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Mária Tomková

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GRAPH ISOMORPHISMS OF MODULAR MULTILATTICES

MÁRIA TOMKOVÁ

1. Preliminaries

A partially ordered set P is said to be of locally finite length if each bounded chain in P is finite. For the elements $a, b \in P$ we write $a \succ b$ (a covers b) if $a > b$ and if there does not exist any element $c \in P$ with $a > c > b$; in this case the interval $[a, b]$ is called prime.

A partially ordered set P is called upper (lower) directed if for each pair of elements $a, b \in P$ there exists an element $h \in P$ ($d \in P$) such that $a \leqq h$, $b \leqq h$ ($d \leqq a$, $d \leqq b$). The upper and lower directed partially ordered set is called directed.

A multilattice [1] is a partially ordered set M in which the conditions (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $v \geqq a$, $v \geqq b$, and (b) $z \in M$, $z \leqq v$, $z \geqq a$, $z \geqq b$ implies $z = v$.

A multilattice M is modular [1] iff for every $a, b, c, u, v \in M$ satisfying the conditions $u \leqq a \leqq v$, $u \leqq b \leqq c \leqq v$, $v \in a \vee b$, $u \in a \wedge c$ we have $b = c$.

A multilattice M is distributive [1] iff for every $a, b, c, u, v \in M$ satisfying the conditions $u \leqq a, b, c \leqq v$, $v \in a \vee b$, $v \in a \vee c$, $u \in a \wedge b$, $u \in a \wedge c$ we have $b = c$.

It is evident that each partially ordered set of locally finite length is a multilattice [1]. All partially ordered sets dealt with in this note are assumed to be of locally finite length.

By a graph $G(S)$ of a subset $S \subset P$ there is meant the unoriented graph (without multiple edges and loops) whose vertices are elements of S; two vertices $a, b \in S$ are joined by the edge (a, b) iff $a \succ b$ or $b \succ a$.

We say that unoriented graphs $G(S_1)$ and $G(S_2)$ are isomorphic if there exists a bijection φ of S_1 onto S_2 satisfying: (x, y) is an edge in $G(S_1)$ iff $(\varphi(x), \varphi(y))$ is an edge in $G(S_2)$.

The following two assertions (T_1), (T_2) were proved in [1] (4.7.4 and Theorem 4.5).

(T_1) *A multilattice M of locally finite length is modular iff it fulfils the following covering condition* (σ') *and the condition* (σ'') *dual to* (σ').

(σ') *If $a, b, u, v \in M$ such that $[u, a]$, $[u, b]$ are prime intervals and $v \in a \vee b$, then $[a, v]$, $[b, v]$ are prime intervals.*

(T_2) *Let C_1, C_2 be two maximal chains from a to b in a modular multilattice M of locally finite length. Then C_1, C_2 are of the same length.*

A set $S=\{a, b, u, v\} \subseteq M$ is called an elementary square if a, b are incomparable elements and $v \succ a$, $v \succ b$, $u \prec a$, $u \prec b$.

Let M_1 and M_2 be directed multilattices of locally finite length and let φ be a graph isomorphism of $G(M_1)$ onto $G(M_2)$. Let $S=\{a, b, u, v\}$ be an elementary square in M_1. We shall say that S breaks by the isomorphism φ if either the elements $\varphi(u)$, $\varphi(v)$ are covered by $\varphi(a)$, $\varphi(b)$ or the elements $\varphi(u)$, $\varphi(v)$ cover $\varphi(a)$ and $\varphi(b)$.

Graph isomorphisms of lattices and multilattices have been studied in the papers [2], [3], [4], [5].

In [2] and [3] the following theorems have been proved:

(A) If L_1 and L_2 are lattices of locally finite length such that (i) L_1 is modular and (ii) the unoriented graphs $G(L_1)$, $G(L_2)$ are isomorphic, then the lattice L_2 is modular as well.

(B) Let M_1 and M_2 be directed distributive multilattices of locally finite length. Then the following conditions are equivalent:

(i) *The unoriented graphs $G(M_1)$, $G(M_2)$ are isomorphic.*

(ii) *There exist multilattices A, B such that M_1 is isomorphic with $A \times B$ and M_2 is isomorphic with $A \times \tilde{B}$ ($\tilde{B}$ is dual to B).*

In the present paper we shall investigate some questions on graph isomorphisms of multilattices analogous to those that have been dealt with in the papers [2], [3], [4], [5].

