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ON GENERALIZED MEASURES OF FUZZY ENTROPY

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ABSTRACT. In the present paper, the existing measures of fuzzy entropy are reviewed. Two new generalized measures of fuzzy entropy are defined and characterized. A new measure of R-normed fuzzy entropy is introduced and characterized. Some generalized measures of fuzzy directed and symmetric divergence are studied and particular cases of the generalized and R-normed fuzzy entropies have also been obtained.

1. Introduction

The concept of entropy was developed to measure the uncertainty of a probability distribution. [15] introduced the concept of fuzzy sets and developed his own theory to measure the ambiguity of a fuzzy set. A fuzzy set A is represented as

$$A = \{x_i | \mu_A(x_i) : \ i = 1, 2, \dots, n\},$$

where $\mu_A(x)$ is a membership function defined as follows:

$$\text{if} \quad \mu_A(x) = \begin{cases} 0, & x \text{ does not belong to } A \text{ and there is no ambiguity,} \\ 1, & x \text{ belongs to } A \text{ and there is no ambiguity,} \\ 0.5, & \text{there is maximum ambiguity whether } x \text{ belongs} \\ & \text{to } A \text{ or not.} \end{cases}$$

In fact $\mu_A(x)$ associates with each $x \in \mathbb{R}^n$ a grade of membership to the set A.

In the area of pattern recognition, image processing, speech recognition, etc., it is often required to get some idea about the degree of fuzziness or ambiguity

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present in a fuzzy set. Thus if $x_1, x_2, \ldots, x_n$ are members of the universe of discourse, then all $\mu_A(x_1), \mu_A(x_2), \ldots, \mu_A(x_n)$ lie between 0 and 1, but these are not probabilities because their sum is not unity. However,

$$\Phi_A(x_i) = \frac{\mu_A(x_i)}{\sum_{i=1}^{n} \mu_A(x_i)}, \qquad i = 1, 2, \ldots, n, \tag{1.1}$$

is a probability distribution. Thus [7] defined entropy of a fuzzy set A having n support points by

$$H(A) = -\frac{1}{\log n} \sum_{i=1}^{n} \Phi_A(x_i) \log \Phi_A(x_i). \tag{1.2}$$

In fuzzy set theory, the entropy is a measure of fuzziness which expresses the amount of average ambiguity/difficulty in making a decision whether an element belongs to a set or not. A measure of fuzziness $H(A)$ in a fuzzy set should have at least the following properties:

(P-1) $H(A)$ is minimum if and only if A is a crisp set, i.e. $\mu_A(x) = 0$ or 1 for all x.

(P-2) $H(A)$ is maximum if and only if A is most fuzzy set, i.e. $\mu_A(x) = 0.5$ for all x.

(P-3) $H(A) \geq H(A^*)$, where A^* is a sharpened version of A.

(P-4) $H(A) = H(\bar{A})$, where $\bar{A}$ is the complement set of A.

Since $\mu_A(x)$ and $1 - \mu_A(x)$ give the same degree of fuzziness, therefore, corresponding to entropy due to [14], [3] suggested the following measure of fuzzy entropy:

$$H(A) = -\sum_{i=1}^{n} \left[\mu_A(x_i) \log \mu_A(x_i) + \left(1 - \mu_A(x_i)\right) \log\left(1 - \mu_A(x_i)\right)\right]. \tag{1.3}$$

As (1.3) satisfies all four properties (P-1) to (P-4), so it is a valid measure of fuzzy entropy. Later on [1] made a survey on information measures on fuzzy sets and gave some measures of fuzzy entropy. Corresponding to [11]'s entropy they suggested the following measure:

$$\frac{1}{1-\alpha} \sum_{i=1}^{n} \log\left[\mu_A^{\alpha}(x_i) + \left(1 - \mu_A(x_i)\right)^{\alpha}\right], \qquad \alpha \neq 1, \ \alpha > 1, \tag{1.4}$$

and corresponding to [10]'s exponential entropy they introduced

$$\frac{1}{n\sqrt{\mathrm{e}} - 1} \sum_{i=1}^{n} \log\left[\mu_A(x_i)\, \mathrm{e}^{1-\mu_A(x_i)} + \left(1 - \mu_A(x_i)\right) \mathrm{e}^{\mu_A(x_i)} - 1\right]. \tag{1.5}$$

