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ON THE CONTINUITY OF STRONGLY NONLINEAR POTENTIAL

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ABSTRACT. The boundedness principle and the continuity principle are given in the case of Orlicz classes. This allows to study the continuity of strongly nonlinear potential and to establish relations between some capacities.

1. Introduction

The nonlinear potential theory has taken its time over the development and unifying its different components. Actually, it finds its applications in some fields, particularly in the theory of partial differential equations.

In [4], [7], [8] we have introduced a theory of potential in Orlicz spaces. In [5], [6] we have treated some development of this theory, in particular the continuity of Bessel potential and the instability of capacity.

The present paper establishes the boundedness principle in Orlicz spaces $\mathbf{L}_A$ when the conjugate N-function A^* satisfies the Δ_2-condition and for any radially decreasing convolution kernel. This principle stated in Theorem 3.6, says that for a positive Radon measure μ, if the potential in Orlicz spaces is bounded in the support of μ, then it is bounded everywhere. In the case of the nonlinear potential defined on L^p Lebesgue spaces, the proof is based essentially on the studying the cases $1 < p \leq 2$ and $p > 2$. For Orlicz spaces, the proof is somewhat complicated and uses some properties of N-functions satisfying the Δ_2-condition. Therefore, this gives another proof of the boundedness principle in L^p Lebesgue spaces, without distinction of the exponent p.

As a corollary, if $\mathbf{L}_A$ is reflexive (respectively uniformly convex for Luxemburg norm), then the $C_{g,A}$-capacitary potential $g * f$ is a bounded function on $\mathbb{R}^N$ for any analytic set (respectively any set) X such that $C_{g,A}(X) < \infty$.

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Another consequence of the boundedness principle is an equivalence of the capacities $V_{g,A}$ and $W_{g,A}$ (see Definition 3.3), stated in Theorem 3.9. Hence these two capacities have the same null sets.

The potential in Orlicz spaces is called *strongly nonlinear*, in distinction from the nonlinear potential defined on L^p Lebesgue spaces.

The boundedness principle is due to Ugaheri [12] for the classical potential theory and for a general kernel. The nonlinear extension defined on L^p Lebesgue spaces is due to Havin and Maz'ya [9] for Riesz kernels and to Adams and Meyers [2] in the general case.

Theorem 3.1 states that in order to have a capacity that is useful for measuring small sets, we should work with kernels $g \notin \mathbf{L}_{A^*}$, but $g \in \mathbf{L}_{A^*,\{|x|>1\}}$. In this case, the capacity is a more sensitive measure than N-dimensional Lebesgue measure. Note that this result is valid for any N-function.

The continuity principle for the nonlinear potential theory is due to Havin and Maz'ya [9] for Riesz kernels. We generalize this principle for strongly nonlinear case. This principle stated in Theorem 3.10, says that in Orlicz spaces $\mathbf{L}_A$ with conjugate N-function A^* satisfying the Δ_2-condition and for any radially decreasing convolution kernel, if μ is a positive Radon measure with compact support and if the restriction of the strongly nonlinear potential to the support of μ is continuous, then this potential is continuous everywhere. The proof uses the boundedness principle.

For a reflexive Orlicz space $\mathbf{L}_A$, Theorem 3.11 gives a positive Radon measure μ with compact support K, such that the strongly nonlinear potential associated to μ and K is continuous everywhere. This extension of a well known result in classical potential theory and in nonlinear potential theory, is a consequence of the continuity principle for strongly nonlinear potential.

2. Preliminaries

2.1. Orlicz spaces.

We recall some definitions and results about Orlicz spaces. For more details, one can consult [3], [10], [11].

Let $A\colon \mathbb{R} \to \mathbb{R}^+$ be an *N-function*, i.e. A is continuous, convex, with $A(t) > 0$ for $t > 0$, $\lim_{t\to 0} \frac{A(t)}{t} = 0$, $\lim_{t\to+\infty} \frac{A(t)}{t} = +\infty$ and A is even.

Equivalently, A admits the representation: $A(t) = \int_0^{|t|} a(x)\,\mathrm{d}x$, where $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $\lim_{t\to+\infty} a(t) = +\infty$.

The N-function A^* *conjugate* to A is defined by $A^*(t) = \int_0^{|t|} a^*(x)\,\mathrm{d}x$, where a^* is given by $a^*(s) = \sup\{t : \; a(t) \le s\}$.

Let A be an N-function and let Ω be an open set in $\mathbb{R}^N$. We denote by $\mathcal{L}_A(\Omega)$ the set of measurable functions f on Ω, called an *Orlicz class*, such that

$$\rho(f, A, \Omega) = \int_\Omega A(f(x))\,\mathrm{d}x < \infty .$$

Let A and A^* be two conjugate N-functions and let f be a measurable function defined almost everywhere in Ω. The *Orlicz norm of* f, $\|f\|_{A,\Omega}$ or $\|f\|_A$ if there is no confusion, is defined by

$$\|f\|_A = \sup\Big\{ \int_\Omega |f(x)g(x)|\,\mathrm{d}x : \; g \in \mathcal{L}_{A^*}(\Omega) \text{ and } \rho(g, A^*, \Omega) \le 1 \Big\} .$$

The set $\mathbf{L}_A(\Omega)$ of measurable functions f such that $\|f\|_A < \infty$ is called an *Orlicz space.* When $\Omega = \mathbb{R}^N$, we set $\mathbf{L}_A$ in place of $\mathbf{L}_A(\mathbb{R}^N)$.

