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NOTE ON MEASURES

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Suppose that X is a non-empty set and $\mathscr{A}$ is a σ-algebra of subsets of it. By a measure on $\mathscr{A}$ we shall mean a countably additive function μ from $\mathscr{A}$ into the set of all complex numbers $\mathbf{C}$. For the measure μ by μ^* we shall denote the total variation of it and by μ^+ and μ^- its positive and negative variation, respectively, whenever μ will be real. Moreover, if T is a self-mapping of $\mathscr{A}$, then we shall say that it has the property (m) iff $a\left(\bigcup_{m=1}^{\infty} A_m\right) = \bigcup_{m=1}^{\infty} T(A_m)$ holds for every sequence $(A_m\colon m \in \mathbf{N})$ of mutually disjoint sets from $\mathscr{A}$ and $T(A) \cap T(B) = \emptyset$, whenever A, $B \in \mathscr{A}$ and $A \cap B = \emptyset$.

Let a non-empty set X and a σ-algebra $\mathscr{A}$ of subsets of it be given together with measures ν_i on $\mathscr{A}$, measurable functions $f_{i,j,k}\colon X \to \mathbf{C}$ and self-mappings S_k and T_k of $\mathscr{A}$ with the property (m), $i, j \in \{1, \ldots, M\}$, $k \in \{1, \ldots, N\}$, where M and N are positive integers. The aim of this note is to give a sufficient condition for the existence of exactly one sequence $(\mu_1, \ldots, \mu_M)$ of measures on $\mathscr{A}$ such that

$$\mu_i(A) = \sum_{j=1}^{M} \sum_{k=1}^{N} \int_{T_k(A)} f_{i,j,k}\, \mathrm{d}\mu_j \circ S_k + \nu_i(A)$$

holds for every $A \in \mathscr{A}$ and $i \in \{1, \ldots, M\}$.

In order to be brief we shall assume permanently that the indexes i and j (with or without affixes) run over the set $\{1, \ldots, M\}$, k (with or without affixes) runs over the set $\{1, \ldots, N\}$, n runs over the set of all non-negative integers and m runs over the set of all positive integers.

Assume that

(i) X is a non-empty set and $\mathscr{A}$ is a σ-algebra of subsets of it.

(ii) $\nu_{i,n}$ are measures on $\mathscr{A}$ such that $\lim_n (\nu_{i,n} - \nu_{i,0})^*(X) = 0$ for every i.

(iii) $f_{i,j,k,n}\colon X \to \mathbf{C}$ are measurable functions such that $\limsup_n \{|f_{i,j,k,n}(x) - f_{i,j,k,0}(x)| : x \in X\} = 0$ for every i, j, k and $\sup\{|f_{i,j,k,n}(x)| : x \in X\} \leqslant a_{i,j,k}$ holds for all i, j, k and n with constants $a_{i,j,k}$ such that all the characteristic roots of the matrix $\left(\sum_k a_{i,j,k}\right)$ are less than one.

(iv) S_k and T_k are self-mappings of $\mathscr{A}$ with the property (m).
We have the following

Theorem. *Under the hypotheses (i)—(iv) there exists for every n exactly one sequence $(\mu_{1,n}, \ldots, \mu_{M,n})$ of measures on $\mathscr{A}$ such that*

$$\mu_{i,n}(A) = \sum_j \sum_k \int_{T_k(A)} f_{i,j,k,n} \, d\mu_{j,n} \circ S_k + \nu_{i,n}(A)$$

holds for every i and $A \in \mathscr{A}$. Moreover,

(I) $\lim_n (\mu_{i,n} - \mu_{i,0})^*(X) = 0$ *for every i;*

(II) *if for a certain n all the $f_{i,j,k,n}$ and $\nu_{i,n}$ are real, so are $\mu_{i,n}$ for that n and all i;*

(III) *if for a certain n all the $f_{i,j,k,n}$ and $\nu_{i,n}$ are non-negative, so are $\mu_{i,n}$ for that n and all i;*

(IV) *if $\mathfrak{M}$ is a subset of $\mathscr{A}$ such that $S_k(T_k(\mathfrak{M})) \subset \mathfrak{M}$ for every k and for a certain n*

$$(*) \qquad \sum_i \nu_{i,n}^*|_{\mathfrak{M}} = 0,$$

then $\sum_i \mu_{i,n}^*|_{\mathfrak{M}} = 0$ for that n.

