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MAXIMUMS OF DARBOUX BAIRE ONE FUNCTIONS

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ABSTRACT. In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson asked whether each Baire one function which can be written as the maximum of Darboux functions, can be written as the maximum of two Darboux Baire one functions. We provide the affirmative answer to this question.

The letters $\mathbb{R}$ and $\mathbb{N}$ denote the real line and the set of positive integers, respectively. For each $A \subset \mathbb{R}$ the symbol $\operatorname{card} A$ stands for the cardinality of A. We write $\mathfrak{c} = \operatorname{card} \mathbb{R}$.

Let $f\colon \mathbb{R} \to \mathbb{R}$. We define the *oscillation of f at a point $x \in \mathbb{R}$* as

$$\operatorname{osc}(f,x) = \lim_{\delta\to 0^+} \sup\bigl\{|f(x_1) - f(x_2)| :\ x_1, x_2 \in (x-\delta, x+\delta)\bigr\}.$$

For each nondegenerate interval $I \subset \mathbb{R}$ we define

$$\mathfrak{c}\text{-}\sup(f,I) = \sup\Bigl\{y \in \mathbb{R} :\ \operatorname{card}\bigl\{x \in I :\ f(x) > y\bigr\} = \mathfrak{c}\Bigr\}.$$

For each $x \in \mathbb{R}$ we denote

$$\mathfrak{c}\text{-}\varlimsup_{t\to x^-} f(t) = \lim_{\delta\to 0^+} \mathfrak{c}\text{-}\sup\bigl(f, (x-\delta, x)\bigr),$$

and similarly we define the symbol $\mathfrak{c}\text{-}\varlimsup_{t\to x^+} f(t)$. We say that f is *Darboux* if it maps intervals onto connected sets.

In 1974 A. M. Bruckner, J. G. Ceder, and T. L. Pearson proved that a function f is the maximum of Darboux functions g_0 and g_1 if and only if

$$\min\Bigl\{\mathfrak{c}\text{-}\varlimsup_{t\to x^-} f(t),\ \mathfrak{c}\text{-}\varlimsup_{t\to x^+} f(t)\Bigr\} \ge f(x) \qquad \text{for each} \quad x \in \mathbb{R}, \tag{1}$$

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and we can conclude that g_0 and g_1 are Lebesgue measurable (belong to Baire class α, $\alpha \geq 2$) provided that f is so ([4; Theorem 3]). They asked also, whether the latter result holds true for $\alpha = 1$. We provide the affirmative answer to this question. (See Corollary 4.)

We will need two auxiliary results. Proposition 1 is an immediate consequence of a lemma proved in 1968 by A. M. Bruckner, J. G. Ceder, and R. Keston [3; Lemma 2].

PROPOSITION 1. *Let $P \subset (a,b)$ be a first category set and let τ be a nonnegative extended real number. Then there exists a Darboux Baire one function $h\colon \mathbb{R} \to [0,\tau)$ such that $h(x) = 0$ except for a first category set disjoint from P and*

$$\overline{\lim_{x\to a^+}}\, h(x) = \overline{\lim_{x\to b^-}}\, h(x) = \tau\,.$$

Proposition 2 contains a condition equivalent to the Darboux property of a Baire one function. It is due to H. Sen and J. L. Massera. (See [2; Theorem 6.1] or [1; p. 9, Theorem 1.1].)

PROPOSITION 2. *Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a Baire one function. The following conditions are equivalent:*

(i) *f is Darboux,*

(ii) *for each $x \in \mathbb{R}$, we have*

$$\max\Big\{\underline{\lim_{t\to x^-}}\, f(t),\ \underline{\lim_{t\to x^+}}\, f(t)\Big\} \leq f(x) \leq \min\Big\{\overline{\lim_{t\to x^-}}\, f(t),\ \overline{\lim_{t\to x^+}}\, f(t)\Big\}\,.$$

Now we are ready to prove the main result.