The *Luxemburg norm* $|||f|||_{A,\Omega}$, or $|||f|||_A$ if there is no confusion, is defined in $\mathbf{L}_A(\Omega)$ by

$$|||f|||_A = \inf\Big\{ r > 0 : \; \int_\Omega A\big(\tfrac{f(x)}{r}\big)\,\mathrm{d}x \le 1 \Big\} .$$

As we have noted in [8], we can suppose a and a^* continuous and strictly increasing. Hence the N-functions A and A^* are strictly convex and $a^* = a^{-1}$.

Let A be an N-function. We say that A *satisfies the* Δ_2*-condition* if there exists a constant $C > 0$ such that $A(2t) \le CA(t)$ for all $t \ge 0$.

Recall that A satisfies the Δ_2-condition if and only if $\mathcal{L}_A = \mathbf{L}_A$. Moreover $\mathbf{L}_A$ is reflexive if and only if A and A^* satisfy the Δ_2-condition.

2.2. Capacity and potential in Orlicz spaces.

We shall need some definitions and results concerning capacities and potentials in Orlicz spaces. For more details, see [4], [7], [8].

DEFINITION 2.1. Let Γ be a σ-additive class of sets which contains compact sets in $\mathbb{R}^N$. Let C be a positive function defined in Γ.

A) C is called *capacity* if it satisfies the following axioms:

(i) $C(\emptyset) = 0$.

(ii) If X and Y are in Γ and $X \subset Y$, then $C(X) \le C(Y)$.

(iii) If X_i, $i = 1, 2, \ldots$, are in Γ, then $C\Big(\bigcup_{i\ge1} X_i\Big) \le \sum_{i\ge1} C(X_i)$.

B) C is called an *outer capacity* if for every $X \in \Gamma$

$$C(X) = \inf\{C(O) : \; O \text{ open}, \; X \subset O\} .$$

C) C is called an *inner capacity* if for every $X \in \Gamma$

$$C(X) = \sup\{C(K) : \ K \text{ compact}, \ K \subset X\}.$$

Let k be a positive and measurable function in $\mathbb{R}^N$, called a *kernel*, and let A be an N-function. For $X \subset \mathbb{R}^N$, we define

$$C_{k,A}(X) = \inf\{A(|||f|||_A) : \ f \in \mathbf{L}_A^+ \text{ and } k * f \geq 1 \text{ on } X\},$$
$$C'_{k,A}(X) = \inf\{|||f|||_A : \ f \in \mathbf{L}_A^+ \text{ and } k * f \geq 1 \text{ on } X\},$$

where $k * f$ is the usual convolution. The sign $+$ deals with positive elements in the considered space. From [7], $C'_{k,A}$ is a capacity.

If a statement holds except on a set X, where $C_{k,A}(X) = 0$, then we say that the statement holds $C_{k,A}$-quasieverywhere (abbreviated $C_{k,A}$-q.e., or (k, A)-q.e. if there is no confusion).

We call a function $f \in \mathbf{L}_A^+$ such that $k * f \geq 1$ on X, a *test function* for $C'_{k,A}(X)$. Moreover, a test function, say f, for $C'_{k,A}(X)$ such that $C'_{k,A}(X) = |||f|||_A$ is called a $C'_{k,A}$*-capacitary function* of X and $k * f$ a $C'_{k,A}$*-capacitary potential* of X.

For the properties of $C'_{k,A}$ and $C_{k,A}$, see [7], and for the existence and uniqueness of $C'_{k,A}$-capacitary function of a set, see [8].

M denotes the vector space of Radon measures. M_1 is the Banach space of measures equipped with the norm $||\mu||$ = total variation of $\mu < \infty$. The cone of positive elements of M is denoted by M^+.

F will stand for the σ-field of sets which are μ-measurable for all $\mu \in M_1^+$.

If $\mu \in M_1^+$, we say that μ is concentrated on X if $\mu(Y) = 0$ for all sets Y which are μ-measurable and such that $Y \subset^c X$.

Let A and A^* be two conjugate N-functions. For $X \in F$, we define

$$D_{k,A}(X) = \sup\{||\mu|| : \ \mu \in M_1^+, \ \mu \text{ concentrated on } X \text{ and } ||k * \mu||_{A^*} \leq 1\},$$

where $k*\mu$ is the convolution of k and μ defined by $(k*\mu)(x) = \int k(x-y) \, \mathrm{d}\mu(y)$.

