Mathematica Slovaca

Juraj Lörinc

The height of the first Stiefel-Whitney class of any nonorientable real flag manifold

Mathematica Slovaca, Vol. 53 (2003), No. 1, 91--95

Persistent URL: http://dml.cz/dmlcz/131212

Math. Slovaca, 53 (2003), No. 1, 91–95

Mathematica Slovaca

THE HEIGHT OF THE FIRST STIEFEL-WHITNEY CLASS OF ANY NONORIENTABLE REAL FLAG MANIFOLD

JURAJ LÖRINC

(*Communicated by Miloslav Duchoň*)

ABSTRACT. Using suitable fiberings, we calculate the height of the first Stiefel-Whitney class of any nonorientable real flag manifold $O(n_1 + \cdots + n_q)/O(n_1) \times \cdots \times O(n_q)$.

1. Introduction

Let $n_1, \dots, n_q$ $(q \geq 2)$ be fixed positive integers, and let $F(n_1, \dots, n_q)$ be the real flag manifold consisting of all q-tuples $(S_1, \dots, S_q)$ of mutually orthogonal vector subspaces in $\mathbb{R}^n$, where $n = n_1 + \cdots + n_q$ and $\dim(S_i) = n_i$. As a homogeneous space, we have

$$F(n_1, \dots, n_q) \cong O(n)/O(n_1) \times \cdots \times O(n_q).$$

In particular, $F(n_1, n_2)$ is the Grassmann manifold of all n_1-dimensional vector subspaces in $\mathbb{R}^n$.

Over the manifold $F(n_1, \dots, n_q)$, there are q canonical vector bundles $\gamma_1, \dots, \gamma_q$ with $\dim(\gamma_i) = n_i$. They are characterized by the fact that the fiber of γ_i over $(S_1, \dots, S_q) \in F(n_1, \dots, n_q)$ is the vector space S_i. The direct sum $\bigoplus_{i=1}^{q} \gamma_i$ is the trivial n-dimensional vector bundle.

By Korbaš [3], the manifold $F(n_1, \dots, n_q)$ is nonorientable, hence has its first Stiefel-Whitney class $w_1(F(n_1, \dots, n_q)) \in H^1(F(n_1, \dots, n_q); \mathbb{Z}_2)$ non-zero, precisely when not all of the numbers $n_1, \dots, n_q$ have the same parity.

2000 Mathematics Subject Classification: Primary 57R19; Secondary 57R20, 57T15.
Keywords: height of cohomology class, Stiefel-Whitney class, real flag manifold.

In 2000, Ilori and Ajayi [2] calculated the height of $w_1(F(n_1,\dots,n_q))$ (denoted $\mathrm{height}(F(n_1,\dots,n_q))$) for some of those flag manifolds $F(n_1,\dots,n_q)$ which are nonorientable. (Recall that $\mathrm{height}(F(n_1,\dots,n_q))$ is the largest c such that $w_1^c(F(n_1,\dots,n_q)) \in H^*(F(n_1,\dots,n_q);\mathbb{Z}_2)$ does not vanish.) Their result is the following.

PROPOSITION 1.1. (Ilori, Ajayi [2]) *Suppose that* $\prod_{i=1}^{q-1} n_i$ *is odd,* $n-k$ *is even, where* $k = \sum_{i=1}^{q-1} n_i$, *and* $4 \le 2k \le n$ *with* $2^s < n \le 2^{s+1}$. *Then*

$$\mathrm{height}(w_1(F(n_1,\dots,n_{q-1},n-k))) = \begin{cases} 2^{s+1}-2 & \text{if } k=2 \text{ or} \\ & \text{if } k=3 \text{ and } n=2^s+1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

Our aim here is to show that a slight modification of the approach used by Ilori and Ajayi leads in fact to the following complete result covering the height of the first Stiefel-Whitney class of *any* nonorientable real flag manifold.

