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OSCILLATION CRITERIA OF THIRD-ORDER NONLINEAR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we consider a certain class of third order nonlinear delay differential equations. By means of the Riccati transformation techniques we establish some new criteria and also Kamenev-type criteria which insure that every solution oscillates or converges to zero. Some examples are considered to illustrate our main results.

1. Introduction

In recent years, the oscillation theory and asymptotic behavior of differential equations and their applications have been and still are receiving intensive attention. In fact, in the last few years several monographs and hundreds of research papers have been written, see for example the monographs [1], [2], [6], [8], [9].

In particular case, determining oscillation criteria for second order differential equations has received a great deal of attention in the last few years, for some contributions we refer to the [17] and the references cited therein.

Compared to the second order differential equations, the study of oscillation and asymptotic behavior of the third order differential equations has received considerably less attention in the literature. Some recent results on the third order differential equations can be found in [3], [4], [5], [10]–[15], [18], [19].

Most of the results of oscillation of the third order differential equations are written on the equations of the forms

$$y'''(t) + a(t)y''(t) + b(t)y'(t) + c(t)y\big(g(t)\big) = 0\,,$$
$$y'''(t) + b(t)y'(t) + c(t)y\big(g(t)\big) = 0\,,$$

under some restrictive conditions on the functions a, b, c and g. The oscillation results are established by the general means and reducing the equations to the second order equations.

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In this paper, by using the Riccati transformation technique which is different from that used in the above mentioned papers, we study the oscillation behavior of the self-adjoint nonlinear delay differential equation

$$\big(c(t)\big(a(t)x'(t)\big)'\big)' + q(t)f\big(x(t-\sigma)\big) = 0\,, \qquad t \geq t_0\,, \tag{1.1}$$

where σ is a nonnegative real number, the functions $c(t)$, $a(t)$, $q(t)$ and the function f satisfy the following conditions:

(h1) $c(t)$, $a(t)$ and $q(t)$ are positive continuous functions and

$$\int_{t_0}^{\infty} \frac{1}{c(t)}\,\mathrm{d}t = \int_{t_0}^{\infty} \frac{1}{a(t)}\,\mathrm{d}t = \infty\,;$$

(h2) $f \in C(\mathbb{R}, \mathbb{R})$ such that $uf(u) > 0$ for $u \neq 0$ and $f(u)/u \geqslant K > 0$.

Our attention is restricted to those solutions of (1.1) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\big\{|x(t)|\,:\ t > T\big\} > 0$ for any $T \geq t_x$. We make a standing hypothesis that (1.1) does possess such solutions. A solution of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros; otherwise it is *nonoscillatory*. The equation itself is called oscillatory if all its solutions are oscillatory.

For the oscillation of second-order differential equation

$$x''(t) + q(t)x(t) = 0\,, \qquad t \geq t_0\,, \tag{1.2}$$

it is known that, due to Kamenev [7], the average function $A_\lambda(t)$ defined by

$$A_\lambda(t) = \frac{1}{t^\lambda}\int_{t_0}^{t} (t-s)^\lambda q(s)\,\mathrm{d}s, \qquad \lambda > 1\,,$$

plays a crucial role in the oscillation of equation (1.2). He proved that every solution of (1.2) oscillates if

$$\lim_{t\to\infty} A_\lambda(t) = \infty\,. \tag{1.3}$$

Since Kamenev have established the condition (1.3), many authors considered some different types of second order differential equations and established some sufficient conditions for oscillations which extended and improved (1.3). For instance, Philos [16] improves Kamenev's result by proving the following:

Suppose there exist continuous functions $H, h : \mathbf{D} := \big\{(t,s) :\ t \geq s \geq t_0\big\} \to \mathbb{R}$ such that

(i) $H(t,t) = 0$, $t \geq t_0$,

(ii) $H(t,s) > 0$, $t > s \geq t_0$, and H has a continuous and nonpositive partial derivative on $\mathbf{D}$ with respect to the second variable and satisfies

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \geq 0\,. \tag{1.4}$$

Further, suppose that

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Big[H(t,s)q(s) - \frac{1}{4}h^2(t,s)\Big]\, ds = \infty\,. \tag{1.5}$$

Then every solution of equation (1.2) oscillates.

Our aim in this paper is, by using the Riccati transformation techniques, to establish some new sufficient conditions which insure that every solution of equation (1.1) oscillates or converges to zero. Our results are different from those in [3], [4], [5], [10] [15], [18], [19]. The paper is organized as follows: In Section 2, we shall present some lemmas which are useful in the proof of our main results. In Section 3, we establish sufficient condition and also conditions of Kamenev-type (1.3) and Philos-type (1.5) for oscillation of equation (1.1). In Section 4, some examples are considered to illustrate our main results.