A measure $\mu \in M_1^+$ such that μ is concentrated on X and $||k * \mu||_{A^*} \leq 1$ is called a *test measure* for $D_{k,A}(X)$. If in addition $D_{k,A}(X) = ||\mu||$, we say that μ is a $D_{k,A}$*-capacitary distribution*, and $k * \mu$ a $D_{k,A}$*-capacitary potential* for X. For the properties of $D_{k,A}$ and the existence of a $D_{k,A}$-capacitary distribution, see [7], [8].

2.3. Bessel kernels.

For $m > 0$, the *Bessel kernel* G_m is most easily defined through its Fourier transform $F(G_m)$ as:

$$[F(G_m)](x) = (2\pi)^{-\frac{N}{2}} (1 + |x|^2)^{-\frac{m}{2}},$$

where $[F(f)](x) = (2\pi)^{-\frac{N}{2}} \int f(y)\, e^{-ixy}\, dy$ for $f \in \mathbf{L}^1$.

G_m is positive in $\mathbf{L}^1$ and satisfies the equality: $G_{r+s} = G_r * G_s$.

We put in the sequel $B_{m,A} = C_{G_m,A}$ and $B'_{m,A} = C'_{G_m,A}$.

3. On the continuity of potentials

3.1. Radially decreasing convolution kernels.

We recall the definition of radially decreasing convolution kernel.

DEFINITION 3.1. A function g defined on $\mathbb{R}^N \times \mathbb{R}^N$ is a *radially decreasing convolution kernel* if $g(x,y) = g_0(|x-y|)$, where g_0 is a positive lower semicontinuous, non-increasing function on $\mathbb{R}^+$ and such that $\int_0^1 g_0(t)t^{N-1}\, dt < \infty$.

We put $g(x) = g_0(|x|)$ for $x \in \mathbb{R}^N$.

THEOREM 3.1. *Let g be a radially decreasing convolution kernel and A any N-function. Then*

1) $|||g|||_{A^*} < \infty$ *implies that* $C_{g,A}(\{x\}) > 0$ *for any* $x \in \mathbb{R}^N$.
2) $|||g|||_{A^*,\{|x|>1\}} = \infty$ *implies that* $C_{g,A}(X) = 0$ *for all* X.
3) $|||g|||_{A^*,\{|x|>1\}} < \infty$ *implies that* $|X| = 0$ *whenever* $C_{g,A}(X) = 0$.

Proof. We follow the method in [1] for Lebesgue case.

1) Let g be such that $|||g|||_{A^*} < \infty$ and set $x = 0$. Let f be a test function for $C_{g,A}(\{0\})$. Then Hölder inequality in Orlicz spaces gives

$$1 \le \int gf\, dx \le |||f|||_A \|g\|_{A^*}.$$

This implies $C'_{g,A}(\{0\}) \ge [\|g\|_{A^*}]^{-1} > 0$.

2) Let g be such that $|||g|||_{A^*,\{|x|>1\}} = \infty$. It suffices to show that $C_{g,A}(B) = 0$ for the unit ball $B = B(0,1)$, since any set X can be covered by a countable number of balls with radius 1, having the same capacity because of the translation invariance of capacity for a convolution kernel. If the positive measure μ is supported on B, then: $g * \mu(x) \ge g_0(|x|+1)\mu(B)$.

Hence $|||g * \mu|||_{A^*} = \infty$.

[7; Theorem 11] gives $C_{g,A}(B) = 0$.

3) Let g be such that $|||g|||_{A^*,\{|x|>1\}} < \infty$. It suffices to consider measurable sets and show that $|X \cap B| = 0$ for the unit ball. Let $Y = X \cap B$ and $f \in \mathbf{L}_A^+$ be such that $g * f \ge 1$ on Y. Then

$$|Y| \le \int_Y g * f\, dx = \int_{\mathbb{R}^N} f(g * \chi_Y)\, dx \le 2|||f|||_A |||g * \chi_Y|||_{A^*},$$

where χ_Y is the characteristic function of Y.

We have $g*\chi_Y(x) \le Cg\left(\frac{x}{2}\right)$ when $|x| \ge 2$, and $g*\chi_Y(x) \le C$ when $|x| < 2$, when C is a constant independent of Y.

Hence $|X \cap B| \le CC'_{g,A}(X)$.

The proof is complete. □

THEOREM 3.2. *Let A be an N-function and $g(x) = g_0(|x|)$ be a radially decreasing convolution kernel. Suppose that g is continuous on $\mathbb{R}^N \setminus \{0\}$, satisfies $|||g|||_{A^*,\{|x|>1\}} < \infty$ and that there exist H and δ positive such that*

$$g_0(r) \le Hg_0(2r) \qquad \textit{for} \quad 0 < r \le \delta .$$

Let $f \in \mathbf{L}_A^+$ be such that $g*f \ge 1$ a.e. on an open set O. Then $g*f \ge 1$ everywhere on O.

Proof. We follow the idea given in [1]. We can assume, without loss of generality, that $g*f \ge 1$ a.e. on a neighborhood of 0, and prove that $g*f(0) \ge 1$. We can suppose that $g*f(0) < \infty$.

