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Dedicated to memory of professor Hans-Jürgen Voss

A KOTZIG TYPE THEOREM FOR NON-ORIENTABLE SURFACES

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(*Communicated by Martin Škoviera*)

ABSTRACT. A. Kotzig in 1955 proved that every polyhedral map on the sphere (i.e., a 3-connected plane graph) contains an edge with degree sum of its endvertices at most 13; this bound being sharp. J. Ivančo in 1992 proved an analogue of Kotzig's theorem for graphs of an orientable genus g. In this note it is proved that every simple graph embeddable in a non-orientable surface of genus q and minimum degree ≥ 3 contains an edge e with degree sum $w(e)$ of its endvertices being

$$w(e) \leq \begin{cases} 2q+11 & \text{if } 1 \leq q \leq 2, \\ 2q+9 & \text{if } 3 \leq q \leq 5, \\ 2q+7 & \text{if } q \geq 6. \end{cases}$$

All the above bounds are tight.

1. Introduction

Throughout this note we use terminology of [MT]. However, we recall some definitions. Informally, an *orientable surface* $\mathbb{S}_g$ of genus g is obtained from the sphere by adding g handles. Correspondingly, a *non-orientable surface* $\mathbb{N}_q$ of genus q is obtained from the sphere by adding q crosscaps. The *Euler characteristic* is defined by

$$\chi(\mathbb{S}_g) = 2 - 2g \qquad \text{and} \qquad \chi(\mathbb{N}_q) = 2 - q\,.$$

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By the *genus* g (the *non-orientable genus* q) of a graph G we mean the smallest integer g (q) such that G has an embedding into $\mathbb{S}_g$ ($\mathbb{N}_q$, respectively). Graphs may have loops or multiple edges. Simple graphs have neither loops nor multiple edges. If a graph G is embedded in a surface $\mathbb{M}$, then the connected components $\mathbb{M}-G$ are called the *faces* of G. If each face is an open disc, then the embedding is called a 2*-cell embedding*. The *facial walk* of a face α in a 2-cell embedding is the shortest closed walk induced by all the edges incident to α. The *degree* of a face α of a 2-cell embedding is the length of its facial walk. Vertices and faces of degree i are called i*-vertices* and i*-faces*, respectively. If in a 2-cell embedding of a graph G in a surface $\mathbb{M}$ each vertex has degree at least three, then we call G a *map* in $\mathbb{M}$. Let $v_i = v_i(G)$ denote the number of i-vertices in G. We denote by $e_{i,j}(G)$ the number of edges in G having endvertices which are an i-vertex and a j-vertex. We will write $e_{i,j}$ instead of $e_{i,j}(G)$ if G is known from the context. Let $e = AB$ be an edge of a map G with endvertices A and B. The *weight* $w(e)$ of the edge is the sum of degrees of its endvertices, i.e. $w(e) = \deg(A) + \deg(B)$. By the weight of a graph G we understand the quantity $w(G) = \min\{w(e) :\ e \in E(G)\}$.

If each face of a map G in $\mathbb{M}$ is a 3-face (a 4-face), then the map G is called a *triangulation* (a *quadrangulation*, respectively).

Kotzig's theorem ([K]) states that every 3-connected planar graph G contains an edge e of the weight $w(e)$ at most 13; and 13 is the best possible bound.

Zaks [Z] proved that the weight of every graph which triangulates $\mathbb{S}_g$ ($g \geq 1$) is at most $n(g)$, where $n(g)$ is the least odd integer which is greater than $6 + \sqrt{48g+1}$.

Ivančo extended the Kotzig's theorem as follows:

THEOREM 1. ([I]) *Let G be a simple graph of minimum degree $\delta(G) \geq 3$ embeddable in an orientable surface of genus g. Then G contains an edge e of the weight*

$$w(e) \leq \begin{cases} 2g+13 & \text{if } 0 \leq g \leq 3, \\ 4g+7 & \text{if } g \geq 3. \end{cases}$$

The bounds are tight.

In a recent survey paper [JV1; p. 403] on light subgraph in graphs on surfaces there is posed a problem to find an analogue to the theorem of Ivančo for maps on non-orientable surfaces. The main purpose of this note is to solve this problem.