2. Some preliminary lemmas

In this section we state and prove some lemmas, which we will use in the proof of our main results. We begin with the following lemma:

Lemma 2.1. *Assume that* (h1) *and* (h2) *hold. Let* $x(t)$ *is an eventually positive solution of* (1.1). *Then there are only the following two cases for* $t \geqslant t_1$ *sufficiently large:*

Case (I): $x(t) > 0\,, \quad x'(t) > 0\,, \quad \big(a(t)x'(t)\big)' > 0\,.$

Case (II): $x(t) > 0\,, \quad x'(t) < 0\,, \quad \big(a(t)x'(t)\big)' > 0\,.$

Proof. Let $x(t)$ be an eventually positive solution of (1.1). Then there exists a $t_1 \geqslant t_0$ such that $x(t-\sigma) > 0$ for $t \geqslant t_1$. From (1.1) we have $\big(c(t)\big(a(t)x'(t)\big)'\big)' \le 0$ for $t \geqslant t_1$. Now, we prove that $x'(t)$ is monotone and eventually of one sign. We assume that this is not true and let $x'(t) = 0$ for $t \ge t_1$. Now, since $q(t)$ is a positive real-valued function, we may let $t_2 \ge t_1$ so that $q(t_2) > 0$. Then in view of (1.1), we have

$$0 = \big(c(t_2)\big(a(t_2)x'(t_2)\big)'\big)' + q(t_2)f\big(x(t_2-\sigma)\big) = q(t_2)f\big(x(t_2-\sigma)\big) > 0\,,$$

which is a contradiction.

We claim that there is $t_2 \geqslant t_1$ such that for $t \geqslant t_2$, $\big(a(t)x'(t)\big)' > 0$. Suppose to the contrary that $\big(a(t)x'(t)\big)' \le 0$ for $t \geqslant t_2$. Since $c(t) > 0$ and $c(t)\big(a(t)x'(t)\big)'$ is nonincreasing, there exists a negative constant C and $t_3 \geqslant t_2$

such that $c(t)\big(a(t)x'(t)\big)' \le C$ for $t \geqslant t_3$. Dividing by $c(t)$ and integrating from t_3 to t, we obtain

$$a(t)x'(t) \le a(t_3)x'(t_3) + C\int_{t_3}^{t} \frac{\mathrm{d}s}{c(s)} \,.$$

Letting $t \to \infty$, then $a(t)x'(t) \to -\infty$ by (h1). Thus, there is an integer $t_4 \geqslant t_3$ such that for $t \geqslant t_4$, $a(t)x'(t) \le a(t_4)x'(t_4) < 0$. Dividing by $a(t)$ and integrating from t_4 to t we obtain

$$x(t) - x(t_4) \le a(t_4)x'(t_4)\int_{t_4}^{t} \frac{\mathrm{d}s}{a(s)} \,,$$

which implies that $x(t) \to -\infty$ as $t \to \infty$ by (h1), a contradiction with the fact that $x(t) > 0$. Then $\big(a(t)x'(t)\big)' > 0$. The proof is complete. □

Lemma 2.2. *Assume that* (h1) *and* (h2) *hold. Let $x(t)$ be an eventually positive solution of* (1.1) *and suppose that Case (I) of Lemma* 2.1 *holds. Then there exists $t_1 \geqslant t_0$ sufficiently large such that*

$$x'(t-\sigma) \geqslant \frac{\delta(t-\sigma)c(t)}{a(t-\sigma)}\big(a(t)x'(t)\big)' \qquad \text{for} \quad t \geqslant t_1 \,, \tag{2.1}$$

where $\delta(t) = \int_{t_0}^{t} \frac{\mathrm{d}s}{c(s)}$.

P r o o f. From Case (I) of Lemma 2.1 and equation (1.1) we have for $t \geqslant t_1$

$$a(t)x'(t) > 0\,, \qquad c(t)\big(a(t)x'(t)\big)' > 0 \qquad \text{and} \qquad \Big(c(t)\big(a(t)x'(t)\big)'\Big)' \le 0\,.$$

Since

$$\int_{t_1}^{t} \big(a(s)x'(s)\big)' \,\mathrm{d}s = a(t)x'(t) - a(t_1)x'(t_1)\,,$$

for $t \geqslant t_1$ we have

$$a(t)x'(t) = a(t_1)x'(t_1) + \int_{t_1}^{t} \frac{c(s)\big(a(s)x'(s)\big)'}{c(s)}\,\mathrm{d}s \geqslant c(t)\delta(t)\big(a(t)x'(t)\big)' . \tag{2.2}$$

Since $\Big(c(t)\big(a(t)x'(t)\big)'\Big)' \le 0$, we get

$$c(t-\sigma)\big(a(t-\sigma)x'(t-\sigma)\big)' \geqslant c(t)\big(a(t)x'(t)\big)' .$$