Proof. Denote by $\mathscr{C}$ (resp. $\mathscr{R}$) the set of all measures (resp. real measures) on $\mathscr{A}$ and define $\|\cdot\|: \mathscr{C} \to [0, \infty)$ by

$$\|\mu\| = \mu^*(X), \quad \mu \in \mathscr{C}.$$

It is known (cf. [3], §§ 43 and 44) that $(\mathscr{C}, \|\cdot\|)$ and $(\mathscr{R}, \|\cdot\||_{\mathscr{R}})$ are Banach spaces. Defining, for every i, j, k and n, the (linear) operator $\mathbf{I}_{i,j,k,n}: \mathscr{C} \to \mathscr{C}$ by

$$\mathbf{I}_{i,j,k,n}(\mu)(A) = \int_{T_k(A)} f_{i,j,k,n} \, d\mu \circ S_k, \quad \mu \in \mathscr{C}, \; A \in \mathscr{A},$$

we see that the inequalities

$$\mathbf{I}_{i,j,k,n}(\mu)^* \leqslant a_{i,j,k} \mu^* \circ S_k \circ T_k, \quad \mu \in \mathscr{C},$$

and

$$(\mathbf{I}_{i,j,k,n}(\mu) - \mathbf{I}_{i,j,k,}0(\mu))^* \leqslant$$
$$\leqslant \sup\{|f_{i,j,k,n}(x) - f_{i,j,k,0}(x)| : x \in X\} \mu^* \circ S_k \circ T_k, \quad \mu \in \mathscr{C},$$

are valid for all i, j, k and n. Therefore

$$\|\mathbf{I}_{i,j,k,n}(\mu)\| \leqslant a_{i,j,k} \|\mu\|, \quad \mu \in \mathscr{C},$$

and

$$\|\mathbf{I}_{i,j,k,n}(\mu) - \mathbf{I}_{i,j,k,0}(\mu) \leqslant \sup\{|f_{i,j,k,n}(x) - f_{i,j,k,0}(x)| : x \in X\} \|\mu\|, \; \mu \in \mathscr{C},$$

for every i, j, k and n. Hence, next, for $\mathscr{F}_{i,n}: \mathscr{C}^M \to \mathscr{C}$ defined for all i and n by

$$\mathscr{F}_{i,n}(\mu_1, \ldots, \mu_M) = \sum_j \sum_k \mathbf{I}_{i,j,k,n}(\mu_j) + \nu_{i,n},$$
$$(\mu_1, \ldots, \mu_M) \in \mathscr{C}^M,$$

we have

$$\left\| \mathscr{F}_{i,n}(\mu_1, \ldots, \mu_M) - \mathscr{F}_{i,n}(\hat{\mu}_1, \ldots, \hat{\mu}_M) \right\| \leqslant \sum_j \sum_k a_{i,j,k} \left\| \mu_j - \hat{\mu}_j \right\|,$$
$$(\mu_1, \ldots, \mu_M),\ (\hat{\mu}_1, \ldots, \hat{\mu}_M) \in \mathscr{C}^M,$$

and

$$\| \mathscr{F}_{i,n}(\mu_1, \ldots, \mu_M) - \mathscr{F}_{i,0}(\mu_1, \ldots, \mu_M) \| \leqslant$$
$$\sum_j \sum_k \sup \{ |f_{i,j,k,n}(x) - f_{i,j,k,0}(x)| : x \in X \} \|\mu_j\| + \|\nu_{i,n} - \nu_{i,0}\|,$$
$$(\mu_1, \ldots, \mu_M) \in \mathscr{C}^M,$$

for every i, j, k and n. The last inequality gives

$$\mathscr{F}_{i,0}(\mu_1, \ldots, \mu_M) = \lim_n \mathscr{F}_{i,n}(\mu_1, \ldots, \mu_M),$$
$$(\mu_1, \ldots, \mu_M) \in \mathscr{C}^M,$$

for all i. Now we see that the first part of the Theorem and the property (I) follows from Lemma in [1] and Lemma 1.2 (ii) in [2]. To obtain (II) observe that if for a certain n all the $f_{i,j,k,n}$ and $\nu_{i,n}$ are real, then $\mathscr{F}_{i,n}(\mathscr{R}) \subset \mathscr{R}$ for that n and all i. Passing to the proof of (III) fix an n such that all the $f_{i,j,k,n}$ and $\nu_{i,n}$ are non-negative. Then $\mu_{i,n} = \mu_{i,n}^+ + \mu_{i,n}^-$ and

$$\mu_{i,n} = \sum_j \sum_k \mathbf{I}_{i,j,k,n}(\mu_{j,n}^+) + \nu_{i,n} - \sum_j \sum_k \mathbf{I}_{i,j,k,n}(-\mu_{j,n}^-)$$

for every i. Therefore

$$-\mu_{i,n}^- \leqslant \sum_j \sum_k \mathbf{I}_{i,j,k,n}(-\mu_{j,n}^-)$$

for all i and so

$$\|\mu_{i,n}^-\| \leqslant \sum_j \sum_k a_{i,j,k} \|\mu_{j,n}^-\|$$

for every i. Hence and from Lemmas 1.3 and 1.2 (*ii*) from [2] we get $\mu_{i,n}^- = 0$ and, consequently, $\mu_{i,n} \geqslant 0$ for all i. In order to obtain the property (IV) fix an n such that (*) is true. Since

$$\mu_{i,n}{}^* = \Big(\sum_j \sum_k \mathbf{I}_{i,j,k,n}(\mu_{j,n}) + \nu_{i,n} \Big)^* \leqslant$$
$$\leqslant \sum_j \sum_k \mathbf{I}_{i,j,k,n}(\mu_{j,n})^* + \nu_{i,n}{}^* \leqslant$$
$$\leqslant \sum_j \sum_k a_{i,j,k} \mu_{j,n}{}^* \circ S_K \circ T_k + \nu_{i,n}{}^*$$