2. Statement of results

Theorem 1. *If M_1 and M_2 are directed multilattices of locally finite length such that* (i) *the unoriented graphs $G(M_1)$, $G(M_2)$ are isomorphic,* (ii) *M_2 is modular and* (iii) *M_1 is distributive, then the multilattice M_2 is distributive as well.*

Theorem 2. *There exist directed finite multilattices M_1 and M_2 such that* (i) *the unoriented graphs $G(M_1)$, $G(M_2)$ are isomorphic,* (ii) *M_1 is modular,* (iii) *M_2 is not modular.*

Theorem 3. *Let M_1 and M_2 be directed modular multilattices of locally finite length. Then the following conditions are equivalent:*

(i) *There exist a graph isomorphism φ of $G(M_1)$ onto $G(M_2)$ such that no elementary square $S \subset M_1$ breaks by the isomorphism φ and no elementary square $S' \subset M_2$ breaks by the isomorphism φ^{-1}.*

(iii) *There are multilattices A, B such that M_1 is isomorphic with $A \times B$ and M_2 is isomorphic with $A \times \tilde{B}$ ($\tilde{B}$ is dual to B).*

3. Proofs of theorems

3.1. Proof of Theorem 1. First we recall the definition of the ternary betwenness relation [6] in the directed multilattices.

Let a, b, $x \in M$. We say that x is between a and b and write axb if

$$(b) \qquad [(a \wedge x) \vee (b \wedge x)]_x = x, \quad (a \wedge x) \wedge (b \wedge x) \subset a \wedge b.$$

Directed multilattices M_1, M_2 are said to be b-equivalent if there exists a bijection f of M_1 onto M_2 such that for each triple $a, b, x \in M$ the relation axb is equivalent with $f(a)f(x)f(b)$.

From Theorems 4.3 and 2.2 of [7] it follows:

($*$) *If M_1 and M_2 are directed modular multilattices of locally finite length such that the unoriented graphs $G(M_1)$, $G(M_2)$ are isomorphic, then M_1, M_2 are b-equivalent.*

In [9] the following assertion is proved:

($**$) *Let M_1 and M_2 be directed b-equivalent multilattices. If the multilattice M_1 is distributive, then M_2 is distributive as well.*

If we assume that M_1 and M_2 are directed multilattices of locally finite length such that M_1 is distributive and M_2 is modular and $G(M_1)$, $G(M_2)$ are isomorphic, then by the assertion ($*$) the multilattices M_1, M_2 are b-equivalent. Thus from ($**$) it follows that the multilattice M_2 is distributive.

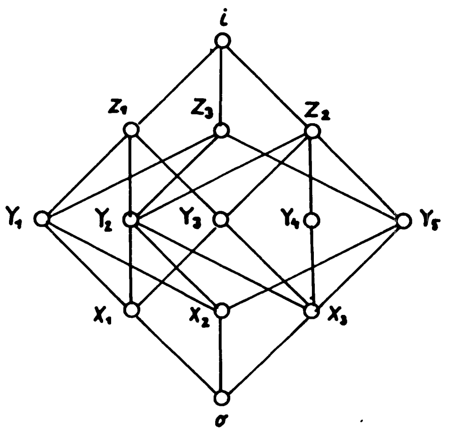

Fig. 1

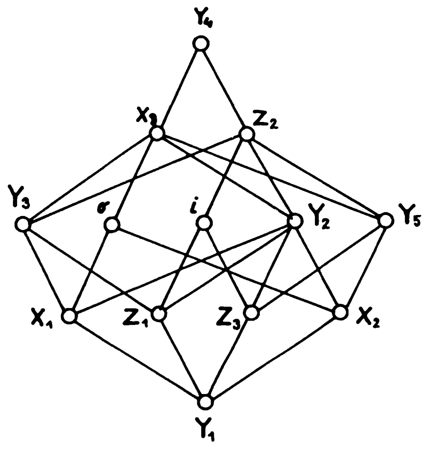

Fig. 2

3.2. Proof of Theorem 2. The partially ordered sets M_1 and M_2 in Fig. 1 and Fig. 2 are of the same length 4 and card M_1 = card $M_2 = 13$. It is obvious that M_1 and M_2 are directed multilattices.

