c^* \wedge c] = [a, b \wedge c^*, 0] = b \wedge c^* \Longrightarrow b \wedge c^* \leq [a,b,c]$;

b) $(a \wedge b) \vee [a,b,c] = [(a \wedge b) \vee a, (a \wedge b) \vee b, c] = [a,b,c] \Longrightarrow a \wedge b \leq [a,b,c]$;

c) $(a \vee b) \vee [a,b,c] = [(a \vee b) \vee a, (a \vee b) \vee b, c] = [a \vee b, a \vee b, c] = a \vee b \Longrightarrow a \vee b \geq [a,b,c]$;

d) $(b \vee c) \wedge [a,b,c] = [a, b \wedge (b \vee c), c \wedge (b \vee c)] = [a,b,c] \Longrightarrow b \vee c \geq [a,b,c]$;

e) $c \wedge a = c \wedge [a,b,1] = [a, c \wedge b, c \wedge 1] = [a, c \wedge b, c \wedge c] = c \wedge [a,b,c] \Longrightarrow a \wedge c \leq [a,b,c]$;

f) $0 = c \wedge c^* = c \wedge [c^*,b,c] \Longrightarrow c^* \geq [c^*,b,c] \Longrightarrow (a \vee c^*) \vee [a,b,c] = [a \vee c^*, a \vee c^* \vee b, c] = (a \vee c^*) \vee [c^*,b,c] = a \vee c^* \Longrightarrow a \vee c^* \geq [a,b,c]$;

g) Finally, $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c^*) \leq [a,b,c] \leq (a \vee b) \wedge (a \vee c^*) \wedge (b \vee c)$ holds, and Proposition 2.2. implies that $[\cdot,\cdot,\cdot]$ is the $*$-median. □

EXAMPLE. Let L be a distributive p-algebra, and $[\cdot,\cdot,\cdot]$ be a ternary operation on L defined in the following way:

$$[a,b,c] = (a \vee b) \wedge (b \vee c) \wedge (a \vee c^*) \wedge (a \vee a^*) \qquad \text{for} \quad a,b,c \in L .$$

Then, $[\cdot\,,\cdot\,,\cdot]$ has properties 1°, 2°, 3°, 4° from Theorem 4, but $[\cdot\,,\cdot\,,\cdot]$ is not the $*$-median on L because $[\cdot\,,\cdot\,,\cdot]$ has not the property $x \vee [a,b,c] = [x \vee a, x \vee b, c]$ from Theorem 8.

Namely, $x \wedge [a,b,c] = x \wedge \{(a \vee b) \wedge (b \vee c) \wedge (a \vee c^*) \wedge (a \vee a^*)\} = (a \vee x) \wedge (a \vee b) \wedge x \wedge (b \vee c) \wedge (a \vee (x \wedge c)^*) \wedge (a \vee a^*) = (a \vee (x \wedge b)) \wedge ((x \wedge b) \vee (x \wedge c)) \wedge (a \vee (x \wedge c)^*) \wedge (a \vee a^*) = [a, x \wedge b, x \wedge c]$, since $x \wedge (a \vee c^*) = x \wedge (a \vee (x \wedge c)^*)$ – see part 2 from Theorem 4. Then property 1° is true. Properties 2°, 3°, 4° are fulfilled trivially. Now, $x \vee [0,1,0] = x \vee 1 = 1$ and $[x \vee 0, x \vee 1, 0] = [x,1,0] = x \vee x^* \neq 1$, for $x \in L$, in the case that L is not a Boolean algebra.

THEOREM 9. *Let L be a distributive p-algebra, and let $[\cdot\,,\cdot\,,\cdot]$ be a ternary operation on L such that $a = [1,0,a]$ and $a^* = [0,1,a]$, for $a \in L$. Then $[\cdot\,,\cdot\,,\cdot]$ is the $*$-median operation on L if and only if for $a,b,c,x \in L$, $x \wedge [a,b,c] = [x \wedge a, x \wedge b, c]$ and $x \vee [a,b,c] = [x \vee a, x \vee b, c]$.*

Proof.