Let $0 < d < b$ and define a function η by

$$\eta(x) = \begin{cases} \dfrac{g(x)}{\int\limits_{|y|<|x|} g(y)\, \mathrm{d}y} & \text{for } d < |x| < b, \\ 0 & \text{otherwise.} \end{cases}$$

We put $G(r) = \int\limits_{|y|<r} g(y)\, \mathrm{d}y$ and obtain $\int\limits_{\mathbb{R}^N} \eta \, \mathrm{d}x = \log G(b) - \log G(d)$.

But $\lim\limits_{d\to 0} G(d) = 0$, so for an arbitrarily small b, we can choose d so that $\int\limits_{\mathbb{R}^N} \eta \, \mathrm{d}x = 1$.

Hence for b small enough, we obtain: $1 \le \int\limits_{\mathbb{R}^N} \eta(g*f)\, \mathrm{d}x = \int\limits_{\mathbb{R}^N} (\eta*g) f \, \mathrm{d}y$.

We fix ρ so that $0 < \rho \le \delta$. Then

$$\lim_{b\to 0} \eta * g(y) = \lim_{b\to 0} \int\limits_{\mathbb{R}^N} \eta(x) g(y-x)\, \mathrm{d}x = g(y) , \qquad \text{uniformly for} \quad |y| \ge \rho .$$

We get $\int\limits_{\mathbb{R}^N} \eta(x) g(y-x)\, \mathrm{d}x \le g_0\big(|y| - b\big)$.

Hence, for any $S < \infty$, we obtain $\lim\limits_{b\to 0} \int\limits_{\rho\le|y|\le S} (\eta*g) f \, \mathrm{d}y = \int\limits_{\rho\le|y|\le S} gf \, \mathrm{d}y$.

On the other hand, Hölder inequality in Orlicz spaces gives $\int\limits_{|y|\ge S} (\eta*g) f \, \mathrm{d}y \le 2|||f|||_A |||g|||_{A^*,\{|x|>S-b\}} < \varepsilon$ for S large enough.

We may estimate $\eta * g(y)$ for $|y| \le \rho$.

Remark that for $|x-y| \le \frac{|y|}{2}$, we get $|x| \ge \frac{|y|}{2}$ and $|x| \ge |x-y|$. So, by the monotonicity of g

$$\int\limits_{|x-y|\le\frac{|y|}{2}} \eta(x)g(y-x)\ \mathrm{d}x = \int\limits_{|x-y|\le\frac{|y|}{2}} \frac{g(x)g(y-x)}{\int\limits_{|t|<|x|} g(t)\ \mathrm{d}t}\ \mathrm{d}x$$

$$\le g\Big(\frac{y}{2}\Big)\int\limits_{|x-y|\le\frac{|y|}{2}} \frac{g(y-x)}{\int\limits_{|t|<|x|} g(t)\ \mathrm{d}t}\ \mathrm{d}x \le g\Big(\frac{y}{2}\Big)\,.$$

Also the monotonicity of g gives $\int\limits_{|x-y|\ge\frac{|y|}{2}} \eta(x)g(y-x)\ \mathrm{d}x \le g(\frac{y}{2})$.

By hypothesis, if $|y| \le 2\delta$, then $\eta * g(y) \le 2Hg(y)$, and thus

$$\int\limits_{|y|\le\rho} (\eta * g)f\ \mathrm{d}y \le 2H \int\limits_{|y|\le\rho} gf\ \mathrm{d}y < \varepsilon$$

if ρ is small enough.

If we let $b \to 0$, we get $1 \le \int\limits_{\rho\le|y|\le S} gf\ \mathrm{d}y + 2\varepsilon$.

We obtain the theorem by letting $\rho \to 0$ and $S \to \infty$. □

As a consequence, we give the following corollary when the considered kernel is Bessel.

Let A be an N-function and $m > 0$. Define the space of Bessel potentials $\mathbf{L}_{m,A}$ by $\mathbf{L}_{m,A} = \{\psi = G_m * f:\ f \in \mathbf{L}_A\}$, and a norm on $\mathbf{L}_{m,A}$ by $|||\psi|||_{m,A} = |||f|||_A$ if $\psi = G_m * f$.

For $E \subset \mathbb{R}^N$, we pose $F_{m,A}(E) = \inf\{|||\psi|||_{m,A}:\ \psi \in \mathbf{L}_{m,A},\ \psi(x) \ge 1$ a.e. on some neighborhood of $E\}$.

COROLLARY 3.3. *Let A be an N-function satisfying the Δ_2-condition and let $m > 0$. Suppose that $m < \frac{N}{\alpha}$, where $\alpha = \alpha(A) = \sup \frac{ta(t)}{A(t)}$. Let $E \subset \mathbb{R}^N$. Then*

$$B'_{m,A}(E) = F_{m,A}(E)\,.$$

P r o o f . We note that G_m is a radially decreasing convolution kernel, that G_m is continuous on $\mathbb{R}^N \setminus \{0\}$, and that there exist H and δ positive such that $G_m(r) \le HG_m(2r)$ for $0 < r \le \delta$.

