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ON THE CONSTRUCTION OF OUTER MEASURES WITH VALUES IN A UNIFORM SEMIGROUP

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ABSTRACT. We give a new construction of a uniform semigroup valued outer measure starting from an exhaustive finitely additive function. The result is obtained using a classical extension theorem.

1. Introduction

In the monograph [5], dedicated to the study of measure and integration for uniform semigroup valued functions, Sion (see Theorem 6.1) has given a method to obtain a Carathéodory type outer measure starting from an exhaustive finitely additive function. Several mathematicians have used this tool (see also [6], [7], [2]). The aim of this note is to give a new construction of the Sion outer measure using a classical extension theorem (see [9]). We obtain this result in a simple way. First we construct an outer measure starting from a countably additive function (see Theorem 2). Then we prove the theorem for an exhaustive finitely additive function using the countably additive case (see Theorem 3).

2. Preliminaries

A triplet $(S,+,\mathcal{U})$, where S is a set, $+$ is a binary operation on S and $\mathcal{U}$ is a uniformity on S, is a *uniform semigroup* if $(S,+)$ is a commutative semigroup and the function

$$S \times S \ni (a,b) \mapsto a+b \in S$$

is uniformly continuous. It is well known that the uniformity $\mathcal{U}$ can be generated by the set $\mathfrak{P}$ of all uniformly continuous pseudometrics p on S such

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that $p(a+c, b+c) \leq p(a, b)$ for all $a, b, c \in S$ (semi-invariant property). For more details on uniform spaces and uniform semigroups we refer to [3], [5], [1]. [8]. Throughout this paper we will denote by $\mathcal{R}$ a ring of subsets of a set Ω and by $\mathcal{R}_\sigma$ the family of all subsets of Ω which are countable unions of elements of $\mathcal{R}$. From now on we will assume that Ω is an element of $\mathcal{R}_\sigma$.

Moreover we will denote by $(S, +, \mathcal{U})$ an Hausdorff uniform semigroup and by $\mathfrak{P}$ the set of all semi-invariant uniformly continuous pseudometrics p on S which generates the uniformity $\mathcal{U}$. Let $p \in \mathfrak{P}$ and $|a|_p = p(a, 0)$ for each $a \in S$. Let $\mu\colon \mathcal{R} \to S$ be a function such that $\mu(\emptyset) = 0$. Set

$$\mu_p(X) = \sup\{|\mu(Y)|_p : \ Y \in \mathcal{R} \text{ and } Y \subseteq X\},$$

for each $X \in \mathcal{R}_\sigma$. Consider the function

$$\mu_p^* : \ \mathcal{P}(\Omega) \ni X \mapsto \inf\{\mu_p(Y) : \ Y \in \mathcal{R}_\sigma \text{ and } X \subseteq Y\}.$$

Note that μ_p^* is monotone, $\mu_p^*(\emptyset) = 0$ and $\mu_p^*(X) = \mu_p(X)$ for each $X \in \mathcal{R}_\sigma$. Moreover it can be proved that if μ is σ-additive, then μ_p^* is an outer measure. Now let $\mu\colon \mathcal{R} \to S$ be a finitely additive function. The function

$$d_\mu^p : \ \mathcal{P}(\Omega) \times \mathcal{P}(\Omega) \ni (X, Y) \mapsto \mu_p^*(X \triangle Y),$$

where $\triangle$ denotes the usual symmetric difference, is a pseudometric on $\mathcal{P}(\Omega)$. Let $\mathcal{U}_\mu$ be the uniformity on $\mathcal{P}(\Omega)$ generated by the family $\{d_\mu^p : \ p \in \mathfrak{P}\}$. If $\mathcal{T}_\mu$ is the topology on $\mathcal{P}(\Omega)$ induced by $\mathcal{U}_\mu$, then $(\mathcal{P}(\Omega), \triangle, \mathcal{T}_\mu)$ is a topological group. Further observe that if $\mu\colon \mathcal{R} \to S$ is σ-additive, then μ is uniformly continuous in $(\mathcal{R}, \mathcal{U}_\mu)$ and, for each $p \in \mathfrak{P}$, we have that μ_p^* is uniformly continuous in $(\mathcal{P}(\Omega), \mathcal{U}_\mu)$. From now on, for each $\mathcal{A} \subseteq \mathcal{P}(\Omega)$, we will denote by $\overline{\mathcal{A}}^\mu$ the closure of $\mathcal{A}$ with respect to $\mathcal{T}_\mu$.