THEOREM 2. *Let G be a simple graph of minimum degree $\delta(G) \geq 3$ embeddable in a non-orientable surface $\mathbb{N}_q$ of genus q, $q \geq 1$. Then G contains an edge e of the weight*

$$w(e) \leq \begin{cases} 2q+11 & \text{if } 1 \leq q \leq 2\,, \\ 2q+9 & \text{if } 3 \leq q \leq 5\,, \\ 2q+7 & \text{if } q \geq 6\,. \end{cases}$$

Moreover, all the above bounds are tight.

2. Proof of Theorem 2 — upper bounds

LEMMA 1. *Let G be a graph of minimum degree $\delta(G) \geq 3$ that triangulates a non-orientable surface $\mathbb{N}_q$ of genus q. Let n denote an odd integer, $n \geq 13$, such that $w(G) \geq n$, and let $v = \sum\limits_{i>\frac{n}{2}} v_i$. Then*

$$w(G) = n = \begin{cases} 13 & \text{if } q = 1\,, \\ 15 + 2\left\lfloor \frac{6(q-2)}{v} \right\rfloor & \text{if } q \geq 2\,. \end{cases}$$

P r o o f . The Euler's formula applied to G yields

$$3v_3 + 2v_4 + v_5 = 6(2-q) + \sum_{k\geq 7} (k-6)v_k\,. \tag{1}$$

Because G is a triangulation with $w(G) \geq n$, the following facts are almost obvious:

(2) At most $\lfloor \frac{k}{2} \rfloor$ neighbours of every k-vertex are of degree $\leq \frac{n-1}{2}$.

(3) $e_{3,k} \leq \lfloor \frac{k}{2} \rfloor v_k$, $e_{3,k} + e_{4,k} \leq \lfloor \frac{k}{2} \rfloor v_k$, $e_{3,k} + e_{4,k} + e_{5,k} \leq \lfloor \frac{k}{2} \rfloor v_k$ for all k.

As $\sum\limits_{k\geq 3} e_{3,k}$ counts all edges with one end 3-vertex and since $e_{3,k} = 0$ for all k, $k \leq n-4$, it follows that

$$3v_3 - e_{3,n-3} = \sum_{k\geq n-2} e_{3,k} \leq \sum_{k\geq n-2} \left\lfloor \frac{k}{2} \right\rfloor v_k\,. \tag{4}$$

Analogously to (4), we obtain

$$3v_3 + 4v_4 - e_{4,n-4} = \sum_{k\geq n-3} (e_{3,k} + e_{4,k}) \leq \sum_{k\geq n-3} \left\lfloor \frac{k}{2} \right\rfloor v_k\,, \tag{5}$$

$$3v_3 + 4v_4 + 5v_5 - e_{5,n-5} = \sum_{k\geq n-4} (e_{3,k} + e_{4,k} + e_{5,k}) \leq \sum_{k\geq n-4} \left\lfloor \frac{k}{2} \right\rfloor v_k\,. \tag{6}$$

By multiplying the inequalities (4), (5) and (6) by 5, 3 and 2, respectively, and adding them together we get

$$\begin{aligned}&30v_3 + 20v_4 + 10v_5 - 2e_{5,n-5} - 3e_{4,n-4} - 5e_{3,n-3}\\ &\leq 2\left\lfloor\frac{n-4}{2}\right\rfloor v_{n-4} + 5\left\lfloor\frac{n-3}{2}\right\rfloor v_{n-3} + 10\sum_{k\geq n-2}\left\lfloor\frac{k}{2}\right\rfloor v_k\,.\end{aligned}\tag{7}$$

Properties (1) and (7), after some manipulations, yield

$$\begin{aligned}&2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3}\\ &\geq 60(2-q) + 10\sum_{k=7}^{n-5}(k-6)v_k + (9n-95)v_{n-4}\\ &\quad + \frac{1}{2}(15n-165)v_{n-3} + 10\sum_{i\geq n-2}\left\lfloor\frac{i-11}{2}\right\rfloor v_i\end{aligned}\tag{8}$$

and

$$2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3} \geq 60(2-q) + (5n-65)\sum_{i\geq \frac{n}{2}} v_i\,.\tag{9}$$

If $q = 1$ and $n = 13$ then, by (9),

$$2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3} \geq 60\,.$$

This implies that $w(G) \leq 13$.