THEOREM 3. *Assume that a Baire one function $f\colon \mathbb{R} \to \mathbb{R}$ fulfills the following condition:*

$$\min\Big\{\overline{\lim_{t\to x^-}}\, f(t),\ \overline{\lim_{t\to x^+}}\, f(t)\Big\} \geq f(x) \qquad \textit{for each} \quad x \in \mathbb{R}. \tag{2}$$

Then f is the maximum of two Darboux Baire one functions.

Proof. First assume that f is nonpositive. Let $\tau_0 = \infty$ and $A_0 = \emptyset$. For each $n \in \mathbb{N}$ define $\tau_n = 2^{-n}$ and

$$A_n = \{x \in \mathbb{R} :\ \operatorname{osc}(f,x) \geq \tau_n\}\,.$$

Then

$$C \stackrel{\text{df}}{=} \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} A_n$$

is the set of points of continuity of f, which is residual by Baire theorem. It is easy to show that the oscillation function is upper semicontinuous. So, each set

A_n is closed and nowhere dense. For each n arrange all components of $\mathbb{R} \setminus A_n$ in a sequence $\{(a_{nk}, b_{nk}) : k \in N_n\}$, where $N_n \subset \mathbb{N}$.

For all $i < 2$, $n \in \mathbb{N}$, and $k \in N_n$ use Proposition 1 to construct a Darboux Baire one function $h_{ink} \colon \mathbb{R} \to [0, \tau_{n-1})$ such that $h_{ink}(x) = 0$ except for a first category set

$$P_{ink} \subset (a_{nk}, b_{nk}) \cap C \setminus \bigcup_{j<2} \bigcup_{m<n} \bigcup_{l \in N_m} P_{jml} \tag{3}$$

and

$$\varlimsup_{x \to a_{nk}^+} h_{ink}(x) = \varlimsup_{x \to b_{nk}^-} h_{ink}(x) = \tau_{n-1}\,; \tag{4}$$

moreover we require that

$$P_{0nk} \cap P_{1nk} = \emptyset\,. \tag{5}$$

Fix an $i < 2$. Put $h_i = \sum_{n \in \mathbb{N}} \sum_{k \in N_n} h_{ink}$. Notice that for each nondegenerate interval I, since P_{ink} are pairwise disjoint first category sets (by condition (3)) and $h_{ink}(x) = 0$ outside of P_{ink}, the image

$$h_i[I] = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in N_n} h_{ink}[I]$$

is the union of a family of connected sets each of which contains 0. Therefore this set is connected as well, and h_i is a nonnegative Darboux function.

Let $U \subset \mathbb{R}$ be an open set. If $0 \notin U$, then the set

$$h_i^{-1}(U) = \bigcup_{n \in \mathbb{N}} \bigcup_{k \in N_n} h_{ink}^{-1}(U)$$

is a countable union of F_σ-sets, whence it is an F_σ-set as well. In the opposite case choose an $n_0 \in \mathbb{N}$ such that $[0, \tau_{n_0}) \subset U$. Then

$$h_i^{-1}(U) = (\mathbb{R} \setminus C) \cup h_i^{-1}(U \setminus \{0\}) \cup \bigcap_{n \le n_0} \bigcup_{k \in N_n} \left((a_{nk}, b_{nk}) \cap h_{ink}^{-1}(U)\right),$$

so $h_i^{-1}(U)$ is an F_σ-set. Therefore h_i is a Baire one function.