This and (2.2) imply that for $t \geqslant t_2 = t_1 + \sigma$ sufficiently large

$$\begin{aligned} a(t-\sigma)x'(t-\sigma) &\geqslant c(t-\sigma)\delta(t-\sigma)\big(a(t-\sigma)x'(t-\sigma)\big)' \\ &\geqslant c(t)\delta(t-\sigma)\big(a(t)x'(t)\big)', \end{aligned}$$

and then we obtain

$$a(t-\sigma)x'(t-\sigma) \geqslant c(t)\delta(t-\sigma)\big(a(t)x'(t)\big)', \qquad t \geqslant t_2 = t_1 + \sigma,$$

and this leads to (2.1). The proof is complete. □

3. Main oscillation results

In this section we establish some sufficient conditions which guarantee that every solution $x(t)$ of (1.1) oscillates. We start with the following theorem:

THEOREM 3.1. *Assume that* (h1)–(h2) *hold and*

(h3) $\limsup\limits_{t\to\infty} \int\limits_{t_0}^{t} \frac{1}{a(s)} \int\limits_{s}^{\infty} \frac{1}{c(u)} \int\limits_{s}^{\infty} q(\tau)\, \mathrm{d}\tau\, \mathrm{d}u\, \mathrm{d}s = \infty$.

Furthermore, assume that there exists a positive function $\rho \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ such that

$$\limsup_{t\to\infty} \int\limits_{t_0}^{t} \left(K\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)} \right) \mathrm{d}s = \infty. \tag{3.1}$$

Then every solution of (1.1) *is oscillatory.*

P r o o f. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \geqslant t_1$ where t_1 is chosen so large that Lemma 2.1 and Lemma 2.3 hold. We shall consider only this case, because the proof when $x(t) < 0$ is similar. According to Lemma 2.1 there are two possible cases.

Case (I): $x'(t) > 0$ for $t \geq t_1 \geq t_0$.

In this case, we define the function $w(t)$ by

$$w(t) = \rho(t)\frac{c(t)\big(a(t)x'(t)\big)'}{x(t-\sigma)}, \qquad t \geq t_1. \tag{3.2}$$

Then by (1.1) and Lemma 2.2, we have

$$w'(t) \leq -K\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t) - \frac{\delta(t-\sigma)}{\rho(t)a(t-\sigma)}w^2(t). \tag{3.3}$$

Thus

$$w'(t) < -K\rho(t)q(t) + \frac{(\rho'(t))^2 a(t-\sigma)}{4\rho(t)\delta(t-\sigma)} - \left[\sqrt{\frac{\delta(t-\sigma)}{\rho(t)a(t-\sigma)}}\,w(t) - \frac{\rho'(t)}{2\rho(t)}\sqrt{\frac{\rho(t)a(t-\sigma)}{\delta(t-\sigma)}}\,\right]^2, \tag{3.4}$$

and hence

$$w'(t) < -\left(K\rho(t)q(t) - \frac{(\rho'(t))^2 a(t-\sigma)}{4\rho(t)\delta(t-\sigma)}\right). \tag{3.5}$$

Integrating (3.5), we have, for $t \ge t_2$,

$$w(t) < w(t_2) - \int_{t_2}^{t} \left(K\rho(s)q(s) - \frac{(\rho'(s))^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)}\right) \mathrm{d}s\,. \tag{3.6}$$

Letting $t \to \infty$, in view of (3.1), we have $w(t) \to -\infty$, a contradiction.
Case (II): $x'(t) < 0$ for $t \ge t_1 \ge t_0$.
This implies that $x(t)$ is positive and decreasing function. Integrating equation (1.1) from t_1 to t $(t \ge t_1)$ we obtain

$$c(t)\big(a(t)x'(t)\big)' - c(t_1)\big(a(t_1)x'(t_1)\big)' + K\int_{t_1}^{t} q(s)x(s-\sigma)\,\mathrm{d}s \le 0\,.$$

From Lemma 2.1, since $c(t)\big(a(t)x'(t)\big)' > 0$ and decreasing, we have

$$-c(t_1)\big(a(t_1)x'(t_1)\big)' + K\int_{t_1}^{\infty} q(\tau)x(\tau-\sigma)\,\mathrm{d}\tau \le 0\,.$$

This implies that

$$-\big(a(t)x'(t)\big)' + K\frac{1}{c(t)}\int_{t}^{\infty} q(\tau)x(\tau-\sigma)\,\mathrm{d}\tau \le 0\,.$$

Integrating again from t to ∞, using $x'(t) < 0$, we have

$$a(t)x'(t) + K\int_{t}^{\infty}\frac{1}{c(u)}\int_{u}^{\infty} q(\tau)x(\tau-\sigma)\,\mathrm{d}\tau\,\mathrm{d}u \le 0\,,$$

so that

$$x'(t) + K\frac{1}{a(t)}\int_{t}^{\infty}\frac{1}{c(u)}\int_{u}^{\infty} q(\tau)x(\tau-\sigma)\,\mathrm{d}\tau\,\mathrm{d}u \le 0\,.$$

Integrating from t_1 to t, we obtain

$$x(t) - x(t_1) + K \int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(\tau)x(\tau - \sigma)\, \mathrm{d}\tau\, \mathrm{d}u\, \mathrm{d}s \le 0 .$$