[5] has given measure of fuzzy entropy corresponding to [4] as

$$H^\alpha(A) = (1-\alpha)^{-1} \sum_{i=1}^{n} \left[\mu_A^\alpha(x_i) + \left(1-\mu_A(x_i)\right)^\alpha - 1\right]. \tag{1.6}$$

Similarly corresponding to [6]'s measure of entropy

$$-\sum_{i=1}^{n} p_i \log p_i = \frac{1}{a}\sum_{i=1}^{n}(1+ap_i)\log(1+ap_i) - \frac{1}{a}(1+a)\log(1+a), \qquad a \geq 0, \tag{1.7}$$

we have the following measure of fuzzy entropy:

$$\begin{aligned} &-\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + \left(1-\mu_A(x_i)\right)\log\left(1-\mu_A(x_i)\right)\right] \\ &\qquad + \frac{1}{a}\sum_{i=1}^{n}\left(1+a\mu_A(x_i)\right)\log\left(1+a\mu_A(x_i)\right) \\ &\qquad + \frac{1}{a}\sum_{i=1}^{n}\left(1+a-a\mu_A(x_i)\right)\log\left(1+a-a\mu_A(x_i)\right) - \frac{1}{a}(1+a)\log(1+a). \end{aligned} \tag{1.8}$$

On the some lines, corresponding to [12]'s entropy of type (α,β), [5] suggested the following measure of fuzzy entropy

$$\frac{1}{\beta-\alpha}\sum_{i=1}^{n}\left[\mu_A^\alpha(x_i) + \left(1-\mu_A(x_i)\right)^\alpha - \mu_A^\beta(x_i) - \left(1-\mu_A(x_i)\right)^\beta\right], \tag{1.9}$$

where $\alpha \geq 1$, $\beta \leq 1$ or $\alpha \leq 1$, $\beta \geq 1$ and $\alpha = \beta$ only if each is equal to 1.

2. Two new generalized measures of fuzzy entropy

[13] characterized the following non-additive entropies of discrete probability distributions:

$$H^\beta(P) = \left(2^{(\beta-1)\sum_{i=1}^{n} p_i \log p_i}\right)\left(2^{1-\beta}-1\right)^{-1}, \quad \beta > 0,\ \beta \neq 1, \tag{2.1}$$

and

$$H_\alpha^\beta(P) = \left[\left(\sum_{i=1}^{n} p_i^\alpha\right)^{\frac{\beta-1}{\alpha-1}} - 1\right]\left(2^{1-\beta}-1\right)^{-1}, \quad \alpha \neq \beta,\ \alpha > 0,\ \beta > 0,\ \alpha \neq 1. \tag{2.2}$$

Corresponding to these entropies, we propose the following measures of fuzzy entropies respectively:

$$H^{\beta}(A)=\frac{1}{1-\beta}\Big[2^{(\beta-1)\sum\limits_{i=1}^{n}\mu_A(x_i)\log\mu_A(x_i)+(1-\mu_A(x_i))\log(1-\mu_A(x_i))}-1\Big],$$
$$\text{where}\quad \beta>0\,,\ \ \beta\neq 1\,, \tag{2.3}$$

and

$$H_{\alpha}^{\beta}(A)=\frac{1}{1-\beta}\sum_{i=1}^{n}\Big[\big(\mu_A^{\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{\alpha}\big)^{\frac{\beta-1}{\alpha-1}}-1\Big],$$
$$\text{where}\quad \alpha\neq\beta\,,\ \ \alpha,\beta>0\,,\ \ \alpha\neq 1\,. \tag{2.4}$$

THEOREM 1. *The expressions* (2.3) *and* (2.4) *represent valid measures of fuzzy entropy.*

Proof. To prove that these measures are valid measures of fuzzy entropy, we shall show that four properties (P-1) to (P-4) are satisfied. The measure (2.3) can be written

$$H^{\beta}(A)=g_{\beta}\big(H(A)\big)\,. \tag{2.5}$$

where

$$H(A)=-\sum_{i=1}^{n}\big[\mu_A(x_i)\log\mu_A(x_i)+\big(1-\mu_A(x_i)\big)\log\big(1-\mu_A(x_i)\big)\big]$$

and $g_{\beta}\colon[0,\infty)\to\mathbb{R}$ is defined as

$$g_{\beta}(x)=(1-\beta)^{-1}\big(2^{(1-\beta)x}-1\big)\,. \tag{2.6}$$

Trivially, function $g_{\beta}(x)$ is increasing function of $x\geq 0$. Hence $H^{\beta}(A)$ is an increasing function of $H(A)$ and shall satisfy four properties (P-1) to (P-4) in view of $H(A)$ is fuzzy entropy corresponding to Shannons's entropy. Thus $H^{\beta}(A)$ is a valid generalized measure of fuzzy entropy.