$\Longrightarrow$: We have $x \wedge [a,b,c] = x \wedge \{(a \vee b) \wedge (b \vee c) \wedge (a \vee c^*)\} = x \wedge (a \vee b) \wedge (x \vee c) \wedge (b \vee c) \wedge (x \vee c^*) \wedge (a \vee c^*) = \{(x \wedge a) \vee (x \wedge b)\} \wedge \{(x \wedge b) \vee c\} \wedge \{(x \wedge a) \vee c^*\} = [x \wedge a, x \wedge b, c]$ and $x \vee [a,b,c] = x \vee \{(a \vee b) \wedge (b \vee c) \wedge (a \vee c^*)\} = (x \vee a \vee b) \wedge (x \vee b \vee c) \wedge (x \vee a \vee c^*) = [x \vee a, x \vee b, c]$.

$\Longleftarrow$: This part has the following steps:

a) $(a \wedge b) \vee [a,b,c] = [(a \wedge b) \vee a, (a \wedge b) \vee b, c] = [a,b,c] \Longrightarrow a \wedge b \leq [a,b,c]$;

b) $(a \vee b) \wedge [a,b,c] = [(a \vee b) \wedge a, (a \vee b) \wedge b, c] = [a,b,c] \Longrightarrow a \vee b \geq [a,b,c]$;

c) $b \wedge c^* = b \wedge [0,1,c] = [0,b,c] \Longrightarrow c^* = c^* \vee (b \wedge c^*) = c^* \vee [0,b,c] = [c^*, b \vee c^*, c] \Longrightarrow 0 = c \wedge c^* = c \wedge [c^*, b \vee c^*, c] = [0, c \wedge (b \vee c^*), c] = [0, b \wedge c, c] = c \wedge [c^*, b, c] \Longrightarrow [c^*, b, c] \leq c^* \Longrightarrow (a \vee c^*) \vee [a,b,c] = [a \vee c^*, a \vee b \vee c^*, c] = (a \vee c^*) \vee [c^*, b, c] = a \vee c^* \Longrightarrow a \vee c^* \geq [a,b,c]$;

d) $a \vee c^* = a \vee [0,1,c] = [a,1,c] \Longrightarrow c^* \wedge [a, c^*, c] = [a \wedge c^*, c^*, c] = c^* \wedge [a,1,c] = c^* \wedge (a \vee c^*) = c^* \Longrightarrow c^* \leq [a, c^*, c] \Longrightarrow (b \wedge c^*) \wedge [a,b,c] = [a \wedge b \wedge c^*, b \wedge c^*, c] = (b \wedge c^*) \wedge [a, c^*, c] = b \wedge c^* \Longrightarrow b \wedge c^* \leq [a,b,c]$;

e) $[1,a,b] = a \vee [1,0,b] = a \vee b \Longrightarrow (a \wedge c) \wedge [a,b,c] = [a \wedge c, a \wedge b \wedge c, c] = (a \wedge c) \wedge [1,b,c] = (a \wedge c) \wedge (b \vee c) = a \wedge c \Longrightarrow a \wedge c \leq [a,b,c]$;

f) $[a,0,b] = a \wedge [1,0,b] = a \wedge b \Longrightarrow (b \vee c) \vee [a,b,c] = [a \vee b \vee c, b \vee c, c] = (b \vee c) \vee [a,0,c] = (b \vee c) \vee (a \wedge c) = b \vee c \Longrightarrow b \vee c \geq [a,b,c]$;

g) Finally, $(a \wedge b) \vee (a \wedge c) \vee (b \wedge c^*) \leq [a,b,c] \leq (a \vee b) \wedge (a \vee c^*) \wedge (b \vee c)$, and Proposition 2.2. implies that $[\cdot\,,\cdot\,,\cdot]$ is the $*$-median. □

PROPOSITION 10. *A distributive p-algebra L is a Boolean algebra if and only if the $*$-median operation $[\cdot\,,\cdot\,,\cdot]$ on L satisfies $x \vee [a,b,c] = [x \vee a, b, x \vee c]$ for $a,b,c,x \in L$.*

Proof.