The multilattice M_2 is not modular because there exist elements $y_3, y_5 \in M_2$ such that $z_2 > y_3$, $z_2 > y_5$ and $y_3 \wedge y_5 = \{y_1\}$, where y_1 is not covered by y_3, y_5.

The modularity of M_1 will be verified as follows:

We define the height $v(x)$ of an element $x \in M_1$ as the maximum of lengths of chains between the least element of M_1 and x. Let us denote by $M(i)$ the set of elements $x \in M_1$ with $v(x) = i$. Then $M(0) = \{o\}$, $M(1) = \{x_1, x_2, x_3\}$, $M(2) = \{y_1, y_2, y_3, y_4, y_5\}$, $M(3) = \{z_1, z_2, z_3\}$, $M(4) = \{i\}$. It is routine to verify that M_1 satisfies the Jordan—Dedekind chain condition and that, whenever i is a positive integer and $x, y \in M(i)$, then $x \wedge y \in M(i-1)$ iff $x \vee y \in M(i+1)$. Hence the conditions (σ') and (σ'') are fulfilled and therefore M_1 is modular.

The multilattices M_1 and M_2 are defined on the same set $M = \{o, x_1, x_2, x_3, y_1, y_2, y_3, y_4, y_5, z_1, z_2, z_3, i\}$. Let φ be the identical mapping on M, then φ is a graph isomorphism of $G(M_1)$ onto $G(M_2)$.

3.3. For proving Theorem 3 we need some results of the papers [8], [3].

(K) [8] *Let A be a quasiordered set. There exists a one-to-one correspondence between the nontrivial direct decompositions of the quasiordered set A into two factors and pairs (R_1, R_2) of nontrivial congruence relations R_1, R_2 on A having the properties:*

(i) $R_1 R_2 = R_2 R_1$

(ii) $R_1 \cup R_2 = I$, $R_1 \cap R_2 = 0$ *(I, 0 are the greatest and the least elements of the lattice of all equivalence relations on the set A).*

(iii) *If $a, b, c \in A$, $a \leqq c$, aR_1b, bR_2c, then $a \leqq b \leqq c$.*

(iv) *Let $a, b, c, d \in A$, aR_1b, cR_1d, aR_2c, bR_2d, then from $a \leqq b$ it follows that $c \leqq d$ and from $a \leqq c$ it follows that $b \leqq d$. To each couple (R_1, R_2) with the mentioned properties there corresponds the decomposition $A \sim A/R_1 \times A/R_2$ and to each element $a \in A$ there corresponds the element (a_1, a_2), where a_i is the equivalence class under R_i $(i = 1, 2)$ containing a.*

In the paper [3] the following two lemmas were proved under the assumption that M_1 and M_2 are directed distributive multilattices and φ is a graph isomorphism of $G(M_1)$ onto $G(M_2)$.

Lemma 1. *For $x, y \in M_1$ let $u \in x \wedge y$, $v \in x \vee y$ such that $[u, x]$, $[u, y]$ are prime intervals and let $\varphi(x) < \varphi(u) < \varphi(y)$. Then $\varphi(x) \in \varphi(u) \wedge \varphi(v)$, $\varphi(y) \in \varphi(u) \vee \varphi(v)$.*

Lemma 2. *For $x, y \in M_1$ let $u \in x \wedge y$, $v \in x \vee y$, such that $[x, v]$, $[y, v]$ are prime intervals and let $\varphi(x) < \varphi(v)$, $\varphi(y) < \varphi(v)$. Then $\varphi(u) \in \varphi(x) \wedge \varphi(y)$.*

From the method of the proof of these lemmas in [3] it follows that they remain valid also when we replace the assumption of the distributivity of the multilattices M_1, M_2 by the conditions: (a) M_1, M_2 are modular multilattices, (b) no elementary square of M_1 breaks by the isomorphism φ and no elementary square of M_2 breaks by the isomorphism φ^{-1}.

For proving the implication (i)$\Rightarrow$(ii) in Theorem (b) in [3] only the modularity of the multilattices M_1, M_2 and the assertion of Lemmas 1 and 2 have been applied. From this it follows that the following assertion is valid.