A function $\mu\colon \mathcal{R} \to S_2$ is said to be *exhaustive* if, for every disjoint sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathcal{R}$, we have $\lim\limits_{n\to\infty} \mu(X_n) = 0$.

For more details on exhaustive functions with values in topological groups and uniform semigroups, see [5], [2], [8]. The following is a result, let us say. internal to the theory of semigroup valued measures and whose proof will not be given here.

PROPOSITION 1. (cf. [8; p. 27, Proposition 2.12]) *Let $\mu\colon \mathcal{R} \to S$ be an exhaustive σ-additive function and $p \in \mathfrak{P}$. Then $\overline{\mathcal{R}}^\mu$ is a σ-ring and, for each decreasing sequence $(X_n)_{n\in\mathbb{N}}$ of $\overline{\mathcal{R}}^\mu$,*

$$\lim_{n\to\infty} \mu_p^*\Big(X_n \setminus \Big(\bigcap_{k\in\mathbb{N}} X_k\Big)\Big) = 0.$$

THEOREM 1. (cf. [9; p. 418, Satz (4.4)(c)]) *If $(S,+,\mathcal{U})$ is complete and $\mu\colon \mathcal{R} \to S$ is an exhaustive σ-additive function, then there is a unique exhaustive σ-additive extension of μ on $\overline{\mathcal{R}}^{\mu}$.*

Remark 1. Let $(S,+,\mathcal{U})$ be complete and let $\mu\colon \mathcal{R} \to S$ be an exhaustive σ-additive function. In the following we will denote by $\overline{\mu}$ the unique exhaustive σ-additive extension of μ on $\overline{\mathcal{R}}^{\mu}$, which exists by Theorem 1. Observe that $\overline{\mu}$ is the $\mathcal{U}_{\mu}$-uniformly continuous extension of μ on $\overline{\mathcal{R}}^{\mu}$ (see also [3; p. 195. Theorem 26]).

3. Generation of an outer measure

Let $\mathcal{H} \subseteq \mathcal{P}(\Omega)$. Denote by $\sigma(\mathcal{H})$ the σ-ring generated by $\mathcal{H}$. A function $\mu\colon \mathcal{P}(\Omega) \to S$ is said to be *$\mathcal{H}$-outer regular*, or *outer regular with respect to $\mathcal{H}$*, if for each $X \in \mathcal{P}(\Omega)$ and for each $U \in \mathcal{U}$ there is $Y \in \mathcal{H}$ such that $X \subseteq Y$ and

$$\big(\mu(X), \mu(Y \cap Z)\big) \in U\,,$$

for each $Z \in \mathcal{H}$ such that $X \subseteq Z$.

A function $\mu\colon \mathcal{P}(\Omega) \to S$ is a *$\mathcal{H}$-outer measure* if μ is σ-additive in $\sigma(\mathcal{H})$ and μ is $\mathcal{H}$-outer regular. We say that μ is an *outer measure* if there exists $\mathcal{H} \subseteq \mathcal{P}(\Omega)$ such that μ is a $\mathcal{H}$-outer measure.

Remark 2. For more details on outer measures in topological groups and uniform semigroups see also [4], [5], [2], [8]. Observe that if $S = \mathbb{R}$ and μ takes values in $[0,+\infty[$, the above definition of outer measure coincides with the usual definition of outer measure in the sense of Carathéodory (see [2; p. 48, (4.5)]).

Let $X \in \mathcal{P}(\Omega)$. Set $\mathcal{R}_X = \{Y \in \mathcal{R}_\sigma :\ X \subseteq Y\}$. The couple $(\mathcal{R}_X, \subseteq)$ is an oriented set.

THEOREM 2. *If $(S,+,\mathcal{U})$ is complete and $\mu\colon \mathcal{R} \to S$ is exhaustive and σ-additive function, then the function*

$$\mu^*:\ \mathcal{P}(\Omega) \ni X \mapsto \begin{cases} \overline{\mu}(X) & \text{if } X \in \overline{\mathcal{R}}^{\mu}\,,\\ \lim\limits_{Y \in (\mathcal{R}_X, \subseteq)} \overline{\mu}(Y) & \text{if } X \in \mathcal{P}(\Omega) \setminus \overline{\mathcal{R}}^{\mu} \end{cases}$$

is an outer measure.