On the other hand, recall that if β is the conjugate of α, i.e. $\alpha^{-1} + \beta^{-1} = 1$, then (see [10]) $(\forall\, t \le 1)\big(A^*(t) \le A^*(1)t^\beta\big)$.

We must show that $|||G_m|||_{A^*,\{|x|>1\}} < \infty$.

We know that there is a constant B such that $(\forall\, x)\big(G_m(x) \le B|x|^{m-N}\big)$.

Hence $\int_{\{|x|>1\}} A^*(G_m(x))\,\mathrm{d}x \le \int_{\{|x|>1\}} A^*(B|x|^{m-N})\,\mathrm{d}x$.

By changes of variables, there are constants $C_1 > 0$ and $0 < B' < 1$ such that $\int_{\{|x|>1\}} A^*(B|x|^{m-N})\,\mathrm{d}x \le C_1 \int_{B'}^{\infty} A^*(t^{m-N})t^{N-1}\,\mathrm{d}t$.

But $C_1 \int_{B'}^{1} A^*(t^{m-N})t^{N-1}\,\mathrm{d}t$ is a constant C_2 and

$$\int_1^{\infty} A^*(t^{m-N})t^{N-1}\,\mathrm{d}t \le A^*(1)\int_1^{\infty} t^{(m-N)\beta}t^{N-1}\,\mathrm{d}t = C\,.$$

Hence, there is a constant C' such that $\int_{\{|x|>1\}} A^*(G_m(x))\,\mathrm{d}x \le C'$.

This implies $|||G_m|||_{A^*,\{|x|>1\}} \le \sup(1, C') < \infty$.

Let $\psi \in \mathbf{L}_{m,A}$ be such that $\psi(x) \ge 1$ a.e. on an open set U containing E. Then $\psi = G_m * f$ for an $f \in \mathbf{L}_A$ and $|||\psi|||_{m,A} = |||f|||_A$.

We have also $G_m * f^+ \ge 1$ a.e. on U. By Theorem 3.2, $G_m * f^+ \ge 1$ on U, and thus $B'_{m,A}(U) \le |||f|||_A$.

By [7; Théorème 2], $B'_{m,A}$ is an outer capacity. Hence $B'_{m,A}(E) \le F_{m,A}(E)$. The opposite inequality is obvious and the proof is complete. □

3.2. The boundedness principle.

The following proposition can be found in [2]. To make the paper self-consistent, we give the proof.

PROPOSITION 3.4. ([2]) *Let g be a radially decreasing convolution kernel and let $\mu \in M^+$. Then there is a constant Q, depending only on N, such that for all $x \in \mathbb{R}^N$*

$$g * \mu(x) \le Q \sup_{y \in \operatorname{supp}\mu} g * \mu(y)\,.$$

Proof. We can assume that $\sup_{y\in\operatorname{supp}\mu} g * \mu(y) = 1$. Let $x \notin \operatorname{supp}\mu$, and let $\Gamma_1, \Gamma_2, \dots, \Gamma_Q$ be closed circular cones with vertices at x and total angular opening at the vertex of $\frac{\pi}{3}$, such that $\bigcup_{i=1}^{Q} \Gamma_i = \mathbb{R}^N$.

Let μ_i be the restriction of μ to Γ_i. Let x_i be a point of $\operatorname{supp}\mu_i$ such that $|x - x_i| = \operatorname{dist}(x, \operatorname{supp}\mu_i)$. Let Π_i be the perpendicular bisector of the line segment from x to x_i, and let Π_i^+ and Π_i^- be the halfspaces determined by Π_i. By elementary geometry, it is seen that if $x_i \in \Pi_i^-$, then $\operatorname{supp}\mu_i \subset \Pi_i^-$. Hence, if $y \in \operatorname{supp}\mu_i$, then $|y - x_i| \le |y - x|$. This implies that $g(x - y) \le g(x_i - y)$, and thus $g * \mu_i(x) \le g * \mu_i(x_i) \le 1$.

Hence $g * \mu(x) \leq \sum g * \mu_i(x) \leq Q$.
This completes the proof. □

Recall the so called *boundedness principle* in the case of Lebesgue L^p spaces.

THEOREM 3.5. ([2]) *Let g, μ and Q as in Proposition 3.4. Let $p > 1$ and p' its conjugate. Define the associated potential to μ by $V^{\mu}_{g,p}(x) = g*(g*\mu)^{p'-1}(x)$.*
Then for all $x \in \mathbb{R}^N$, $V^{\mu}_{g,p}(x) \leq \max\{Q, Q^{p'-1}\} \sup_{y \in \operatorname{supp} \mu} V^{\mu}_{g,p}(y)$.

Now we introduce a principle, which we also call a *boundedness principle* in the case of Orlicz spaces $\mathbf{L}_A$ such that A^* satisfies the Δ_2-condition.