Particular Case:

It may be noted that when $\beta=1$, (2.5) reduces to

$$H^{1}(A)=-\sum_{i=1}^{n}\big[\mu_A(x_i)\log\mu_A(x_i)+\big(1-\mu_A(x_i)\big)\log\big(1-\mu_A(x_i)\big)\big]\,, \tag{2.7}$$

which is (1.4). The measure (2.4) can be written

$$H_{\alpha}^{\beta}(A)=\sum_{i=1}^{n}g_{\beta}\big(f_{\alpha}\big(\mu_A(x_i)\big)\big)\,, \tag{2.8}$$

where

$$f_{\alpha}\big(\mu_A(x_i)\big)=\frac{1}{1-\alpha}\log\big(\mu_A^{\alpha}(x_i)+\big(1-\mu_A(x_i)\big)^{\alpha}\big) \tag{2.9}$$

and $g_\beta\colon [0,\infty) \to \mathbb{R}$ is defined by

$$g_\beta(x) = (1-\beta)^{-1}\big[2^{(1-\beta)x} - 1\big] \qquad \text{for all} \quad x \geq 0\,, \tag{2.10}$$

where $\beta > 0$, $\beta \neq 1$. It can be easily verified that $g_\beta(x)$ is an increasing function for $x \geq 0$.

Now let $H_\alpha^\beta(A) = 0$, then

$$2^{(1-\beta)f_\alpha(\mu_A(x_i))} - 1 = 0 \implies f_\alpha\big(\mu_A(x_i)\big) = 0\,.$$

It implies that

$$\mu_A^\alpha(x_i) + \big(1-\mu_A(x_i)\big)^\alpha = 1\,. \tag{2.11}$$

Since $\alpha > 0$ and $\alpha \neq 1$, therefore, (2.11) is satisfied when either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$ for all $i = 1, 2, \ldots, n$.

Conversely, let A be non-fuzzy set, then either $\mu_A(x_i) = 0$ or $\mu_A(x_i) = 1$, this implies

$$\mu_A^\alpha(x_i) + \big(1-\mu_A(x_i)\big)^\alpha = 1 \qquad \text{for} \quad \alpha > 0\,, \quad \alpha \neq 1\,,$$

from which

$$f_\alpha\big(\mu_A(x_i)\big) = 0$$

and then

$$H_\alpha^\beta(A) = 0\,.$$

Hence $H_\alpha^\beta(A) = 0$ if and only if A is non-fuzzy set or crisp set.

Differentiating $H_\alpha^\beta(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{\partial H_\alpha^\beta}{\partial \mu_A(x_i)} = \left(\frac{\alpha}{1-\alpha}\right) \frac{2^{(\beta-1)f_\alpha(\mu_A(x_i))}}{\mu_A^\alpha(x_i) + \big(1-\mu_A(x_i)\big)^\alpha} \Big[\mu_A^{\alpha-1}(x_i) - \big(1-\mu_A(x_i)\big)^{\alpha-1}\Big]\,. \tag{2.12}$$

Let $0 \leq \mu_A(x_i) \leq 0.5$, then two cases arise

Case 1.

When $\alpha > 1$, $\frac{\alpha}{1-\alpha} < 0$ and $\mu_A^{\alpha-1}(x_i) - \big(1-\mu_A(x_i)\big)^{\alpha-1} > 0$, then we have

$$\frac{\partial H_\alpha^\beta}{\partial \mu_A(x_i)} < 0\,.$$

Case 2.

When $\alpha < 1$, $\frac{\alpha}{1-\alpha} > 0$ and $\mu_A^{\alpha-1}(x_i) - \big(1-\mu_A(x_i)\big)^{\alpha-1} < 0$, then we get

$$\frac{\partial H_\alpha^\beta}{\partial \mu_A(x_i)} < 0\,.$$

Hence $H_\alpha^\beta(A)$ is an increasing function of $\mu_A(x_i)$ satisfying $0 \le \mu_A(x_i) < 0.5$.