Hence, using the fact that $x(t)$ is decreasing, we have

$$x(u) - x(t_1) + Kx(t_1) \int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(\tau)\, \mathrm{d}\tau\, \mathrm{d}u\, \mathrm{d}s \le 0 .$$

$$Kx(t_1) \int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{t} \frac{1}{c(u)} \int_{u}^{\infty} q(\tau)\, \mathrm{d}\tau\, \mathrm{d}u\, \mathrm{d}s \le -x(t) + x(t_1) \le x(t_1) .$$

This implies that

$$\int_{t_1}^{t} \frac{1}{a(s)} \int_{s}^{\infty} \frac{1}{c(u)} \int_{u}^{\infty} q(\tau)\, \mathrm{d}\tau\, \mathrm{d}u\, \mathrm{d}s \le \frac{1}{K} ,$$

which contradicts (h3). The proof is complete. □

Next, we present some new oscillation results for equation (1.1), by using integral averages condition of Kamenev-type.

THEOREM 3.2. *Let all the assumptions of Theorem* 3.1 *hold except the condition* (3.1)*, which is changed to*

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_0}^{t} (t-s)^n \left[K\rho(s)q(s) - \frac{(\rho'(s))^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)} \right] \mathrm{d}s = \infty . \tag{3.7}$$

Then every solution $x(t)$ *of* (1.1) *is oscillatory.*

P r o o f . Proceeding as in the proof of Theorem 3.1, we assume that equation (1.1) has a nonoscillatory solution, say $x(t) > 0$ and $x(t-\sigma) > 0$ for all $t \ge t_1$ where t_1 is chosen so large that Lemma 2.1 and Lemma 2.3 hold and there are two possible cases.

If the Case (I) holds, then by defining again $w(t)$ by (3.2) as in Theorem 3.1, we have $w(t) > 0$ and (3.5) holds. From (3.5) we have for $t \ge t_1$

$$\int_{t_1}^{t} (t-s)^n \left[K\rho(s)q(s) - \frac{(\rho'(s))^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)} \right] \mathrm{d}s < - \int_{t_1}^{t} (t-s)^n w'(s)\, \mathrm{d}s . \tag{3.8}$$

Since

$$\int_{t_1}^{t} (t-s)^n w'(s)\, ds = n \int_{t_0}^{t} (t-s)^{n-1} w(s)\, ds - w(t_1)(t-t_1)^n\,, \tag{3.9}$$

we get

$$\frac{1}{t^n} \int_{t_1}^{t} (t-s)^n Q(s)\, ds \le w(t_1)\left(\frac{t-t_1}{t}\right)^n - \frac{n}{t^n} \int_{t_1}^{t} (t-s)^{n-1} w(s)\, ds\,, \tag{3.10}$$

where

$$Q(s) = K\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)}\,.$$

Hence

$$\frac{1}{t^n} \int_{t_1}^{t} (t-s)^n Q(s)\, ds \le w(t_1)\left(\frac{t-t_1}{t}\right)^n\,. \tag{3.11}$$

Then

$$\limsup_{t\to\infty} \frac{1}{t^n} \int_{t_1}^{t} (t-s)^n Q(s)\, ds \to w(t_1)\,,$$

which contradicts the condition (3.7).

If the Case (II) holds, we come back to the proof of the second part of Theorem 3.1 and hence it is omitted. The proof is complete. □

Next, we present some new oscillation results for equation (1.1) by using integral averages condition of Philos-type. Following Philos [16], we introduce a class of functions $\Re$. Let

$$\mathbf{D}_0 = \big\{(t,s):\ t > s \ge t_0\big\} \qquad \text{and} \qquad \mathbf{D} = \big\{(t,s):\ t \ge s \ge t_0\big\}\,.$$

The function $H \in C(\mathbf{D}, \mathbb{R})$ is said *to belong to the class* $\Re$ if

(i) $H(t,t) = 0$ for $t \ge t_0$; $H(t,s) > 0$ for $(t,s) \in \mathbf{D}_0$;

(ii) H has a continuous and nonpositive partial derivative on $\mathbf{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(t,s)}{\partial s} = h(t,s)\sqrt{H(t,s)} \qquad \text{for all} \quad (t,s) \in \mathbf{D}_0\,.$$

THEOREM 3.3. *Assume that* (h1) (h3) *hold. Furthermore, assume that there exist functions* $\rho \in C^1([t_0,\infty),\mathbb{R}^+)$ *and* $H \in \Re$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\left(K\rho(s)q(s) - \frac{\rho(s)a(s-\sigma)Q^2(t,s)}{4\delta(s-\sigma)}\right) \mathrm{d}s = \infty\,, \tag{3.12}$$

where

$$Q(t,s) = \frac{h(t,s)}{\sqrt{H(t,s)}} - \frac{\rho'(s)}{\rho(s)}\,. \tag{3.13}$$

Then every solution $x(t)$ *of* (1.1) *is oscillatory.*

P r o o f. Let x be a nonoscillatory solution of (1.1). Let us first assume that x is eventually positive and that $x(t) > 0$ and $x(t-\sigma) > 0$ for $t \ge t_1$. The case where x is eventually negative is dealt with similarly and is omitted. As in the proof of Lemma 2.1 there are two possible cases.