Recall that if A satisfies the Δ_2-condition, then for all $Q > 1$, there is a constant $C(Q)$ such that for all x, $a(Qx) \leq C(Q)a(x)$.

Let $\alpha = \alpha(A) = \sup \frac{ta(t)}{A(t)}$.

DEFINITION 3.2. Let A be an N-function, $\mu \in M^+$ and let g be a kernel. We define the associated potential to μ by $V^{\mu}_{g,A}(x) = g * a^{-1}(g * \mu)(x)$.

We call this potential a *strongly nonlinear potential.*

We have the following result:

THEOREM 3.6 (BOUNDEDNESS PRINCIPLE). *Let g, μ and Q be as in Proposition 3.4 and let A be an N-function such that A^* satisfies the Δ_2-condition. Define the strongly nonlinear potential $V^{\mu}_{g,A}(x) = g*a^{-1}(g*\mu)(x)$. Then for all $x \in \mathbb{R}^N$*

$$V^{\mu}_{g,A}(x) \leq QC(Q)\alpha(A^*) \sup_{y \in \operatorname{supp} \mu} V^{\mu}_{g,A}(y).$$

Proof. Let $x \notin \operatorname{supp} \mu$, and let $x_0 \in \operatorname{supp} \mu$ minimize the distance from x to $\operatorname{supp} \mu$. We suppose that $\operatorname{supp} \mu \subset \Pi^+$, one of the halfspaces determined by the perpendicular bisector Π of the segment from x to x_0. If a point y_- belongs to the halfspace Π^-, we denote its reflected point in Π by $y_+ \in \Pi^+$. Then, for all $z \in \operatorname{supp} \mu$, we get $|z - y_-| > |z - y_+|$. Thus $g * \mu(y_-) \leq g * \mu(y_+)$, and by monotony, $f(y_-) \leq f(y_+)$, where $f(y) = a^{-1}(g * \mu)(y)$.

Now we claim that $g * f(x_-) \leq g * f(x_+)$ for all $x_- \in \Pi^-$.

We first suppose that all terms below are finite. Then the claim holds if and only if

$$\int_{\Pi^-} [g(x_- - y) - g(x_+ - y)] f(y)\,\mathrm{d}y \leq \int_{\Pi^+} [g(x_+ - y) - g(x_- - y)] f(y)\,\mathrm{d}y \quad (\star)$$

for all $x_- \in \Pi^-$.

Note that $g(x_- - y_-) - g(x_+ - y_-) = g(x_+ - y_+) - g(x_- - y_+)$.

For $y_- \in \Pi^-$, the inequality $|x_- - y_-| \le |x_+ - y_-|$ implies

$$g(x_- - y_-) - g(x_+ - y_-) = g(x_+ - y_+) - g(x_- - y_+) \ge 0.$$

Hence $f(y_-)[g(x_- - y_-) - g(x_+ - y_-)] \le f(y_+)[g(x_+ - y_+) - g(x_- - y_+)]$.

We obtain $(\star)$ by integrating the members of this inequality over points and their reflections.

If one or more terms in $(\star)$ are infinite, we replace g by a truncated kernel, for example by g_n defined by $g_n(x) = \max\{0, \min\{g(x) - n^{-1}, n\}\}$ for $n = 1, \dots$, and apply monotone convergence.

In the case of arbitrary measure $\mu \in M^+$, we choose $x \notin \operatorname{supp}\mu$, and consider the subdivision $\bigcup_{i=1}^{Q} \Gamma_i = \mathbb{R}^N$ as in Proposition 3.4. Let μ_i and x_i be as in this Proposition. Then we get $V^{\mu_i}_{g,A}(x) = V^{\mu_i}_{g,A}(x_i)$.

Now we have

$$a^{-1}(g*\mu)(x) \le a^{-1}\left[\sum_{i=1}^{Q}(g*\mu_i)(x)\right] = a^{-1}\left[Q\sum_{i=1}^{Q}Q^{-1}(g*\mu_i)(x)\right].$$

Since A^* satisfies the Δ_2-condition, there is a constant $C(Q)$ such that for all x, $a^{-1}(Qx) \le C(Q)a^{-1}(x)$.

Moreover, from the inequality $xa^{-1}(x) \le \alpha(A^*)A^*(x)$, we obtain

$$\begin{aligned} &a^{-1}\left[Q\sum_{i=1}^{Q}Q^{-1}(g*\mu_i)(x)\right] \\ &\le QC(Q)\alpha(A^*)A^*\left[\sum_{i=1}^{Q}Q^{-1}(g*\mu_i)(x)\right]\left[\sum_{i=1}^{Q}(g*\mu_i)(x)\right]^{-1}. \end{aligned}$$

The convexity of A^* yields

$$a^{-1}\left[Q\sum_{i=1}^{Q}Q^{-1}(g*\mu_i)(x)\right] \le C(Q)\alpha(A^*)\left[\sum_{i=1}^{Q}A^*(g*\mu_i)(x)\right]\left[\sum_{i=1}^{Q}(g*\mu_i)(x)\right]^{-1}.$$

The inequalities $A^*(x) \le xa^{-1}(x)$, and

$$\left[\sum_{i=1}^{Q}(g*\mu_i)(x)a^{-1}(g*\mu_i)(x)\right]\left[\sum_{i=1}^{Q}(g*\mu_i)(x)\right]^{-1} \le \left[\sum_{i=1}^{Q}a^{-1}(g*\mu_i)(x)\right]$$

imply $a^{-1}(g*\mu)(x) \le C(Q)\alpha(A^*)\left[\sum_{i=1}^{Q}a^{-1}(g*\mu_i)(x)\right]$.