Similarly, we can prove that $H_\alpha^\beta(A)$ is a decreasing function of $\mu_A(x_i)$ satisfying $0.5 < \mu_A(x_i) \le 1$. It may be noted that (2.12) vanishes at $\mu_A(x_i) = 0.5$.

Hence $H_\alpha^\beta(A)$ is a concave function and has a global maximum at $x = 0.5$. Thus $H_\alpha^\beta(A)$ is maximum if and only if A is the most fuzzy set. Next we prove that sharpening reduces the value of fuzzy entropy.

Let A^* be sharpened version of A, i.e.:

(i) If $\mu_A(x_i) < 0.5$, then $\mu_{A^*}(x_i) \le \mu_A(x_i)$.
(ii) If $\mu_A(x_i) > 0.5$, then $\mu_{A^*}(x_i) \ge \mu_A(x_i)$.

Since $H_\alpha^\beta(A)$ is increasing function of $\mu_A(x_i)$ in $[0, 0.5)$ and decreasing function in $(0.5, 1]$,

$$\mu_{A^*}(x_i) \le \mu_A(x_i) \implies H_\alpha^\beta(A^*) \le H_\alpha^\beta(A) \quad \text{in} \quad [0, 0.5) \tag{2.13}$$

and

$$\mu_{A^*}(x_i) \ge \mu_A(x_i) \implies H_\alpha^\beta(A^*) \le H_\alpha^\beta(A) \quad \text{in} \quad (0.5, 1]. \tag{2.14}$$

(2.13) together with (2.14) gives

$$H_\alpha^\beta(A^*) \le H_\alpha^\beta(A). \tag{2.15}$$

It is evident from the definition that $H_\alpha^\beta(A) = H_\alpha^\beta(\bar{A})$. Hence $H_\alpha^\beta(A)$ satisfies all the essential properties of a fuzzy entropy and it is a valid measure of fuzzy entropy.

Particular Cases:

(a) In case $\beta = 1$ in (2.4), it reduces to

$$\frac{1}{1-\alpha} \sum_{i=1}^{n} \log\big(\mu_A(x_i) + (1 - \mu_A(x_i))^\alpha\big), \tag{2.16}$$

which is (1.6).

(b) In case $\alpha = \beta$ except each is unity, then (2.4) also reduces to (1.6).

(c) When $\alpha \to 1$, then (2.4) tends to (2.3).

□

3. R-normed fuzzy entropy

[2] suggested the following R-normed entropy:

$$H_R(P) = \frac{R}{R-1}\left(1 - \left(\sum_{i=1}^{n} p_i^R\right)^{\frac{1}{R}}\right), \qquad \text{where} \quad R > 0, \quad R \ne 1. \tag{3.1}$$

Correspondingly, we propose the following fuzzy entropy and characterize the same in next theorem:

$$H_R(A) = \frac{R}{R-1}\sum_{i=1}^{n}\Big[1 - \big(\mu_A^R(x_i) + (1-\mu_A(x_i))^R\big)^{\frac{1}{R}}\Big], \tag{3.2}$$

where $R > 0$, $R \neq 1$.

THEOREM 2. *$H_R(A)$ given by (3.2) is a valid measure of fuzzy entropy.*

Proof. The proposed measure (3.2) will be valid if and only if it satisfies four properties (P-1) to (P-4). So we shall verify these properties:

(P-1): $H_R(A) = 0$ if and only if A is crisp set, i.e. $\mu_A(x_i) = 0$ or 1 for all $i = 1, 2, \dots, n$.

Let $H_R(A) = 0$, i.e.

$$\frac{R}{R-1}\sum_{i=1}^{n}\Big[1 - \big(\mu_A^R(x_i) + (1-\mu_A(x_i))^R\big)^{\frac{1}{R}}\Big] = 0. \tag{3.3}$$

Since $R > 0$ and $R \neq 1$, therefore, (3.3) holds if and only if $\mu_A(x_i) = 0$ or 1. Hence $H_R(A) = 0$ if and only if A is a crisp set.

(P-2): $H_R(A)$ is maximum if and only if A is the most fuzzy set, i.e. $\mu_A(x_i) = 0.5$ for all $i = 1, 2, \dots, n$.