Proof. We start observing that, since $\overline{\mu}$ is exhaustive in $\overline{\mathcal{R}}^{\mu}$ and $(S,+,\mathcal{U})$ is complete, for each $X \in \mathcal{P}(\Omega)$ the net

$$\big(\overline{\mu}(Y)\big)_{Y \in (\mathcal{R}_X, \subseteq)}$$

is convergent in S (see [2; p. 26, (1.2)]). This implies that the function μ^* is well defined. Now we want to show that μ^* is $\mathcal{R}_\sigma$-outer regular. Let $p \in \mathfrak{P}$, $\varepsilon > 0$ and $X \in \overline{\mathcal{R}}^\mu$. By Proposition 1 we can find $Y \in \mathcal{R}_\sigma$ such that $X \subseteq Y$ and $\mu_p^*(Y \setminus X) < \varepsilon$. Then, if $Z \in \mathcal{R}_\sigma$ and $X \subseteq Z \subseteq Y$, we have that

$$p\big(\mu^*(X), \mu^*(Z)\big) \leq |\overline{\mu}(Z \setminus X)|_p \\ \leq \mu_p^*(Z \setminus X) \leq \mu_p^*(Y \setminus X) < \varepsilon\,.$$

Now let $X \in \mathcal{P}(\Omega) \setminus \overline{\mathcal{R}}^\mu$. By the definition of μ^* there exists $Y \in \mathcal{R}_\sigma$ such that $X \subseteq Y$ and such that, for each $Z \in \mathcal{R}_\sigma$ with $X \subseteq Z \subseteq Y$,

$$p\big(\mu^*(X), \mu^*(Z)\big) = p\big(\mu^*(X), \overline{\mu}(Z)\big) < \varepsilon\,.$$

Then μ^* is $\mathcal{R}_\sigma$-outer regular. Moreover, since μ^* is clearly a σ-additive extension of μ on $\overline{\mathcal{R}}^\mu$ and $\overline{\mathcal{R}}^\mu$ is a σ-ring, we have that μ^* is an $\mathcal{R}_\sigma$-outer measure. □

Now we want to construct an outer measure starting from a finitely additive function. Define

$$\mathcal{P}_X = \Big\{P \subseteq \mathcal{R} :\ P \text{ is countable, disjoint and } X \subseteq \bigcup_{Y \in P} Y\Big\}$$

for each $X \in \mathcal{P}(\Omega)$ and, for every $P, Q \in \mathcal{P}_X$, let $P \leq Q$ if Q is finer than P (i.e. every element of Q is contained in some element of P). The couple $(\mathcal{P}_X, \leq)$ is an oriented set. Let $\mathfrak{D}$ be the set of all functions which associate to each countable disjoint subfamily P of $\mathcal{R}$ a finite subset $\Delta(P)$ of P. Set

$$\mathcal{D}_X = \big\{(P, \Delta) :\ P \in \mathcal{P}_X\,,\ \ \Delta \in \mathfrak{D}\big\}\,,$$

and, for every $(P, \Delta), (Q, \Gamma) \in \mathcal{D}_X$, let $(P, \Delta) \trianglelefteq (Q, \Gamma)$ if $P \leq Q$ and $\Delta(R) \subseteq \Gamma(R)$ for all countable disjoint subfamily R of $\mathcal{R}$. The couple $(\mathcal{D}_X, \trianglelefteq)$ is an oriented set. We need the following proposition.

Proposition 2. (cf. [5; p. 28, Theorem 5.2]) *Let μ: $\mathcal{R} \to S$ be an exhaustive finitely additive function. If $(S, +, \mathcal{U})$ is complete, then for each $X \in \mathcal{P}(\Omega)$ the net*

$$\Big(\sum_{Y \in \Delta(P)} \mu(Y)\Big)_{(P,\Delta) \in (\mathcal{D}_X, \trianglelefteq)}$$

is convergent in S.

Now we are able to prove our main result (see also [5; p. 34, Theorem 6.1]).

THEOREM 3. *Let $\mu\colon \mathcal{R} \to S$ be an exhaustive finitely additive function. If $(S,+,\mathcal{U})$ is complete, then there exists an extension of the function*

$$\mu^*:\ \mathcal{R} \ni X \mapsto \lim_{(P,\Delta)\in(\mathcal{D}_X,\trianglelefteq)} \sum_{Y\in\Delta(P)} \mu(Y)$$

on $\mathcal{P}(\Omega)$ which is an outer measure.

