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DENSITIES IN DISJOINT UNIONS

GEORGES GREKOS

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ABSTRACT. Let A, B, C be sets of positive integers such that $A \cap B = \emptyset$ and $A \cup B = C$. We establish necessary and sufficient conditions satisfied by the lower and upper asymptotic densities of the three sets.

Let A be an infinite subset (sequence) of $\mathbb{N} = \{1, 2, 3, \dots\}$. The same symbol A will denote the *counting function* of the set; that is, for each integer n, we let $A(n)$ be the number of elements of A not exceeding n. We define the *lower* and the *upper asymptotic densities* of A as

$$\alpha' = \underline{d}A = \liminf_{n\to+\infty} \frac{A(n)}{n},$$
$$\alpha = \overline{d}A = \limsup_{n\to+\infty} \frac{A(n)}{n}.$$

For sets B and C of positive integers we denote by β', β and γ', γ the corresponding lower and upper densities, respectively.

Suppose that A and B are disjoint and let $C = A \cup B$. Then $C(n) = A(n) + B(n)$ for all n. It is easy to prove that the following two conditions are valid:

$$\alpha' + \beta' \le \gamma' \le \min\{\alpha' + \beta, \alpha + \beta'\}, \tag{C.1}$$
$$\max\{\alpha' + \beta, \alpha + \beta'\} \le \gamma \le \alpha + \beta. \tag{C.2}$$

In this note we establish the sufficiency of these conditions.

THEOREM. *Given six real numbers* $\alpha', \alpha, \beta', \beta, \gamma', \gamma$ *such that* $0 \le \alpha' \le \alpha \le 1$, $0 \le \beta' \le \beta \le 1$, $0 \le \gamma' \le \gamma \le 1$, *satisfying the conditions* (C.1) *and* (C.2), *there*

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exist subsets A, B, C *of* $\mathbb{N}$ *such that* $A \cap B = \emptyset$, $C = A \cup B$, *and* $\underline{d}A = \alpha'$, $\overline{d}A = \alpha$, $\underline{d}B = \beta'$, $\overline{d}B = \beta$, $\underline{d}C = \gamma'$, $\overline{d}C = \gamma$.

Remark. If $\mathbb{N}$ is replaced by the interval $[0,1[$, the upper density by the exterior Lebesgue measure and the lower density by the interior Lebesgue measure, then, as it was pointed out by Max Shiffman [1], the conditions (C.1) and (C.2) are necessary but not sufficient. In that case, in order to obtain a complete set of necessary and sufficient conditions, one has to add the following inequality:

$$\alpha + \beta - \gamma \geq \gamma' - \alpha' - \beta'. \tag{C.3}$$

Proof of the theorem. First we shall define on $[0,+\infty[$ two real increasing and continuous functions a and b, taking values in $[0,+\infty[$, such that

$$\begin{aligned} \liminf_{x\to+\infty} \frac{a(x)}{x} &= \alpha', & \limsup_{x\to+\infty} \frac{a(x)}{x} &= \alpha, \\ \liminf_{x\to+\infty} \frac{b(x)}{x} &= \beta', & \limsup_{x\to+\infty} \frac{b(x)}{x} &= \beta, \\ \liminf_{x\to+\infty} \frac{c(x)}{x} &= \gamma', & \limsup_{x\to+\infty} \frac{c(x)}{x} &= \gamma, \end{aligned} \tag{1}$$

where $c = a + b$. In the second part of the proof, we determine two disjoint sets A and B having counting functions neighbouring a and b.

First part of the proof.

We define sequences of abscissas

$$1 = x_1 = y_1 = z_1 = w_1 < x_2 < y_2 < z_2 < w_2 < \dots$$
$$\dots < x_n < y_n < z_n < w_n < x_{n+1} < \dots$$

tending to infinity, and the two functions a and b as follows.

Firstly, it is easy to find two real numbers $a(1)$ and $b(1)$, belonging to $[0,1]$ such that

$$\alpha' \leq a(1) \leq \alpha, \qquad \beta' \leq b(1) \leq \beta \qquad \text{and} \qquad \gamma' \leq a(1) + b(1) \leq \gamma.$$

To see this, let us observe that when x varies from α' to α and y from β' to β, then $x+y$ varies from $\alpha'+\beta'$ to $\alpha+\beta$. As $\alpha'+\beta' \leq \gamma' \leq \gamma \leq \alpha+\beta$, it is possible to choose $a(1) = x$, $b(1) = y$ such that $a(1) + b(1) = \frac{\gamma+\gamma'}{2}$, $a(1) \in [\alpha', \alpha]$ and $b(1) \in [\beta', \beta]$.