Hence $V^{\mu}_{g,A}(x) \le C(Q)\alpha(A^*) \sum_{i=1}^{Q} V^{\mu_i}_{g,A}(x) \le QC(Q)\alpha(A^*) \sup_{y\in \operatorname{supp}\mu} V^{\mu}_{g,A}(y)$.
This completes the proof. □

Remark that Theorem 3.6 remains valid if we deal with a strongly nonlinear potential of the form $g_1 * a^{-1}(g_2 * \mu)$, where both g_1 and g_2 are radially decreasing convolution kernels.

Note also that for strongly nonlinear potential, the constant $QC(Q)\alpha(A^*)$ is perhaps not the best possible. This fact is of no importance for us, since all we need is an inequality of the form $V^{\mu}_{g,A}(x) \le K \sup_{y\in \operatorname{supp}\mu} V^{\mu}_{g,A}(y)$, where K is a constant depending only on N and A.

COROLLARY 3.7.

1) *Let A be an N-function such that A and A^* satisfy the Δ_2-condition. Let g be a radially decreasing convolution kernel and let X be an analytic set such that $C_{g,A}(X) < \infty$. Then the $C_{g,A}$-capacitary potential $g * f$ is a bounded function on $\mathbb{R}^N$.*

2) *Let A be an uniformly convex N-function for the Luxemburg norm, satisfying the Δ_2-condition. Let g be a radially decreasing convolution kernel and let X be any set such that $C_{g,A}(X) < \infty$. Then the $C_{g,A}$-capacitary potential $g * f$ is a bounded function on $\mathbb{R}^N$.*

Proof.

1) From [4] (see also [8; Théorème 6]), we have $g * f \le 1$ on $\operatorname{supp}\gamma$, where γ is the $D_{g,A}$-capacitary distribution measure for X and f the $C_{g,A}$-capacitary function of X. Moreover, we have the following relation

$$g * \gamma = \left[\left\| a \circ \left(\frac{f}{|||f|||_A} \right) \right\|_{A^*} \right]^{-1} a \circ \left(\frac{f}{|||f|||_A} \right) \qquad \text{a.e.}$$

This implies $|||f|||_A [g * a^{-1}(g * \gamma')] = g * f$, where $\gamma' = \left[\left\| a \circ \left(\frac{f}{|||f|||_A} \right) \right\|_{A^*} \right] \gamma$.

Theorem 3.6 gives

$$g * f(x) \le C \sup_{y\in \operatorname{supp}\gamma'} \{|||f|||_A [g * a^{-1}(g * \gamma')](y)\} \le C.$$

2) Let X be any set such that $C_{g,A}(X) < \infty$, and let f be the $C_{g,A}$-capacitary function of X. From [8; Théorème 3], there is a sequence of open sets $(O_i)_i$ for which the sequence of capacitary functions $(f_i)_i$ converges strongly to f in $\mathbf{L}_A$. For a subsequence, the semicontinuity of positive functions gives $g * f(x) \le \liminf g * f_i(x) \le C$.

The proof is finished. □

DEFINITION 3.3. Let A be an N-function and let g be a radially decreasing convolution kernel. On F define $V_{g,A}$ and $W_{g,A}$ by

$$\begin{aligned} V_{g,A}(X) = \sup\{\|\mu\| :\ \mu \in M^+,\ \mu \text{ concentrated on } X \text{ and} \\ (\forall x)\big(g * a^{-1}(g*\mu)(x) \le 1\big)\}, \\ W_{g,A}(X) = \sup\{\|\mu\| :\ \mu \in M^+,\ \mu \text{ concentrated on } X \text{ and} \\ (\forall x \in \operatorname{supp}\mu)\big(g * a^{-1}(g*\mu)(x) \le 1\big)\}. \end{aligned}$$

PROPOSITION 3.8. *$V_{g,A}$ and $W_{g,A}$ are inner capacities.*

Proof. The proof is identical with that given in [7; Théorème 10]. □

THEOREM 3.9. *Let A be an N-function such that A and A^* satisfy the Δ_2-condition. Let g be a radially decreasing convolution kernel and let $X \in F$. Then there is a constant C, depending only on A and N, such that*

$$V_{g,A}(X) \le W_{g,A}(X) \le C V_{g,A}(X).$$

Proof. It is obvious that $V_{g,A}(X) \le W_{g,A}(X)$. We must prove the last inequality.