Let the Case (I) hold: Again, defining $w(t)$ as in (3.1), we obtain (3.3). Let us denote

$$\gamma(s) = \frac{\rho'(s)}{\rho(s)} \qquad \text{and} \qquad W(s) = \frac{\delta(s-\sigma)}{\rho(s)a(s-\sigma)}\,.$$

Then from (3.3), we get

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)K\rho(s)q(s)\,\mathrm{d}s \\
&\le \int_{t_1}^{t} H(t,s)\left[-w'(s) + \gamma(s)w(s) - W(s)w^2(s)\right] \mathrm{d}s \\
&= -H(t,s)w(s)\Big|_{t_1}^{t} + \int_{t_1}^{t} \left\{\frac{\partial H(t,s)}{\partial s} w(s) + H(t,s)\left[\gamma(s)w(s) - W(s)w^2(s)\right] \mathrm{d}s\right. \\
&= H(t,t_1)w(t_1) - \int_{t_1}^{t} \Big[\sqrt{H(t,s)}\left(h(t,s) - \sqrt{H(t,s)}\gamma(s)\right) w(s) \\
&\qquad\qquad + H(t,s)W(s)\,w^2(s)\Big]\,\mathrm{d}s \\
&= H(t,t_1)w(t_1) - \int_{t_1}^{t} \left[\sqrt{H(t,s)W(s)}w(s) + \frac{1}{2}\frac{Q(t,s)}{\sqrt{W(s)}}\right]^2 + \frac{Q^2(t,s)}{4W(s)}\,\mathrm{d}s\,.
\end{aligned} \tag{3.14}$$

It follows that

$$\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left(KH(t,s)\rho(s)q(s)-\frac{Q^2(t,s)}{4W(s)}\right)\,\mathrm{d}s \le w(t_1)\,. \tag{3.15}$$

This contradicts (3.12).

If the Case (II) holds, we come back to the proof of the second part of Theorem 3.1 and hence it is omitted. The proof is complete. □

The following two results provide alternative oscillation criteria when (3.12) is difficult to verify. In these results, we make use of the techniques of Y a n [20], [21]. The notations of Theorem 3.3 and its proof will be used.

THEOREM 3.4. *Let all the assumptions, except* (3.12), *of Theorem* 3.3 *hold. Further, let*

$$0<\inf_{s\ge t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le\infty\,, \tag{3.16}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{Q^2(t,s)}{W(s)}\,\mathrm{d}s<\infty\,. \tag{3.17}$$

Let $\psi\in C\big([t_0,\infty),\mathbb{R}\big)$ *such that for* $t\ge t_0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\psi_+^2(s)W(s)\,\mathrm{d}s=\infty\,, \tag{3.18}$$

where $\psi_+(t)=\max\{\psi(t),0\}$, *and*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[KH(t,s)\rho(s)q(s)-\frac{Q^2(t,s)}{4W(s)}\right]\,\mathrm{d}s\ge\sup_{t>t_0}\psi(t)\,. \tag{3.19}$$

Then every solution $x(t)$ *of* (1.1) *is oscillatory.*

P r o o f . As in the proof of Theorem 3.3, we have (3.14). It follows that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[KH(t,s)\rho(s)q(s)-\frac{Q^2(t,s)}{4W(s)}\right]\,\mathrm{d}s$$

$$\le w(t_1)-\liminf_{t\to\infty}\frac{1}{H(t,t_1)}\int_{t_1}^{t}\left[\sqrt{H(t,s)W(s)}\,w(s)+\frac{Q(t,s)}{2\sqrt{W(s)}}\right]^2\,\mathrm{d}s\,.$$

By (3.19), it follows that

$$w(t_1) \geq \psi(t_1) + \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 \mathrm{d}s\,, \tag{3.20}$$

and hence

$$\begin{aligned} 0 &\leq \liminf_{t\to\infty} \frac{1}{H(t,t_1)} \int_{t_1}^{t} \left[\sqrt{H(t,s)W(s)}w(s) + \frac{Q(t,s)}{2\sqrt{W(s)}} \right]^2 \mathrm{d}s \\ &\leq w(t_1) - \psi(t_1) < \infty\,. \end{aligned} \tag{3.21}$$

Define the functions α and β by

$$\begin{aligned} \alpha(t) &= \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s)W(s)w^2(s)\,\mathrm{d}s\,, \\ \beta(t) &= \frac{1}{H(t,t_1)} \int_{t_1}^{t} \sqrt{H(t,s)}Q(t,s)w(s)\,\mathrm{d}s\,. \end{aligned} \tag{3.22}$$