for every i,

$$\mu_{i,n}\Big|_{\mathfrak{M}} \leq \sum_j \sum_k a_{i,j,k}\mu_{j,n}{}^* \circ S_k \circ T_k\Big|_{\mathfrak{M}}$$

holds for all i. By induction

$$\mu_{i,n}{}^*|_{\mathfrak{M}} \leq \sum_{j_1, \dots, j_{m+1}} \sum_{k_1, \dots, k_{m+1}} a_{i,j_1,k_1} \cdot a_{j_1,j_2,k_2} \cdot \dots \cdot$$
$$\cdot a_{j_m, j_{m+1}, k_{m+1}} \mu_{j_{m+1},n}{}^* \circ (S_{k_{m+1}} \circ T_{k_{m+1}}) \circ \dots \circ (S_{k_1} \circ T_{k_1})|_{\mathfrak{M}}$$

is valid for every i and m. Hence, recalling Lemmas 1.1 and 1.2 (ii) in [2] and choosing a $\vartheta \in (0, 1)$ and $r_i \in (\mu_{i,n}{}^*(X), \infty)$ for all i in such a manner that

$$\sum_j \sum_k a_{i,j,k} r_j \leq \vartheta r_i$$

holds for every i, $\mu_{i,n}{}^*|_{\mathfrak{M}} \leq \vartheta^m r_i$ for all i and m. Consequently $\sum_i \mu_{i,n}{}^*|_{\mathfrak{M}} = 0$ and the proof is finished.

The just proved Theorem leads to the following

Corollary. *Suppose that the hypotheses (i) and (ii) are fulfilled and T_k are self-mappings of $\mathcal{A}$ with the property (m). If for the complex numbers $s_{i,j,k,n}$ we have $\lim_n s_{i,j,k,n} = s_{i,j,k,0}$ for every i, j, k and $|s_{i,j,k,n}| \leq a_{i,j,k}$ for every i, j, k and n, where all the characteristic roots of the matrix $\left(\sum_k a_{i,j\ k}\right)$ are less than one, then for every n there exists exactly one sequence $(\mu_{1,n}, \dots, \mu_{M,n})$ of measures on $\mathcal{A}$ such that*

$$\mu_{i,n} = \sum_j \sum_k s_{i,j,k,n} \mu_{j,n} \circ T_k + \nu_{i,n}$$

holds for every i. Moreover, we have (I) *and*

(II′) *if for a certain n all the $s_{i,j,k,n}$ and $\nu_{i,n}$ are real so are $\mu_{i,n}$ for that n and all i;*

(III′) *if for a certain n all the $s_{i,j,k,n}$ and $\nu_{i,n}$ are non-negative, so are $\mu_{i,n}$ for that n and all i;*

(IV′) *if $\mathfrak{M}$ is a subset of $\mathcal{A}$ such that $T_k(\mathfrak{M}) \subset \mathfrak{M}$ for every k and for a certain n we have $(*)$, then $\sum_i \mu_{i,n}{}^*|_{\mathfrak{M}} = 0$ for that n.*

Remark. Using our Corollary we may strengthen Theorem 6.6a) from [2]. In fact, suppose (i), assume ν to be a measure on $\mathcal{A}$ and s to be a complex number such that $|s| < 1$. If f is a one-to-one self-mapping of X such that the image $f(A)$ of every set $A \in \mathcal{A}$ is in $\mathcal{A}$, then, by the Corollary, there exists exactly one measure μ on $\mathcal{A}$ such that

$$\mu(A)=s\mu[f(A)]+\nu(A), \quad A\in\mathscr{A}.$$

This measure is real provided s and ν are real and it is non-negative whenever s and ν are non-negative. Moreover, if $\mathfrak{M}$ is a subset of $\mathscr{A}$ such that $f(\mathfrak{M})\subset\mathfrak{M}$ and $\nu|_{\mathfrak{M}}=0$, then $\mu|_{\mathfrak{M}}=0$.

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ЗАМЕТКА О МЕРАХ

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Резюме

Предположим, что данно непустое множество X, σ-алгебра $\mathscr{A}$ его подмножеств, меры ν_i на $\mathscr{A}$, измеримые функции $f_{i,j,k}$: $X\to\mathbf{C}$, далее фунции S_k и T_k отображающие семейство $\mathscr{A}$ в себя, i, $j\in\{1, ..., M\}$, $k\in\{1, ..., N\}$, где M и N являются некоторыми натуральными числами.

Доказывается теорема о существовании, единственности и некоторых свойствах решений системы

$$\mu_i(A)=\sum_{j=1}^{M}\sum_{k=1}^{N}\int_{T_k(A)} f_{i,j,k}\,\mathrm{d}\mu_j\circ S_k+\nu_i(A), \quad i\in\{1, ..., M\},$$

в которой неизвестными функциями являются меры $\mu_1, \ldots, \mu_M$.