Functions a and b are defined on $[0,1]$ as linear functions:

$$a(t) = t a(1), \quad b(t) = t b(1), \qquad 0 \leq t \leq 1.$$

These two functions will be continuous on $[0,+\infty[$ and affine on each interval $[1, x_2], [x_2, y_2], \dots, [w_n, x_{n+1}], \dots$ and so on. The reader may find it helpful to see the slopes at each interval from the following table.

TABLE.

Slope of the functions			Between abscissas		
a	b	$a+b$			
				$\cdots$	
α'	$\gamma'-\alpha'$	γ'	w_{k-1}	and	x_k
$\gamma'-\beta'$	β'	γ'	x_k	and	y_k
α	$\gamma-\alpha$	γ	y_k	and	z_k
$\gamma-\beta$	β	γ	z_k	and	w_k
α'	$\gamma'-\alpha'$	γ'	w_k	and	x_{k+1}
				$\cdots$	

The functions a and b are essentially determined by these slopes, and by the relations (2) and (3-1) to (3-4) below. We give full details only for the first interval $[w_{k-1}, x_k]$. The functions a and b will satisfy the conditions

$$\alpha' \leq \frac{a(t)}{t} \leq \alpha\,, \qquad \beta' \leq \frac{b(t)}{t} \leq \beta\,, \qquad \gamma' \leq \frac{c(t)}{t} \leq \gamma\,, \tag{2}$$

for any real number $t > 0$. In order to satisfy equalities (1), we shall require that, for each $n \geq 1$,

$$0 \leq \frac{a(x_n)}{x_n} - \alpha' \leq \frac{1}{n}\,, \qquad 0 \leq \frac{c(x_n)}{x_n} - \gamma' \leq \frac{1}{n}\,, \tag{3-1}$$

$$0 \leq \frac{b(y_n)}{y_n} - \beta' \leq \frac{1}{n}\,, \qquad 0 \leq \frac{c(y_n)}{y_n} - \gamma' \leq \frac{1}{n}\,, \tag{3-2}$$

$$0 \leq \alpha - \frac{a(z_n)}{z_n} \leq \frac{1}{n}\,, \qquad 0 \leq \gamma - \frac{c(z_n)}{z_n} \leq \frac{1}{n}\,, \tag{3-3}$$

$$0 \leq \beta - \frac{b(w_n)}{w_n} \leq \frac{1}{n}\,, \qquad 0 \leq \gamma - \frac{c(w_n)}{w_n} \leq \frac{1}{n}\,. \tag{3-4}$$

Conditions (3-1) to (3-4) obviously hold when $n = 1$. We suppose that they hold up to $n = k-1$, for some integer $k \geq 2$. For $w_{k-1} \leq t \leq x_k$, we define

$$a(t) = a(w_{k-1}) + (t - w_{k-1})\alpha'$$

and

$$b(t) = b(w_{k-1}) + (\gamma' - \alpha')(t - w_{k-1}),$$

and we choose x_k sufficiently large so that conditions (3-1) hold with $n = k$. We prove that the three inequalities in (2) are valid for t belonging to the interval $[w_{k-1}, x_k]$. We have

$$\frac{a(t)}{t} = \frac{a(w_{k-1})}{t} + \alpha' - \alpha' \frac{w_{k-1}}{t}$$

and

$$\alpha' w_{k-1} \leq a(w_{k-1}) \leq \alpha w_{k-1}.$$

Therefore

$$\frac{a(t)}{t} \geq \frac{\alpha' w_{k-1}}{t} + \alpha' - \alpha' \frac{w_{k-1}}{t} = \alpha',$$

and

$$\frac{a(t)}{t} - \alpha \leq \frac{\alpha w_{k-1}}{t} + \alpha' - \alpha' \frac{w_{k-1}}{t} - \alpha = (\alpha' - \alpha)\left(1 - \frac{w_{k-1}}{t}\right) \leq 0.$$

We also have

$$b(t) = b(w_{k-1}) + (\gamma' - \alpha')(t - w_{k-1})$$

and, by (C.1),

$$\beta' \leq \gamma' - \alpha' \leq \beta.$$

It follows that

$$\beta' t \leq b(w_{k-1}) + (t - w_{k-1})\beta' \leq b(t) \leq b(w_{k-1}) + (t - w_{k-1})\beta \leq \beta t,$$

and hence

$$\beta' \leq \frac{b(t)}{t} \leq \beta.$$

For t belonging to $[w_{k-1}, x_k]$, we have

$$c(t) = a(t) + b(t) = a(w_{k-1}) + b(w_{k-1}) + (t - w_{k-1})\gamma'$$

and the third inequality in (2) is deduced in the same manner as the first one. More precisely, we have

$$c(t) = c(w_{k-1}) + (t - w_{k-1})\gamma' \geq \gamma' w_{k-1} + (t - w_{k-1})\gamma' = t\gamma'$$

and also

$$c(t) \leq \gamma w_{k-1} + (t - w_{k-1})\gamma' - \gamma t + \gamma t = (t - w_{k-1})(\gamma' - \gamma) + \gamma t \leq \gamma t.$$