THEOREM 1.2. *Let* $F(n_1,\dots,n_q)$, *for* $q \ge 2$, *be any nonorientable real flag manifold; hence not all of* $n_1,\dots,n_q$ *have the same parity. Letting* p *be the sum of all even numbers among* $n_1,\dots,n_q$, *put* $k = \min\{p, n-p\}$. *If* s *is the uniquely determined integer such that* $2^s < n \le 2^{s+1}$, *then we have*

$$\mathrm{height}(w_1(F(n_1,\dots,n_q))) = \begin{cases} n-1 & \text{if } k=1, \\ 2^{s+1}-2 & \text{if } k=2 \text{ or} \\ & \text{if } k=3 \text{ and } n=2^s+1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

The knowledge of $\mathrm{height}(F(n_1,\dots,n_q))$ is useful for several reasons. For instance, Ilori and Ajayi [2] show how it can be used for deriving a result on immersions of real flag manifolds in Riemannian manifolds. Of course, $\mathrm{height}(F(n_1,\dots,n_q))$ also gives a lower bound for the cup-length. Results of our study of the cup-length for real flag manifolds will be postponed to a forthcoming paper [4].

2. Proof of Theorem 1.2

We intend to make the proof of Theorem 1.2 as selfcontained as possible.

Let $w_i(\gamma_j)$ be the ith Stiefel-Whitney class of the canonical vector bundle γ_j over $F(n_1,\dots,n_q)$. Then according to Borel [1; Theorem 11.1], we have

$$H^*(F(n_1,\dots,n_q);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1(\gamma_1),\dots,w_{n_1}(\gamma_1),\dots,w_1(\gamma_q),\dots,w_{n_q}(\gamma_q)]/I,$$

where the ideal I is given by the identity

$$\prod_{j=1}^{q}\left(1+w_1(\gamma_j)+\cdots+w_{n_j}(\gamma_j)\right)=1\,.$$

Let σ be any permutation of the set $\{1,\dots,q\}$. The map $\widetilde{\sigma}\colon F(n_1,\dots,n_q)\to F\left(n_{\sigma(1)},\dots,n_{\sigma(q)}\right)$ given by $\widetilde{\sigma}(S_1,\dots,S_q)=\left(S_{\sigma(1)},\dots,S_{\sigma(q)}\right)$ is a diffeomorphism. Thus we may and shall suppose that there is $t\in\{1,\dots,q\}$ such that $n_1,\dots,n_t$ are odd, and $n_{t+1},\dots,n_q$ are even. Then the map

$$\begin{aligned}\pi\colon F(n_1,\dots,n_q)&\to F(n_1,\dots,n_t,n_{t+1}+\cdots+n_q)\,,\\ \pi(S_1,\dots,S_q)&=(S_1,\dots,S_t,S_{t+1}\oplus\cdots\oplus S_q)\,,\end{aligned}\tag{1}$$

defines a smooth fiber bundle (cf. [5; 7.4]) with fiber $F(n_{t+1},\dots,n_q)$. We obviously have $\gamma_i=\pi^*(\gamma_i)$ for $i=1,\dots,t$.

For the inclusion of the fiber, $i\colon F(n_{t+1},\dots,n_q)\hookrightarrow F(n_1,\dots,n_q)$, one has $\gamma_j=i^*(\gamma_j)$, $j=t+1,\dots,q$, and the classes

$$w_m(\gamma_j)=i^*\left(w_m(\gamma_j)\right)\qquad\text{for}\quad j=t{+}1,\dots,q\,,\quad m=1,\dots,n_j$$

generate $H^*\left(F(n_{t+1},\dots,n_q)\right)$ as a vector space over $\mathbb{Z}_2$. Choosing an appropriate basis we see that the assumptions of the Leray-Hirsch theorem (see, e.g. [7]) are satisfied. This implies that π^* is a monomorphism.

From the Korbaš formula ([3; Theorem 1.1]) for the first Stiefel-Whitney class of $F(n_1,\dots,n_q)$, we obtain

$$\begin{aligned}&\pi^*\left(w_1\left(F(n_1,\dots,n_t,n_{t+1}+\cdots+n_q)\right)\right)\\ &\quad=\pi^*\left(w_1(\gamma_1)+\cdots+w_1(\gamma_t)\right)\\ &\quad=\pi^*\left(w_1(\gamma_1)\right)+\cdots+\pi^*\left(w_1(\gamma_t)\right)\\ &\quad=w_1(\gamma_1)+\cdots+w_1(\gamma_t)\\ &\quad=w_1\left(F(n_1,\dots,n_q)\right).\end{aligned}\tag{2}$$

It is needed to analyse two cases.