Then, it follows from (3.21) that

$$\liminf_{t\to\infty} \big[\alpha(t) + \beta(t)\big] < \infty\,. \tag{3.23}$$

Now we claim that

$$\int_{t_1}^{\infty} W(s)w^2(s)\,\mathrm{d}s < \infty\,. \tag{3.24}$$

Suppose to the contrary that

$$\int_{t_1}^{\infty} W(s)w^2(s)\,\mathrm{d}s = \infty\,. \tag{3.25}$$

By (3.16), there is a positive constant ζ such that

$$\inf_{s\geq t_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right] > \zeta\,. \tag{3.26}$$

Let μ be an arbitrary positive number, then by (3.25) there exists $t_2 \geq t_1$ such that

$$\int_{t_1}^{t} W(s)w^2(s)\,\mathrm{d}s \geq \frac{\mu}{\zeta}\,, \qquad t \geq t_2\,,$$

and therefore, for $t \geq t_2$,

$$\begin{aligned}
\alpha(t) &= \frac{1}{H(t,t_1)} \int_{t_1}^{t} H(t,s) \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{t_1}^{s} W(u)w^2(u)\,\mathrm{d}u\right] \\
&= \frac{1}{H(t,t_1)} \int_{t_1}^{t} -\frac{\partial H(t,s)}{\partial s}\left[\int_{t_1}^{s} W(u)w^2(u)\,\mathrm{d}u\right] \mathrm{d}s \\
&\geq \frac{1}{H(t,t_1)} \int_{t_2}^{t} -\frac{\partial H(t,s)}{\partial s}\left[\int_{t_1}^{s} W(u)w^2(u)\,\mathrm{d}u\right] \mathrm{d}s \\
&\geq \frac{\mu}{\zeta}\frac{1}{H(t,t_1)} \int_{t_1}^{t} -\frac{\partial H(t,s)}{\partial s}\,\mathrm{d}s = \frac{\mu}{\zeta}\frac{H(t,t_2)}{H(t,t_1)}\,.
\end{aligned}$$

By (3.26), there exists $t_3 \geq t_2$ such that

$$\frac{H(t,t_2)}{H(t,t_1)} \geq \zeta\,, \qquad t \geq t_3\,.$$

This implies that $\alpha(t) \geq \mu$ for all $t \geq t_3$. As μ is arbitrary, we have

$$\lim_{t\to\infty} \alpha(t) = \infty\,. \tag{3.27}$$

The reminder of the proof of this case is similar to that of the proof of [17; Theorem 5.2] and hence is omitted.

If the Case (II) holds, we came back to the proof of the second part of Theorem 3.1 and hence it is omitted. The the proof is complete. □

THEOREM 3.5. *Let all the assumptions of Theorem* 3.4 *hold except the condition* (3.17)*, which is changed to*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)\,\mathrm{d}s < \infty\,. \tag{3.28}$$

Then every solution $x(t)$ of (1.1) *is oscillatory or $x(t) \to 0$ as $t \to \infty$.*

The proof of Theorem 3.5 is similar to that of Theorem 3.4 and hence it is omitted.

Remark 3.1. For the choice $H(t,s) = (t-s)^n$ and $h(t,s) = n(t-s)^{(n-2)/2}$, the Philos-type condition reduces to the Kamenev-type condition. Other choices of H include

$$H(t,s) = \left(\ln\frac{t}{s}\right)^n, \qquad h(t,s) = \frac{n}{s}\left(\ln\frac{t}{s}\right)^{n/2-1},$$

and

$$H(t,s) = \left(\mathrm{e}^t - \mathrm{e}^s\right)^n, \qquad h(t,s) = n\,\mathrm{e}^s\left(\mathrm{e}^t - \mathrm{e}^s\right)^{(n-2)/2},$$

or more generally,

$$H(t,s) = \left(\int_s^t \frac{\mathrm{d}u}{\theta(u)}\right)^n, \qquad h(t,s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{\mathrm{d}u}{\theta(u)}\right)^{n/2-1},$$

where $n > 1$ is an integer, and $\theta \in C\big([t_0,\infty),\mathbb{R}^+\big)$ satisfies

$$\lim_{t\to\infty} \int_{t_0}^t \frac{\mathrm{d}u}{\theta(u)} = \infty .$$

4. Examples

In this section, we give several examples to illustrate our main results.

EXAMPLE 4.1. Consider the third order nonlinear delay differential equation

$$\left(t\left(\frac{1}{\ln t}x'\right)'\right)' + t^\alpha x(t-1)\big(1 + x^2(t-1)\big) = 0 \qquad \text{for} \quad t \geq 1, \tag{4.1}$$

where $\alpha \geq 1$. Here $c(t) = t$, $a(t) = \frac{1}{\ln t}$, $q(t) = t^\alpha$ and $f(u) = u(1+u^2) \geq u$ with $K = 1$. From this we have $\delta(t) = \int_1^t \frac{1}{c(s)}\,\mathrm{d}s = \ln t$. It is clear that the conditions (h1)–(h3) are satisfied. It remains to satisfy the condition (3.1). Now, by choosing $\rho(t) = t$ we have

$$\limsup_{t\to\infty} \int_{t_0}^t \left(K\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2 a(s-\sigma)}{4\rho(s)\delta(s-\sigma)}\right) \mathrm{d}s$$

$$= \limsup_{t\to\infty} \int_1^t \left[t^{\alpha+1} - \frac{1}{4t\big(\ln(t-1)\big)^2}\right] \mathrm{d}s$$

$$\geq \limsup_{t\to\infty} \int_1^t \left[t^{\alpha+1} - \frac{1}{4t(\ln t)^2}\right] \mathrm{d}s = \infty .$$