If we differentiate $H_R(A)$ with respect to $\mu_A(x_i)$, we have

$$\frac{\partial H_R(A)}{\partial \mu_A(x_i)} =$$

$$= \frac{R}{R-1}\sum_{i=1}^{n}\Big[\Big(\mu_A^R(x_i) + \big(1-\mu_A(x_i)\big)^R\Big)^{\frac{1-R}{R}}\Big(\mu_A^{R-1}(x_i) - \big(1-\mu_A(x_i)\big)^{R-1}\Big)\Big]. \tag{3.4}$$

Let $0 \le \mu_A(x_i) < 0.5$, then

$$\frac{\partial H_R(A)}{\partial \mu_A(x_i)} \ge 0.$$

Similarly for all $\mu_A(x_i) \in (0.5, 1]$ we can prove

$$\frac{\partial H_R(A)}{\partial \mu_A(x_i)} \le 0.$$

Thus $H_R(A)$ is a concave function which has a global maximum at $\mu_A(x_i) = 0.5$.

(P-3): Let A^* be sharped version of A, i.e.

(i) If $\mu_A(x_i) \le 0.5$, then $\mu_{A^*}(x_i) \le \mu_A(x_i)$.
(ii) If $\mu_A(x_i) \ge 0.5$, then $\mu_{A^*}(x_i) \ge \mu_A(x_i)$.

Since $H_R(A)$ is increasing function of $\mu_A(x_i) \in [0, 0.5)$ and is decreasing function of $\mu_A(x_i) \in (0.5, 1]$, if $\mu_A(x_i) \leq 0.5$, then

$$\mu_{A^*}(x_i) \leq \mu_A(x_i) \implies H_R(A^*) \leq H_R(A),$$

and if $\mu_A(x_i) \geq 0.5$, then

$$\mu_{A^*}(x_i) \geq \mu_A(x_i) \implies H_R(A^*) \leq H_R(A).$$

Hence $H_R(A^*) \leq H_R(A)$.
(P-4): $H_R(A) = H_R(\bar{A})$ is obvious from the definition.
Particular Case:
When $R = 1$, (3.2) reduces to

$$-\sum_{i=1}^{n} [\mu_A(x_i) \log \mu_A(x_i) + (1 - \mu_A(x_i)) \log(1 - \mu_A(x_i))],$$

which is (1.3). □

4. Generalized measures of fuzzy directed and symmetric divergence

[9] defined the directed divergence measure of a probability distribution $P = (p_1, p_2, \ldots, p_n)$ from an other probability distribution $Q = (q_1, q_2, \ldots, q_n)$ as

$$D(P : Q) = \sum_{i=1}^{n} p_i \log \frac{p_i}{q_i}. \tag{4.1}$$

The following measure of symmetric divergence was proposed by [8]:

$$J(P : Q) = D(P : Q) + D(Q : P) = \sum_{i=1}^{n} (p_i - q_i) \log \frac{p_i}{q_i}, \tag{4.2}$$

which is also called a distance measure.

Corresponding to the measures (4.1) and (4.2), [1] suggested the following measures between two fuzzy sets A and B respectively:

$$I(A, B) = \sum_{i=1}^{n} \left[\mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} + (1 - \mu_A(x_i)) \log \frac{(1 - \mu_A(x_i))}{(1 - \mu_B(x_i))} \right] \tag{4.3}$$

and

$$D(A, B) = I(A, B) + I(B, A). \tag{4.4}$$

Measure $I(A,B)$ can be valid only if it is non-negative. So we prove that $I(A,B) \geq 0$ with equality if $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1,2,\ldots,n$.

Let $\sum_{i=1}^{n} \mu_A(x_i) = s$ and $\sum_{i=1}^{n} \mu_B(x_i) = t$, then

$$\sum_{i=1}^{n} \mu_A(x_i) \log \frac{\mu_A(x_i)}{\mu_B(x_i)} \geq s \log \frac{s}{t}\,. \tag{4.5}$$

Similarly, we can prove that

$$\sum_{i=1}^{n} (1-\mu_A(x_i)) \log \frac{(1-\mu_A(x_i))}{(1-\mu_B(x_i))} \geq (n-s) \log \frac{n-s}{n-t}\,. \tag{4.6}$$

Adding (4.5) and (4.6), we have

$$I(A,B) \geq s \log \frac{s}{t} + (n-s) \log \frac{n-s}{n-t}\,.$$

Let $f(s) = s \log \frac{s}{t} + (n-s) \log \frac{n-s}{n-t}$, then

$$f'(s) = \log \frac{s}{t} - \log \frac{n-s}{n-t}$$

and

$$f''(s) = \frac{1}{s} + \frac{1}{n-s} = \frac{n}{s(n-s)} > 0\,.$$

This shows that $f(\alpha_0)$ is a convex function of α_0, which has its minimum value when

$$\frac{s}{t} = \frac{n-s}{n-t} = \frac{n}{n} = 1 \qquad \text{or} \quad s = t\,.$$