For $x_k \leq t \leq y_k$, we let

$$a(t) = a(x_k) + (\gamma' - \beta')(t - x_k),$$
$$b(t) = b(x_k) + (t - x_k)\beta'.$$

The real number y_k is chosen large enough to satisfy (3-2) with $n = k$. We have

$$\frac{b(t)}{t} = \frac{b(x_k)}{t} + \beta' - \beta' \frac{x_k}{t}$$

and

$$\beta' x_k \le b(x_k) \le \beta x_k .$$

Therefore

$$\frac{b(t)}{t} \ge \frac{\beta' x_k}{t} + \beta' - \beta' \frac{x_k}{t} = \beta' ,$$

and

$$\frac{b(t)}{t} - \beta \le \frac{\beta x_k}{t} + \beta' - \beta' \frac{x_k}{t} - \beta = (\beta' - \beta)\left(1 - \frac{x_k}{t}\right) \le 0 ,$$

because $x_k \le t$ and $\beta' \le \beta$. The definition of $a(t)$, for t belonging to the interval $[x_k, y_k]$, and the inequality

$$\alpha' \le \gamma' - \beta' \le \alpha ,$$

which is a consequence of (C.1), give

$$\alpha' t \le a(x_k) + (t - x_k)\alpha' \le a(t) \le a(x_k) + (t - x_k)\alpha \le \alpha t$$

and hence

$$\alpha' \le \frac{a(t)}{t} \le \alpha$$

for $x_k \le t \le y_k$. We also have

$$c(t) = a(t) + b(t) = a(x_k) + b(x_k) + (t - x_k)\gamma'$$

and we get that

$$t\gamma' \le c(t) \le t\gamma$$

for all t in $[x_k, y_k]$ in the same manner as for t belonging to $[w_{k-1}, x_k]$.

When $y_k \le t \le z_k$, we define a and b by

$$a(t) = a(y_k) + (t - y_k)\alpha ,$$
$$b(t) = b(y_k) + (\gamma - \alpha)(t - y_k) ,$$

choosing z_k sufficiently large, such that (3-3) with $n = k$ holds. Let us prove that inequalities (2) are valid for $t \in [y_k, z_k]$. We have

$$a(t) \le \alpha y_k + (t - y_k)\alpha = t\alpha$$

and

$$a(t) \ge \alpha' y_k + (t - y_k)\alpha = (t - y_k)(\alpha - \alpha') + \alpha' t \ge \alpha' t .$$

Also, by (C.2),

$$\beta' \le \gamma - \alpha \le \beta$$

and hence

$$b(t) \le b(y_k) + \beta(t - y_k) \le \beta y_k + \beta(t - y_k) = \beta t$$

and

$$b(t) \ge b(y_k) + \beta'(t - y_k) \ge \beta' y_k + \beta'(t - y_k) = \beta' t .$$

Adding $a(t)$ and $b(t)$, we get

$$c(t) = c(y_k) + \gamma(t - y_k)$$

and we easily verify the third inequality of (2).

Finally, for $z_k \le t \le w_k$, we put

$$a(t) = a(z_k) + (\gamma - \beta)(t - z_k) ,$$
$$b(t) = b(z_k) + (t - z_k)\beta ,$$

and we choose w_k sufficiently large enough, so that (3-4) is satisfied with $n = k$. Similarly we prove (2).

Thus we have defined recurrently the sequences $(x_n), (y_n), (z_n), (w_n)$ of abscissas and the two functions a and b verifying relations (1).

Second part of the proof.

In the second and last part of the proof, we explain how one can determine two disjoint sets A and B such that their counting functions $A(n)$ and $B(n)$, $n \in \mathbb{N}$, are close to a and b, respectively.

We note C the set defined recurrently as

$$C = \{ n \in \mathbb{N}; \ \ C(n-1) + 1 \le c(n) \} .$$

Thus for any integer $n \ge 1$, we have that $n \in C$ if and only if $C(n-1)+1 \le c(n)$. We recall that $c = a + b$.