P r o o f . Since Proposition 2 holds, the function μ^* is well defined. By Theorem 2 it is sufficient to show that μ^* is σ-additive and exhaustive. We start to show that μ^* is σ-additive in $\mathcal{R}$. Let $(X_n)_{n\in\mathbb{N}}$ be a disjoint sequence of $\mathcal{R}$ and set $X = \bigcup\limits_{n\in\mathbb{N}} X_n$. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ and $(V_n)_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $V \circ V \subseteq U$ and

$$\Big(\sum_{n\in I} a_n, \sum_{n\in I} b_n\Big) \in V_0$$

for each finite $I \subseteq \mathbb{N}$ and $(a_n, b_n) \in V_n$ $(n \in I)$. Let $(P,\Delta) \in (\mathcal{D}_X, \trianglelefteq)$ such that $\{X_n :\ n \in \mathbb{N}\} \le P$ and

$$\Big(\mu^*(X),\ \sum_{Y\in\Gamma(Q)} \mu(Y)\Big) \in V$$

for each $(Q,\Gamma) \in (\mathcal{D}_X, \trianglelefteq)$ with $(P,\Delta) \trianglelefteq (Q,\Gamma)$. Moreover, for each $n \in \mathbb{N}$ let $(P_n, \Delta_n) \in \mathcal{D}_{X_n}$ such that $\{Y \in P :\ Y \subseteq X_n\} \le P_n$ and

$$\Big(\mu^*(X_n),\ \sum_{Y\in\Gamma(Q)} \mu(Y)\Big) \in V_n$$

for each $(Q,\Gamma) \in (\mathcal{D}_{X_n}, \trianglelefteq)$ with $(P_n,\Delta_n) \trianglelefteq (Q,\Gamma)$. Set $P' = \bigcup\limits_{n\in\mathbb{N}} P_n$, let $I_0 = \{n \in \mathbb{N} :\ \Delta(P') \cap P_n \ne \emptyset\}$. Besides, fixed a finite $I \subseteq \mathbb{N}$ such that $I_0 \subseteq I$, consider the function

$$\Delta'\colon Q \mapsto \begin{cases} \Delta(P') \cup \Big(\bigcup\limits_{n\in I} \Delta_n(P_n)\Big) & \text{if } Q = P', \\ \Delta(Q) & \text{if } Q \ne P', \end{cases}$$

and, for each $n \in I$, let

$$\Delta'_n\colon Q \mapsto \begin{cases} \Delta(P') \cap P_n & \text{if } Q = P_n, \\ \Delta_n(Q) & \text{if } Q \ne P_n. \end{cases}$$

Then $(P,\Delta) \trianglelefteq (P',\Delta')$, $(P_n,\Delta_n) \trianglelefteq (P_n,\Delta'_n)$ $(n \in \mathbb{N})$ and we have that

$$\Big(\mu^*(X),\ \sum_{Y\in\Delta'(P')} \mu(Y)\Big) \in V,$$

$$\Big(\mu^*(X_n),\ \sum_{Y\in\Delta'_n(P_n)} \mu(Y)\Big) \in V_n.$$

Hence, since

$$\sum_{Y\in\Delta'(P')}\mu(Y)=\sum_{n\in I}\sum_{Y\in\Delta'_n(P_n)}\mu(Y),$$

it follows that

$$\Big(\mu^*(X),\sum_{n\in I}\mu^*(X_n)\Big)\in V\circ V\subseteq U.$$

Then

$$\mu^*(X)=\sum_{n\in\mathbb{N}}\mu^*(X_n).$$

Now we prove that the function μ^* is exhaustive in $\mathcal{R}$. Let $(X_n)_{n\in\mathbb{N}}$ be a disjoint sequence in $\mathcal{R}$ and let $U\in\mathcal{U}$. Since μ is exhaustive in $\mathcal{R}$, there is $m\in\mathbb{N}$ such that $\big(\mu(Y),0\big)\in U$ for every $n\geq m$ and $Y\in\mathcal{R}$ with $Y\subseteq X_n$. For each $n\geq m$ choose $(P_n,\Delta_n)\in\mathcal{D}_{X_n}$ such that $Y\subseteq X_n$ for each $Y\in P_n$. Then, since μ is finitely additive, for each $n\geq m$, we have

$$\Big(\sum_{Y\in\Delta_n(P_n)}\mu(Y),0\Big)=\Big(\mu\Big(\bigcup_{Y\in\Delta_n(P_n)}Y\Big),0\Big)\in U.$$

Hence, for each $n\geq m$,

$$\big(\mu^*(X_n),0\big)\in U.$$

This show that μ^* is exhaustive in $\mathcal{R}$ and the proof is now finished. □

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