Consequently condition (3.1) is satisfied. Hence, by Theorem 3.1, every solution of equation (4.1) oscillates or converges to zero.

EXAMPLE 4.2. Consider the third order nonlinear delay differential equation

$$\big((t+1)^{-\gamma}x'\big)'' + t^{\lambda}\Big(\lambda\frac{2-\cos t}{t} + (2+\sin t)\Big)x(t-1)\big(1+x^2(t-1)\big) = 0 \quad \text{for} \quad t \geq 1\,, \tag{4.2}$$

where γ and λ are positive constants. Here $c(t) = 1$, $a(t) = (t+1)^{-\gamma}$, $q(t) = t^{\lambda}\big(\lambda\frac{2-\cos t}{t} + \sin^2 t\big)$ and $f(u) = u(1+u^2) \geq u$ with $K = 1$. Then, for any $t \geq 1$ we have

$$\int_1^{\infty} \frac{1}{c(s)}\,\mathrm{d}s = \infty\,, \qquad \int_1^{\infty} \frac{1}{a(s)}\,\mathrm{d}s = \infty\,, \qquad \delta(t) = \int_1^{t} \frac{1}{c(s)}\,\mathrm{d}s = t-1\,. \tag{4.3}$$

Also,

$$\begin{aligned} \int_{t_0}^{t} q(s)\,\mathrm{d}s &= \int_{t_0}^{t} s^{\lambda}\Big(\lambda\frac{2-\cos s}{s} + (2+\sin s)\Big)\,\mathrm{d}s \\ &\geq \int_{t_0}^{t} s^{\lambda}\Big(\lambda\frac{2-\cos s}{s} + \sin s\Big)\,\mathrm{d}s \\ &= \int_{t_0}^{t} \mathrm{d}\big[s^{\lambda}(2-\cos s)\big] \\ &= t^{\lambda}(2-\cos t) - (2-\cos 1) \\ &\geq t^{\lambda} - K_0 \to \infty \qquad \text{as} \quad t \to \infty\,. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4) we see that (h1)–(h3) hold. To apply Theorem 3.3, it remains to satisfy the condition (3.12). Taking $H(t,s) = (t-s)^2$, $\rho(t) = 1$ for $t \geq s \geq 1$, we have

$$\begin{aligned} &\frac{1}{t^2}\int_1^{t} \left[(t-s)^2 q(s) - \frac{(t-s)^2}{s^{\gamma}(s-2)}\right]\mathrm{d}s \\ &= \frac{1}{t^2}\int_1^{t} \left[2(t-s)\bigg(\int_{t_0}^{s} q(u)\,\mathrm{d}u\bigg) - \frac{(t-s)^2}{s^{\gamma}(s-2)}\right]\mathrm{d}s \\ &\geq \frac{1}{t^2}\int_1^{t} \left[2(t-s)\bigg(\int_{t_0}^{s} q(u)\,\mathrm{d}u\bigg) - \frac{(t-s)^2}{s^{\gamma+1}}\right]\mathrm{d}s \\ &= \frac{1}{t^2}\int_1^{t} 2(t-s)\bigg(\int_{t_0}^{s} q(u)\,\mathrm{d}u\bigg)\,\mathrm{d}s - \frac{1}{t^2}\int_1^{t} \big[t^2 s^{-(\gamma+1)} - 2ts^{-\gamma} + s^{-\gamma+1}\big]\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{2}{t^2}\int_1^t (t-s)\left(s^\lambda - K_0\right) \mathrm{d}s - \frac{1}{t^2}\int_1^t \left[t^2 s^{-(\gamma+1)} - 2ts^{-\gamma} + s^{-\gamma+1}\right] \mathrm{d}s \\
&= \frac{2t^\lambda}{(\lambda+1)(\lambda+2)} + \frac{K_1}{t^2} + \frac{K_2}{t} - K_0 \\
&\quad - \frac{1}{t^2}\left[-\frac{t^2}{\gamma s^\gamma} - \frac{2t}{(-\gamma+1)s^{\gamma-1}} + \frac{1}{(-\gamma+2)s^{\gamma-2}}\right]_1^t \\
&= \frac{2t^\lambda}{(\lambda+1)(\lambda+2)} + \frac{K_1}{t^2} + \frac{K_2}{t} - K_0 + \frac{1}{\gamma t^\gamma} + \frac{2}{(-\gamma+1)t^\gamma} - \frac{1}{(-\gamma+2)t^\gamma} + K_3\,,
\end{aligned}
$$

where K_i, $i = 0, 1, 2, 3$, are constants. Consequently, condition (3.12) is satisfied. Hence, every solution of equation (4.2) oscillates or converges to zero.