Let us prove by induction that, for each $n \in \mathbb{N}$,

$$c(n) - 1 < C(n) \le c(n) . \tag{4}$$

The double inequality is valid when $n = 1$. Because $c(1) = \frac{\gamma+\gamma'}{2} \le 1$; if $c(1) = 1$, then $1 \in C$ and $C(1) = 1$; if $c(1) < 1$, then $1 \notin C$ and $C(1) = 0$. Now, suppose that (4) is valid up to k belonging to $\mathbb{N}$. We shall prove that (4) is also true for $n = k + 1$. We consider two cases:

(i) If $C(k) + 1 \le c(k+1)$, then, by the definition of C, $k + 1 \in C$ and $C(k+1) = C(k) + 1 \le c(k+1)$. On the other hand, $c(k) - 1 < C(k)$ implies $c(k) + 1 - 1 < C(k) + 1$. Thus

$$C(k+1) = C(k) + 1 > c(k) + 1 - 1 \ge c(k+1) - 1 .$$

The last inequality is equivalent to $c(k+1) - c(k) \le 1$, which is true because the nondecreasing piecewise linear continuous function c, for $x < y$, satisfies

$c(y) - c(x) \leq (y-x)\lambda$, where λ is the maximal angular coefficient of c on $[x, y]$; here $y = k+1$, $x = k$ and $\lambda \leq \gamma \leq 1$.

(ii) If $C(k) + 1 > c(k+1)$, then $k+1 \notin C$ and $C(k+1) = C(k) \leq c(k) \leq c(k+1)$, c being increasing. On the other hand, the first inequality of (4) follows directly from the hypothesis $C(k) + 1 > c(k+1)$ of the present case.

From (4) follows that

$$\overline{d}C = \limsup_{n \to +\infty} \frac{C(n)}{n} = \limsup_{n \to +\infty} \frac{c(n)}{n} = \gamma$$

and, similarly, $\underline{d}C = \gamma'$.

The set A is defined as a subset of C such that its counting function $A(n)$ is close to $a(n)$. Thus we stipulate that an integer $n \in \mathbb{N}$ shall be in A if and only if n is in C and $A(n-1) + 1 \leq a(n)$. We shall prove that, for each $n \in \mathbb{N}$,

$$a(n) - 2 < A(n) \leq a(n)\,. \tag{5}$$

Then this yields $\overline{d}A = \alpha$ and $\underline{d}A = \alpha'$. Let $B = C \setminus A$. It follows that, for each $n \in \mathbb{N}$, the quantity $B(n) = C(n) - A(n)$ satisfies

$$b(n) - 1 < B(n) < b(n) + 2\,,$$

so that $\overline{d}B = \beta$ and $\underline{d}B = \beta'$.

Now let us prove the inequality (5). It is obvious that $A(n) \leq a(n)$, so we have to prove only the first inequality in (5). There are integers y, $0 \leq y \leq n$, such that $a(y) - A(y) < 1$; for instance, $y = 0$. Call m the largest one:

$$m = \max\{y \in \mathbb{N} \cup \{0\}\,;\ y \leq n\,,\ a(y) - A(y) < 1\}\,.$$

If $m = n$, then $a(n) - A(n) < 1 < 2$, so that the first inequality in (5) holds. Suppose $m < n$. We have $a(y) - A(y) \geq 1$, that is $A(y) + 1 \leq a(y)$, for $y = m+1, \dots, n$. As $A(y-1) \leq A(y)$, it follows that $A(y-1) + 1 \leq a(y)$ for $y = m+1, \dots, n$. In view of the definition of the set A, this means that for $y = m+1, \dots, n$, we have that $y \in A$ if and only if $y \in C$. Therefore $C \cap\,]m, n] = A \cap\,]m, n]$ and hence $A(n) - A(m) = C(n) - C(m)$. We have

$$\begin{aligned} a(n) - A(n) &= a(n) - a(m) + a(m) - A(n) + A(m) - A(m) \\ &< 1 + a(n) - a(m) - \big(A(n) - A(m)\big) \\ &= 1 + a(n) - a(m) - \big(C(n) - C(m)\big)\,. \end{aligned}$$

The last member is less or equal to

$$1 + c(n) - c(m) - C(n) + C(m)$$

because $c = a + b$, so that

$$c(n) - c(m) - \big(a(n) - a(m)\big) = b(n) - b(m)$$

and b is increasing. Now, by (4), we conclude that

$$a(n) - A(n) \leq 1 + c(n) - c(m) - C(n) + C(m) \leq 1 + c(n) - C(n) < 2\,.$$

This completes the proof of inequality (5) and of the theorem. □

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Département de Mathématiques
Faculté des Sciences et Techniques
23, rue du Dr. Paul Michelon
F-42023 Saint-Étienne Cédex 2
FRANCE
E-mail: grekos@univ-st-etienne.fr