Let $q \geq 2$. The smallest odd integer n for which the right side of (9) is positive has form $n = 15 + 2\left\lfloor\frac{6(q-2)}{v}\right\rfloor$. We conclude that $w(G) \leq n$. □

P r o o f o f t h e u p p e r b o u n d . Suppose we have a cellular embedding of a simple graph G of $\delta(G) \geq 3$ in a non-orientable surface $\mathbb{N}_q$ of genus q. Note that such embeddings exist, see [MT; p. 95]. Each face α of G which is not a triangle is split into triangles by inserting new edges into α joining only vertices of degree $> \frac{1}{2}\big(w(G)-1\big)$ of α. This is always possible because each edge of α contains a vertex of degree $> \frac{1}{2}\big(w(G)-1\big)$. The result is a triangulation G^* with the property that $w(G) \leq w(G^*)$.

Next we apply our Lemma 1 to G^*. It is sufficient to consider $q \geq 2$.

Let e be an edge of G^* with an i-vertex and a j-vertex as its endvertices, where $i + j = w(G^*)$ and let every edge of the weight $w(G^*)$ have its ends of degree $\geq i$; note that $3 \leq i \leq v$ and $i \leq j$. Then e is also an edge of G and its endvertices are an i-vertex and a j-vertex, $j \leq \left\lfloor\frac{j}{2}\right\rfloor + v - 1$, in G. Therefore

$$w(G) \leq w(e) \leq i + \min\left\{j,\ \left\lfloor\tfrac{j}{2}\right\rfloor + v - 1\right\} = \min\left\{w(G^*),\ \left\lfloor\tfrac{w(G^*)-i}{2}\right\rfloor + v + i - 1\right\}.\tag{10}$$

Hence

$$w(G) \le \max_{3\le i\le v} \min\left\{15 + 2\left\lfloor\tfrac{6(q-2)}{v}\right\rfloor,\ v + \left\lfloor\tfrac{13+i}{2}\right\rfloor + \left\lfloor\tfrac{6(q-2)}{v}\right\rfloor\right\}. \tag{11}$$

To finish the proof it is enough to show that at least one of the terms of the right side in (11) is bounded above by $2q+7$ for $q \ge 6$, $2q+9$ for $3 \le q \le 5$ and $2q+11$ for $1 \le q \le 2$, respectively. It is easy to verify that $15+2\lfloor\frac{6(q-2)}{v}\rfloor \le 2q+7$ for any pair q and v, $q \ge 7$ and $v \ge 8$, and that for any $q \ge 7$ and $3 \le v \le 7$ the second term of (11) is $\le 2q+7$. Similarly for $q \in \{4,5\}$ the first term of (11) is $\le 2q+9$ if $v \ge 7$ and the second one is $\le 2q+9$ if $3 \le v \le 6$.

For $q = 2$ the first term of (11) is bounded with 15 for any $v \ge 3$.

For $q = 3$ and $q = 6$ we obtain better bounds as those given by (11). These exceptional cases will now be treated separately.

Let $q = 3$. For $v = 6$ and $i \in \{3,4\}$ or for $v \ne 6$ the formula (11) gives $w(G) \le 15$. For $i = 5$ and $v = 6$ there is $w(G^*) \le 17$. If $w(G^*) = n = 17$, then $v_{12} \ge 1$ and, by (8), $2e_{5,12} \ge -60+60v_{12}+20\sum_{i\ge 13} v_i = -60+60v_{12}+20(v-v_{12}) = -60+40v_{12}+20v \ge 100$. Hence $e_{5,12} \ge 50$. But every 12-vertex has at most six neighbours of degree ≤ 8. Because $v = 6$ there is, in G^*, $e_{5,12} \le 36$, which is a contradiction. In the case $n \le 16$, by (10), we have $w(G) \le 15$.