EXAMPLE 4.3. Consider the third order nonlinear delay differential equation

$$
\left((t+1)^{-\gamma}\left(\frac{1}{t}x'\right)'\right)' + t^\lambda(2+\cos t)x(t-1)\left(1+x^2(t-1)\right) = 0 \qquad \text{for} \quad t \geq 1\,, \tag{4.5}
$$

where γ and λ are positive constants. Here $c(t) = 1/t$, $a(t) = (t+1)^{-\gamma}$, $q(t) = t^\lambda(2+\cos t)$ and $f(u) = u(1+u^2) \geq u$ with $K = 1$. Then, for any $t \geq 1$ we have

$$
\int_1^\infty \frac{1}{c(s)}\,\mathrm{d}s = \infty\,, \qquad \int_1^\infty \frac{1}{a(s)}\,\mathrm{d}s = \infty\,, \qquad \delta(t) = \int_1^t \frac{1}{c(s)}\,\mathrm{d}s = \frac{t^2-1}{2}\,. \tag{4.6}
$$

Also,

$$
\int_{t_0}^t q(s)\,\mathrm{d}s = \int_{t_0}^t s^\lambda(2+\cos s)\,\mathrm{d}s \geq \int_{t_0}^t s^\lambda\,\mathrm{d}s \to \infty \qquad \text{as} \quad t \to \infty\,. \tag{4.7}
$$

From (4.6) and (4.7) we see that (h1)–(h3) hold. To apply Theorem 3.4, it remains to satisfy the conditions (3.17)–(3.19). Taking $H(t,s) = (t-s)^2$ and $\rho = 1$, we have

$$
\frac{1}{t^2}\int_{t_0}^t \frac{Q^2(t,s)}{W(s)}\,\mathrm{d}s = \frac{1}{t^2}\int_1^t \frac{(t-s)^2}{t^{\gamma+1}(t-2)}\,\mathrm{d}s < \infty\,. \tag{4.8}
$$

Therefore, condition (3.17) is satisfied and for arbitrary small constant $\varepsilon > 0$, there exists a $t_1 \geq 1$ such that for $T \geq t_1$

$$\frac{1}{t^2}\int_{t_0}^{t}\left[H(t,s)q(s)-\frac{Q^2(t,s)}{4W(s)}\right]\,\mathrm{d}s = \frac{1}{t^2}\int_{t_0}^{t}\left[(t-s)^2 s^\lambda(2+\cos s)-\frac{(t-s)^2}{2t^{\gamma+1}(t-2)}\right]\,\mathrm{d}s$$
$$\geq \frac{1}{t^2}\int_{t_0}^{t}\left[(t-s)^2 s^\lambda\cos s-\frac{(t-s)^2}{2t^{\gamma+1}(t-2)}\right]\,\mathrm{d}s$$
$$\geq -T^\lambda\cos T-\varepsilon=\psi(T)\,.$$

Then, there exists an integer N such that $(2N+1)\pi-\pi/4 > t_1$, and if $n \geq N$,

$$(2n+1)\pi-\frac{\pi}{4}\leq T\leq(2n+1)\pi+\frac{\pi}{4}\,,\qquad \psi(T)\geq\beta T^\lambda\,,$$

where β is small constant. Now, we have

$$\int_{t_0}^{\infty}\psi_+^2(s)W(s)\,\mathrm{d}s=\int_{t_0}^{\infty}\psi_+^2(s)\frac{\delta(s-1)}{a(s-1)}\geq\frac{\beta^2}{2}\sum_{n=N}^{\infty}\int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}}\frac{s^{2\lambda}s(s-2)}{s^{-\gamma}}\,\mathrm{d}s$$
$$\geq\frac{\beta^2}{2}\sum_{n=N}^{\infty}\int_{(2n+1)\pi-\frac{\pi}{4}}^{(2n+1)\pi+\frac{\pi}{4}}(s-2)^{2\lambda+2+\gamma}\,\mathrm{d}s=\infty\,.$$

Accordingly, all conditions of Theorem 3.4 are satisfied, and hence every solution of equation (4.5) oscillates or converges to zero.

We note that none of the above mentioned papers of oscillation of third order differential equations can be applied to the delay equations (4.1), (4.2) and (4.5).

Remark 4.1. It remains, as an open problem, to study the oscillation behavior of equation (1.1) when

$$\int_{t_0}^{\infty}\frac{1}{c(t)}\,\mathrm{d}t<\infty\,,\qquad\int_{t_0}^{\infty}\frac{1}{a(t)}\,\mathrm{d}t<\infty\,,\qquad\int_{t_0}^{\infty}q(t)\,\mathrm{d}t<\infty\,.$$

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