Let $\mu \in M^+$ be concentrated on X and such that $g * a^{-1}(g*\mu)(x) \le 1$ for all x in $\operatorname{supp}\mu$. Theorem 3.6 gives a constant $C > 1$ depending only on A and N such that: $(\forall x \in \mathbb{R}^N)\big(g * a^{-1}(g*\mu)(x) \le C\big)$.

This means that $(\forall x \in \mathbb{R}^N)\big(g * C^{-1}a^{-1}(g*\mu)(x) \le 1\big)$.

On the other hand, remark that if $0 < C' < 1$, then there is a C'' such that for all t, $C'a^{-1}(t) \ge a^{-1}(C''t)$.

In fact, put $C'a^{-1}(t) = y$. Then $t = a(C'^{-1}y)$, and since A satisfies the Δ_2-condition, there is a constant K depending only on C' such that $t \le Ka(y)$. This implies that $tK^{-1} < a(y)$. So $a^{-1}(tK^{-1}) \le C'a^{-1}(t)$.

Hence, there is a constant K' such that $(\forall x \in \mathbb{R}^N)\big(g*a^{-1}(g*K'\mu)(x) \le 1\big)$.

Thus $\gamma = K'\mu$ is a positive measure concentrated on X and such that

$$(\forall x \in \mathbb{R}^N)\big(g * a^{-1}(g*\gamma)(x) \le 1\big).$$

Whence $W_{g,A}(X) \le K'V_{g,A}(X)$.

The proof is complete. □

3.3 The continuity principle.

The continuity principle for the nonlinear potential theory has been established by Havin and Maz'ya [9] for Riesz kernels. We propose an extension to the strongly nonlinear case.

THEOREM 3.10 (CONTINUITY PRINCIPLE). *Let A be an N-function such that A^* satisfies the Δ_2-condition. Let g be a radially decreasing convolution kernel continuous on $\mathbb{R}^N \setminus \{0\}$, and let $\mu \in M^+$ be a measure with compact support K. Suppose that the restriction of $V^{\mu}_{g,A}$ to K belongs to $C(K)$. Then $V^{\mu}_{g,A}$ is continuous in $\mathbb{R}^N$.*

Proof. Let $f(y) = a^{-1}(g * \mu)(y)$. Then, by Dini's Theorem on monotone convergence, the integral $g * f(x)$ converges uniformly on K in the sense that for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\int\limits_{|x-y|<\delta} g(x-y)f(y)\,\mathrm{d}y < \varepsilon \qquad \text{for all} \quad x \in K .$$

Let the kernel h_δ be defined by: $h_\delta(x) = g(x)$ for $|x| < \delta$, and $h_\delta(x) = 0$ otherwise.

From Theorem 3.6 and the remark following it applied to the kernels g and h_δ, there is a constant C depending only on A and N, such that

$$h_\delta * f(x) = \int\limits_{|x-y|<\delta} g(x-y)f(y)\,\mathrm{d}y < C\varepsilon \qquad \text{for all} \quad x \in \mathbb{R}^N .$$

Let $x_0 \in K$ and $(x_n)_n$ be a sequence such that $x_n \to x$. Then, by the continuity of g away from 0,

$$\limsup_{n\to\infty} V^{\mu}_{g,A}(x_n) \le \left[(g - h_\delta) * f\right](x_0) + C\varepsilon \le V^{\mu}_{g,A}(x_0) + C\varepsilon .$$

Since $V^{\mu}_{g,A}$ is semicontinuous, we get $V^{\mu}_{g,A}(x_0) \le \liminf\limits_{n\to\infty} V^{\mu}_{g,A}(x_n)$.

This implies that $V^{\mu}_{g,A}$ is continuous at all $x \in K$. The continuity off K is a consequence of the continuity of g.

The proof is finished. □

THEOREM 3.11. *Let A be an N-function such that A and A^* satisfy the Δ_2-condition. Let g be a radially decreasing convolution kernel continuous on $\mathbb{R}^N \setminus \{0\}$. Let K be a compact set such that $C_{g,A}(K) > 0$. Then there is a non null measure $\mu \in M^+(K)$ such that the potential $g * \left[a^{-1} \circ (g * \mu)\right]$ is continuous in $\mathbb{R}^N$.*

Proof. From the hypothesis it follows that there is a nonzero measure $\mu \in M^+(K)$ such that: $(\forall x \in K)\left(g * \left[a^{-1} \circ (g * \mu)\right] \le 1\right)$.

Pose $f(y) = a^{-1}(g * \mu)(y)$. Egorov's Theorem gives a compact $K' \subset K$ such that $\mu(K') > \frac{1}{2}\mu(K)$ and $g * f(x)$ converges uniformly on K'.

Denote by μ' the restriction of μ to K'. Then the integral $g * \left[a^{-1} \circ (g * \mu')\right](x)$ converges uniformly on K'. It follows as in the proof of Theorem 3.10, that $g * \left[a^{-1} \circ (g * \mu')\right]$ is continuous.

The proof is finished. □

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