Case of $k=1$:

Now certainly $n_1=1$ and $n_2,\dots,n_q$ are even, hence $t=1$. The relevant fiber bundle (see (1)) is now

$$\pi\colon F(1,n_2,\dots,n_q)\to F(1,n_2+\cdots+n_q)\,;$$

its base is the (nonorientable) $(n-1)$-dimensional real projective space $F(1,n_2+\cdots+n_q)=\mathbb{R}P^{n-1}$. As it is well known,

$$\operatorname{height}\left(w_1(\mathbb{R}P^{n-1})\right)=n-1\,,$$

and by (2), $\pi^*\big(w_1(\mathbb{R}P^{n-1})\big) = w_1\big(F(1,n_2,\dots,n_q)\big)$. Since π^* is a ring monomorphism, we have

$$0 \neq \pi^*\big(w_1^{n-1}(\mathbb{R}P^{n-1})\big) = w_1^{n-1}\big(F(1,n_2,\dots,n_q)\big),$$

while

$$0 = \pi^*\big(w_1^{n}(\mathbb{R}P^{n-1})\big) = w_1^{n}\big(F(1,n_2,\dots,n_q)\big).$$

This proves the theorem in case of $k=1$.

Case of $k \geq 2$:

From (2) and the fact that π^* is a monomorphism, we know that $w_1^c\big(F(n_1,\dots,n_q)\big) = 0$ if and only if $w_1^c\big(F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)\big) = 0$. Therefore the height of $w_1\big(F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)\big)$ is the same as the height of $w_1\big(F(n_1,\dots,n_q)\big)$.

We know that now $w_1\big(F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)\big) \neq 0$. Further consider the fiber bundle

$$p\colon F(n_1,\dots,n_t,n_{t+1}+\dots+n_q) \to F(n_1+\dots+n_t,n_{t+1}+\dots+n_q),$$
$$p(S_1,\dots,S_{t+1}) = (S_1 \oplus \dots \oplus S_t, S_{t+1}),$$

with fiber $F(n_1,\dots,n_t)$. Of course, its base space is nothing but the Grassmann manifold $F(n-p,p)$. In addition to this, the Korbaš formula (cf. [3]) yields

$$\begin{aligned} w_1\big(F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)\big) &= w_1(\gamma_1)+\dots+w_1(\gamma_t) \\ &= w_1(\gamma_1\oplus\dots\oplus\gamma_t) \\ &= \pi^*\big(w_1(\gamma_1)\big) \\ &= \pi^*\big(w_1(\gamma_2)\big). \end{aligned}$$

Note that for the Grassmann manifolds the Whitney sum of their two canonical vector bundles is trivial, hence their first Stiefel-Whitney classes coincide. The Leray-Hirsch theorem now again applies, and it implies that the height of $w_1\big(F(n_1,\dots,n_t,n_{t+1}+\dots+n_q)\big)$ coincides with the height of $w_1(\gamma_1) = w_1(\gamma_2) \in H^*\big(F(n-p,p)\big)$ (γ_1 is the $(n-p)$-plane bundle over $F(n-p,p)$). But the height of $w_1(\gamma_1) = w_1(\gamma_2) \in H^*\big(F(n-p,p)\big)$ is known (Stong [6]):

$$\operatorname{height}\big(w_1(\gamma_1)\big) = \begin{cases} n-1 & \text{if } k=1, \\ 2^{s+1}-2 & \text{if } k=2 \text{ or} \\ & \text{if } k=3 \text{ and } n=2^s+1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases}$$

This completes the proof of Theorem 1.2.

REFERENCES

[1] BOREL, A.: *La cohomologie* mod 2 *de certains espaces homogènes*, Comment. Math. Helv. **27** (1953), 165–197.

[2] ILORI, S. A.—AJAYI, D. O.: *The height of the first Stiefel-Whitney class of the real flag manifolds*, Indian J. Pure Appl. Math. **36** (2000), 621–624.

[3] KORBAŠ, J.: *Vector fields on real flag manifolds*, Ann. Global Anal. Geom. **3** (1985), 173–184.

[4] KORBAŠ, J.—LÖRINC, J.: *On the* $\mathbb{Z}_2$*-cohomology cup-length of real flag manifolds* (In preparation).

[5] STEENROD, N.: *The Topology of Fibre Bundles*, Princeton Univ. Press, Princeton, NJ, 1951.

[6] STONG, R. E.: *Cup products in Grassmannians*, Topology Appl. **13** (1982), 103–113.

[7] SWITZER, R.: *Algebraic Topology – Homotopy and Homology*, Springer, Berlin, 1975.

Received January 30, 2002

Národná banka Slovenska
I. Karvaša 1
SK–813 25 Bratislava 1
SLOVAKIA
E-mail: juraj.lorinc@nbs.sk