If $i = v = 6$, then $w(G^*) \le 17$. Suppose $w(G^*) = n \in \{16,17\}$. The formula (9) gives $2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3} \ge -60 + (5n-65)v \ge 30$. This is the contradiction with the choice $i = 6$. Hence $w(G) \le w(G^*) \le 15$.

Let $q = 6$. For $v \le 6$ or $v \ge 9$ the formula (11) gives $w(G) \le 19$. If $v = 7$ and $i \le 6$, then the second term of (11) gives $w(G) \le 19$.

Let $i = v = 7$. Then $w(G^*) \le 21$. If $w(G^*) \le 20$, then, by (10), we get $w(G) \le 19$. If $w(G^*) = 21$, then, by (9), $2e_{5,16} + 3e_{4,17} + 5e_{3,18} \ge 40$, which is a contradiction with the assumption that $i = 7$.

For $i \le 4$ and $v = 8$ the second term of (11) yields $w(G) \le 19$.

Let $i = 5$ and $v = 8$. Again $w(G^*) = n \le 21$. If $n = 21$, then $v_{16} \ge 1$ and, by (8), $2e_{5,16} \ge -240 + 100v_{16} + 40(v - v_{16}) \ge 140$. Hence $e_{5,16} \ge 70$. On the other hand every 16-vertex has at most eight neighbours of degree ≤ 10. Since $v = 8$, $e_{5,16} \le 64$, which is a contradiction. If $w(G^*) = n \le 20$, then the formula (10) provides $w(G) \le 19$.

Consider the case $q = 6$, $6 \le i \le v = 8$. Suppose $w(G^*) = n \in \{20,21\}$. This means that there is an edge e in G^* with $w(e) = n$. By (9), there holds $2e_{5,n-5} + 3e_{4,n-4} + 5e_{3,n-3} \ge -240 + (5n-65)v \ge 40$. This means that there is an edge h in G^* with $w(h) = n$ of which one endvertex has degree ≤ 5, a contradiction with $i \ge 6$. So $w(G) \le w(G^*) \le 19$. □

3. Tightness of bounds

The goal of this section is to prove that for every non-orientable surface $\mathbb{N}_q$ there exists a 2-cell embedding of a graph G into $\mathbb{N}_q$ such that the weight of G is equal to the bounds of Theorem 2.

For $q = 1,3,4,5$, Ringel [R2] proved the existence of minimum embeddings of complete graphs K_n, $n = 6,7,8,9$, respectively, into non-orientable surfaces $\mathbb{N}_q$. By [MT] these embeddings are 2-cell. It is easy to see that this embedding of K_n contains a 3-face. Into every face a new vertex X is inserted and joint with every vertex incident with this face. The result is a new triangulation G which has a new 3-vertex X. Because any "new" vertex in G has degree at least 3 and any "old" vertex of G has degree exactly $2n-2$, the weight of G is then $w(G) \geq 2n+1$.

Let $q = 2$. It is well known (see e.g. [T]) that there is a 6-regular triangulation G of the non-orientable surface $\mathbb{N}_2$. Analogously as above a new triangulation G^* is constructed by inserting new vertex X to each 3-face α.

Let $q \geq 6$. By Ringel [R1] the complete bipartite graph $K_{3,2q+2}$ has a non-orientable genus q. Because $q < 2g+1$, where g is the orientable genus of the graph $K_{3,2q+2}$, by [MT], there exists a cellular quadrangulation of $\mathbb{N}_q$ such that every face α is incident with exactly two vertices X, Y, $\deg(X) = \deg(Y) = 2q+2$. Let X, Y and Z be all three vertices of degree $2q+2$ of this quadrangulation. Inserting edges XY, YZ and XZ into three suitable faces we obtain an embedded graph H of the non-orientable genus q having weight $w(H) = 2q+7$.

This completes the proof of Theorem 2.

4. Triangle-free graphs

For triangle-free graphs of orientable genera Ivančo proved the following result.