(T_3) *Let M_1 and M_2 be directed modular multilattices of locally finite length and let φ be a graph isomorphism of $G(M_1)$ onto $G(M_2)$. If no elementary square of M_1 breaks by the isomorphism φ and no elementary square of M_2 breaks by the isomorphism φ^{-1}, then the condition* (ii) *from Theorem 3 is valid.*

Now let us suppose that A, B are modular multilattices fulfiling the condition (ii) from Theorem 3. Let f_1 be an isomorphism M_1 onto $A \times B$, f_2 an isomorphism M_2 onto $A \times \tilde{B}$ and let h be the identical mapping on the underlying set of $A \times B$ (this set is clearly equal to the underlying set of $A \times \tilde{B}$). Then the mapping $\varphi = f_2^{-1} h f_1$ is a graph isomorphism of $G(M_1)$ onto $G(M_2)$. From the definition φ it follows immediately that $f_1(x) = f_2(\varphi(x))$ for each $x \in M_1$. Further there exist relations R_1 and R_2 on M_1 such that A is isomorphic with M_1/R_1, B is isomorphic with M_1/R_2 and R_1, R_2 fulfils the conditions (i)—(iv) from the assertion (K) (where we take the multilattice M_1 instead of the quasiordered set A). Then for each x, $y \in M_1$ we have:

(j) If $x \prec y$ and $\varphi(x) \prec \varphi(y)$ ($\varphi(x) \succ \varphi(y)$), then $x \equiv y(R_2)$ ($x \equiv y(R_1)$).

In fact suppose that $x \prec y$ and $\varphi(x) \prec \varphi(y)$; then there are elements a_1, $a_2 \in A$, b_1, $b_2 \in B$ with $f_1(x) = (a_1, b_1)$, $f_1(y) = (a_2, b_2)$. At the same time we have $f_2(\varphi(x)) = (a_1, b_1)$, $f_2(\varphi(y)) = (a_2, b_2)$. From $x \prec y$ it follows that we have either

$$a_1 \prec a_2 \quad \text{and} \quad b_1 = b_2 \tag{1}$$

or

$$a_1 = a_2 \quad \text{and} \quad b_1 \prec b_2. \tag{2}$$

If (2) were valid, then we would have $\varphi(x) \succ \varphi(y)$, which is a contradiction. Therefore the relation (1) holds. From $b_1 = b_2$ we obtain $x \equiv y(R_2)$. Similarly we can verify that if $x \prec y$ and $\varphi(x) \succ \varphi(y)$, then $x \equiv y(R_1)$.

Assume that an elementary square $(a, b, u, v) \subset M_1$ would break by the isomorphism φ. Hence either $\varphi(u)$, $\varphi(v)$ cover the elements $\varphi(a)$ and $\varphi(b)$, or $\varphi(u)$, $\varphi(v)$ are covered by $\varphi(a)$ and $\varphi(b)$. Let us consider the first case (the second case being dual). Then $a \equiv u(R_1)$, $b \equiv u(R_1)$ by (j). Hence $a \equiv b(R_1)$. At the same time $a \equiv v(R_2)$, $b \equiv v(R_2)$ by (j), and hence $a \equiv b(R_2)$. From this it follows according to the property (ii) in the assertion (K) that $a = b$, which is a contradiction. Similarly we can show that no elementary square of the multilattice M_2 breaks by the isomorphism φ^{-1}. Thus we have proved that (ii) implies (i).

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Katedra matematiky
Strojníckej fakulty VŠT
Švermova 9
040 01 Košice

О ГРАФОВОМ ИЗОМОРФИЗМЕ МУЛЬТИСТРУКТУР

Мария Томкова

Резюме

В даной статье доказаны три теоремы о направленных мультиструктурах локально конечной длины. Если графы мультиструктур M_1, M_2 изоморфны, причём M_1 дистрибутивна и M_2 модулярна, тогда M_2 также должна быть дистрибутивна. Однако существуют мультиструктуры M_1, M_2, графы которых изоморфны, причём M_1 модулярна и M_2 немодулярна. Третья теорема говорит о условиях, при которых из изоморфизма графов модулярных мультиструктур M_1, M_2 вытекает существование мультиструктур A, B таких что $M_1 \sim A \times B$, $M_2 \sim A \times \bar{B}$ (где $\bar{B}$ является дуальна мультиструктура к B).