Define $g_i = f - h_i$. To prove that g_i is Darboux, we will use Proposition 2. Fix an $x \in \mathbb{R}$. By (2), there is a sequence $x_n \nearrow x$ such that

$$\varlimsup_{n \to \infty} f(x_n) \ge f(x)\,.$$

For each $n \in \mathbb{N}$, if $x_n \in C$, then choose a point

$$t_n \in (x_n - 1/n, x_n) \setminus \bigcup_{m \in \mathbb{N}} \bigcup_{k \in N_m} P_{imk}$$

such that

$$|f(t_n) - f(x_n)| < 1/n\,;$$

otherwise set $t_n = x_n$. Then

$$\varlimsup_{t\to x^-} g_i(t) \geq \varlimsup_{n\to\infty} g_i(t_n) = \varlimsup_{n\to\infty} f(t_n) \geq f(x) \geq g_i(x)\,.$$

To prove that $\varliminf_{t\to x^-} g_i(t) \leq g_i(x)$, we consider two cases.

If $x \in C$, then let $x_n \nearrow x$ be such that $\varlimsup_{n\to\infty} h_i(x_n) \geq h_i(x)$. (Recall that h_i is Darboux.) Then

$$\begin{aligned}\varliminf_{t\to x^-} g_i(t) \leq \varliminf_{n\to\infty} g_i(x_n) &= \lim_{n\to\infty} f(x_n) - \varlimsup_{n\to\infty} h_i(x_n)\\ &\leq f(x) - h_i(x) = g_i(x)\,.\end{aligned}$$

Now let $x \notin C$. Then $x \in A_m \setminus A_{m-1}$ for some $m \in \mathbb{N}$. There is a sequence (b_{mk_n}) (maybe constant) such that $b_{mk_n} \to x$ and $b_{mk_n} \leq x$ for each n. By (4), for each $n \in \mathbb{N}$ there is an $x_n \in (b_{mk_n} - 1/n, b_{mk_n})$ such that

$$h_i(x_n) \geq \min\{\tau_{m-1} - 1/n,\ n\}\,.$$

Then

$$\begin{aligned}\varliminf_{t\to x^-} g_i(t) \leq \varliminf_{n\to\infty} g_i(x_n) &\leq \varlimsup_{n\to\infty} f(x_n) - \varlimsup_{n\to\infty} h_i(x_n)\\ &\leq \varlimsup_{t\to x} f(t) - \tau_{m-1} \leq f(x) = g_i(x)\,.\end{aligned}$$

(We used the fact that the function f is nonpositive and $\operatorname{osc}(f,x) \leq \tau_{m-1}$.)

Similarly we can show that $\varliminf_{t\to x^+} g_i(t) \leq g_i(x) \leq \varlimsup_{t\to x^+} g_i(t)$. So, g_i is Darboux.

Finally observe that by (3) and (5),

$$\{x \in \mathbb{R}:\ g_0(x) \neq f(x)\} \subset \{x \in \mathbb{R}:\ g_1(x) = f(x)\}\,.$$

So, since h_0 and h_1 are nonnegative, $f = \max\{g_0, g_1\}$ on $\mathbb{R}$. This completes the proof in case f is nonpositive.

Finally let f be an arbitrary Baire one function fulfilling condition (2). Let $\varphi\colon (-\infty, 0) \to \mathbb{R}$ be an increasing homeomorphism. By the first part of the proof, there are Darboux Baire one functions $\bar{g}_0$ and $\bar{g}_1$ such that $\varphi^{-1}\circ f = \max\{\bar{g}_0, \bar{g}_1\}$ on $\mathbb{R}$. Define $g_i = \varphi \circ \bar{g}_i$ $(i < 2)$. One can easily see that g_0 and g_1 fulfill the requirements of the theorem. □

COROLLARY 4. *Let $f\colon \mathbb{R} \to \mathbb{R}$. The following are equivalent:*

(i) *f is the maximum of Darboux Baire one functions,*
(ii) *f is a Baire one function which fulfills condition* (1),
(iii) *f is a Baire one function which fulfills condition* (2).

P r o o f . The implication (ii) $\Longrightarrow$ (iii) is evident, the implication (iii) $\Longrightarrow$ (i) follows from Theorem 3, and the implication (i) $\Longrightarrow$ (ii) follows from [4; Theorem 3]. $\square$

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