THEOREM 3. *Let G be a simple triangle-free graph of minimum degree $\delta(G) \geq 3$ embedded in an orientable surface $\mathbb{S}_g$ of genus $g \geq 0$. Then G contains an edge e of the weight*

$$w(e) \leq \begin{cases} 8 & \text{if } g = 0, \\ 4q+5 & \text{if } g \geq 1. \end{cases}$$

These bounds are sharp.

We supplement Theorem 3 as follows:

THEOREM 4. *Let G be a simple triangle-free graph of minimum degree $\delta(G) \geq 3$ embeddable in a non-orientable surface of genus $q \geq 1$. Then G contains an edge e of the weight*

$$w(e) \leq \begin{cases} 8 & \text{if } q = 1\,, \\ 2q+5 & \text{if } q \geq 2\,. \end{cases}$$

Moreover, the bounds are tight.

Proof. If G contains no 3-cycles, then Euler's formula implies

$$v_3 \geq 4(2-q) + \sum_{j \geq 5} (j-4) v_j\,.$$

Every k-vertex is incident with k edges, therefore $kv_k = e_{k,k} + \sum_{i \geq 3} e_{i,k}$. Then

$$\frac{1}{3}\Big(e_{3,3} + \sum_{i\geq 3} e_{i,3}\Big) \geq 4(2-q) + \sum_{j\geq 5}\Big(1 - \frac{4}{j}\Big)\Big(e_{j,j} + \sum_{i \geq 3} e_{i,j}\Big)\,.$$

Hence

$$\frac{2}{3}e_{3,3} + \frac{1}{3}e_{3,4} + \frac{2}{15}e_{3,5} \geq 4(2-q) + \sum_{i\geq 7}\Big(\frac{2}{3} - \frac{4}{j}\Big)e_{3,j} + \sum_{i\geq j\geq 4}\Big(2 - 4\frac{i+j}{ij}\Big)e_{i,j}\,. \tag{12}$$

Let h be the number of edges of G and $e^* = \frac{2}{3}e_{3,3} + \frac{1}{3}e_{3,4} + \frac{2}{15}e_{3,5}$.

If $q = 1$, then every coefficient at $e_{i,j}$ in (12) is non-negative, therefore $e^* > 8$ and $w(G) \leq 8$.

If $q \geq 2$ and $w(G) \geq 2q+6$, then every coefficient at $e_{i,j}$ in (11) is at least $\frac{4q-6}{6q+9}$ for every i and j with $i+j \geq 2q+6$. Then

$$e^* \geq h\frac{4q-6}{6q+9} - 4(q-2) \geq 2\,, \tag{13}$$

because every simple graph G without 3-cycles must have $\geq 3\big(w(G)-3\big)$ edges. Together with (13) we obtain $w(G) \leq 8$, a contradiction.

Embeddings of the complete bipartite graphs $K_{3,2q+2}$ into $\mathbb{N}_q$ (see the previous section) are good examples for the sharpness of the bound in the case of triangle-free graphs. □

5. Large graphs

For graphs with many vertices of large degree we have:

THEOREM 5. *Let G be a simple graph of minimum degree $\delta(G) \geq 3$ embeddable in a non-orientable surface of genus $q \geq 2$. If $\sum_{i\geq 8} v_i(G) > 6(q-2)$, then G contains an edge e of the weight $w(e) \leq 15$. The bound is tight.*

Proof. The upper bound 15 immediately follows from (11) of the proof of Theorem 2. To prove the lower bound 15 it is sufficient to consider graphs of minimum degree 6 that triangulate surfaces $\mathbb{N}_q$, $q \geq 2$. Such triangulations exist for all q; for $q = 2$ see [T], for $q \geq 3$ see [JV2]. Into each triangle τ of such triangulation a new vertex is inserted and joint with every vertex of τ. The resulting triangulation T has weight $w(T) = 15$. □

6. Remark

In the proof of Theorem 2 we have used similar ideas to those of Ivančo [I]. If we put $2-2q$ instead of $2-q$ in Lemma 1, (1), (8), (9) and (11) and then continue in an analogous way as in our proof case by case analysis depending on g and v, we obtain a proof of upper bounds of Theorem 1.

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