$\Longrightarrow$: We have $x\vee[a,b,c] = x\vee\{(a\vee b)\wedge(b\vee c)\wedge(a\vee c^*)\} = \{x\vee(a\vee b)\}\wedge\{x\vee(b\vee c)\}\wedge\{x\vee(a\vee c^*)\}\wedge\{x\vee(x^*\vee a)\} = (x\vee a\vee b)\wedge(x\vee b\vee c)\wedge(x\vee a\vee(x^*\wedge c^*)) = (x\vee a\vee b)\wedge(x\vee b\vee c)\wedge(x\vee a\vee(x\vee c)^*) = [x\vee a, b, x\vee c]$.

$\Longleftarrow$: For all $a\in L$ it holds $a\vee a^* = a\vee[0,1,a] = [a,1,a] = a\vee[0,1,0] = a\vee 1 = 1$. □

PROPOSITION 11. *Let L be a Boolean algebra, and let $[\cdot,\cdot,\cdot]$ be a ternary operation on L. Then $[\cdot,\cdot,\cdot]$ is the $*$-median operation on L if and only if for all $a,b,c,x\in L$, $x\wedge[a,b,c] = [a, x\wedge b, x\wedge c]$, $x\vee[a,b,c] = [x\vee a, b, x\vee c]$, $1 = [a,1,0]$ and $0 = [0,a,1]$.*

Proof.

$\Longrightarrow$: With regard to Theorem 4.2, we have $x\wedge[a,b,c] = [x,0,[a,b,c]] = [a,[x,0,b],[x,0,c]] = [a, x\wedge b, x\wedge c]$. The rest follows from the first part of the proof of Proposition 10.

$\Longleftarrow$: This part has the following steps:

a) $(b\vee c)\wedge[a,b,c] = [a,(b\vee c)\wedge b,(b\vee c)\wedge c] = [a,b,c] \Longrightarrow b\vee c\geq[a,b,c]$;

b) $b = b\wedge 1 = b\wedge[a,1,0] = [a,b,0] \Longrightarrow (b\wedge c^*)\wedge[a,b,c] = [a,(b\wedge c^*)\wedge b, (b\wedge c^*)\wedge c] = [a, b\wedge c^*, 0] = b\wedge c^* \Longrightarrow b\wedge c^*\leq[a,b,c]$;

c) $(a\wedge c)\vee[a,b,c] = [(a\wedge c)\vee a, b, (a\wedge c)\vee c] = [a,b,c] \Longrightarrow a\wedge c\leq[a,b,c]$;

d) $a = a\vee 0 = a\vee[0,1,b] = [a,b,1] \Longrightarrow (a\vee c^*)\vee[a,b,c] = [a\vee c^*, b, a\vee c^*\vee c] = [a\vee c^*, b, 1] = a\vee c^* \Longrightarrow a\vee c^*\geq[a,b,c]$;

e) $a^* = [b,a^*,0] = [b,a^*,a^*\wedge a] = a^*\wedge[b,1,a] \Longrightarrow a^*\leq[b,1,a] \Longrightarrow [b,1,a] = a^*\vee[b,1,a] = [a^*\vee b,1,1] = a^*\vee b \Longrightarrow (a\wedge b)\wedge[a,b,c] = [a, a\wedge b, a\wedge b\wedge c] = (a\wedge b)\wedge[a,1,c] = (a\wedge b)\wedge(a\vee c^*) = a\wedge b \Longrightarrow a\wedge b\leq[a,b,c]$;

f) $a = [a,b,1] = [a,b,a\vee a^*] = a\vee[0,b,a^*] \Longrightarrow a\geq[0,b,a^*] \Longrightarrow [0,b,c] = c^*\wedge[0,b,c] = [0, b\wedge c^*, 0] = b\wedge c^* \Longrightarrow (a\vee b)\vee[a,b,c] = [a\vee b, b, a\vee b\vee c] = (a\vee b)\vee[0,b,c] = (a\vee b)\vee(b\wedge c^*) = a\vee b \Longrightarrow a\vee b\geq[a,b,c]$;

g) Finally, $(a\wedge b)\vee(a\wedge c)\vee(b\wedge c^*)\leq[a,b,c]\leq(a\vee b)\wedge(a\vee c^*)\wedge(b\vee c)$, and Proposition 2.2. implies that $[\cdot,\cdot,\cdot]$ is the $*$-median. □

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