Thus $I(A,B)$ is always greater than zero unless $\mu_A(x_i) = \mu_B(x_i)$ for each $i = 1,2,\ldots,n$. Hence $I(A,B)$ is a valid measure of fuzzy directed divergence and consequently, $D(A,B)$ defined by (4.3) is also a valid measure of fuzzy symmetric divergence or distance between two fuzzy sets.

Corresponding to [4]'s measure of directed divergence

$$\sum_{i=1}^{n} \frac{p_i^{\alpha} q_i^{1-\alpha} - 1}{\alpha - 1}\,, \qquad \alpha > 0\,,\ \ \alpha \neq 1\,, \tag{4.7}$$

we define the following measures of fuzzy directed divergence

$$I_\alpha(A,B) = \frac{1}{\alpha-1} \sum_{i=1}^{n} \Big[\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}(1-\mu_B(x_i))^{1-\alpha} - 1\Big] \tag{4.8}$$

and

$$D_\alpha(A,B) = I_\alpha(A,B) + I_\alpha(B,A)\,. \tag{4.9}$$

Next we shall prove that $I_\alpha(A,B) \geq 0$.

Let $\sum_{i=1}^{n} \mu_A(x_i) = s$, $\sum_{i=1}^{n} \mu_B(x_i) = t$, then

$$\frac{1}{\alpha-1}\left[\sum_{i=1}^{n}\left(\frac{\mu_A(x_i)}{s}\right)^{\alpha}\left(\frac{\mu_B(x_i)}{t}\right)^{1-\alpha} - 1\right] \geq 0$$

or

$$\frac{1}{\alpha-1}\sum_{i=1}^{n}\mu_A^{\alpha}(x_i)\mu_B^{1-\alpha}(x_i) \geq \frac{1}{\alpha-1}s^{\alpha}t^{1-\alpha}. \tag{4.10}$$

Similarly

$$\frac{1}{1-\alpha}\sum_{i=1}^{n}\left(1-\mu_A(x_i)\right)^{\alpha}\left(1-\mu_B(x_i)\right)^{1-\alpha} \geq \frac{1}{\alpha-1}(n-s)^{\alpha}(n-t)^{1-\alpha}. \tag{4.11}$$

Adding (4.10) and (4.11), we get

$$I_\alpha(A,B) \geq \frac{1}{\alpha-1}\left[s^{\alpha}t^{1-\alpha} + (n-s)^{\alpha}(n-t)^{1-\alpha} - n\right].$$

Further let $\phi(s) = \frac{1}{\alpha-1}\left[s^{\alpha}t^{1-\alpha} + (n-s)^{\alpha}(n-t)^{1-\alpha} - n\right]$, then

$$\phi'(s) = \frac{1}{\alpha-1}\left[\alpha\left(\frac{s}{t}\right)^{\alpha-1} - \alpha\left(\frac{n-s}{n-t}\right)^{\alpha-1}\right]$$

and

$$\phi''(s) = \frac{\alpha}{t}\left(\frac{s}{t}\right)^{\alpha-2} + \frac{\alpha}{n-t}\left(\frac{n-s}{n-t}\right)^{\alpha-2} > 0.$$

This shows that $\phi(s)$ is a convex function of s whose minimum value arises when $\frac{s}{t} = \frac{n-s}{n-t} = 1$ and is equal to zero. Hence $\phi(s) > 0$ and vanishes only when $s = t$.

Hence for all $\alpha > 0$, $I_\alpha(A,B) \geq 0$ and it vanishes only when $A = B$. Thus (4.8) is a valid measure of directed divergence of fuzzy sets A and B and consequently, $D_\alpha(A,B)$ is a valid measure of symmetric divergence. It can be easily seen that

$$\lim_{\alpha\to 1} I_\alpha(A,B) = I(A,B) \qquad \text{and} \qquad \lim_{\alpha\to 1} D_\alpha(A,B) = D(A,B).$$

Remarks. Following the same lines we can obtain more generalized measures of fuzzy directed divergence and symmetric divergence between fuzzy sets A and B corresponding to known probability measures of directed divergence and symmetric divergence.

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