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GENERATION OF OPERATION-INVARIANT CLASSES OF SETS

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Classes of sets with invariance under certain operations play an important role in measure theory and related fields. Given a set X and a class $\mathscr{E} \subset \mathscr{P}(X)$ it is a typical problem to find the smallest class $\mathscr{F} \subset \mathscr{P}(X)$ containing $\mathscr{E}$ and being invariant under one or several operations defined on $\mathscr{E}$. It is well known that the class $\mathscr{E}$ of open intervals in $X = R^1$ generates in this way the class $\mathscr{F}$ of the Borel sets in R^1 that is invariant under countable union and complementation and of course $\mathscr{F}$ is the σ-algebra generated by $\mathscr{E}$.

There are several techniques to "construct" such classes. One of them consists in the repeated and possibly transfinite addition of sets that arise from elements of $\mathscr{E}$ applying the desired operation(s). In some interesting cases (e.g. generated algebras) it is sufficient to repeat this procedure of enlarging $\mathscr{E}$ at most as often as finite ordinals exist and in other cases (e.g. generated σ-algebras) this repetition has to be done more frequently. These differences can be explained in two ways. The first is connected with the kind of complication of the operation(s). There is, e.g., a stronger demand on a system $\mathscr{E}$ to be invariant under a countable than under a finite union and therefore "in general" the first operation requires more steps than the second. A further reason is connected with the cardinality of X: in the case that X is finite there is no difference between the algebra and the σ-algebra generated by a class $\mathscr{E} \subset \mathscr{P}(X)$.

In the present paper we wish to clarify this generation of operation-invariant classes of sets with regard to the number of the steps mentioned above. This seems to be of interest since non-standard classes of sets like σ-classes (as important for quantum mechanics; see e.g. [1], [8]), σ-algebras $\mathscr{F} \subset \mathscr{P}(\mathscr{P}(X))$ of sets of configurations (as important for random fields; see e.g. [9]) and others become more and more important.

We consider general mappings T: $\mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$ that cover the most interesting set operations and introduce notions of isotony and expansiveness of T being essential for our purposes. The condition of $\boldsymbol{m}$-boundedness (see sect. 4) turne out to be essential for the (ordinal) number of steps in the transfinite sequence

$$\mathscr{E},\ T(\mathscr{E}),\ T^2(\mathscr{E}),\ \dots,\ T^\alpha(\mathscr{E}),\ \dots,$$

to get T-invariance. In preparation for these results we consider the possibility of reducing invariance under several operations $T_1, \dots, T_n$ to only one operation T that is needed to justify a further restriction to this case.

1. Notations

If X is a set, let $\mathscr{P}(X)$ denote the class of all subsets. In this paper we consider mappings T defined on the class $\mathscr{P}(\mathscr{P}(X))$ of all subsets $\mathscr{E} \subset \mathscr{P}(X)$ into itself. Every such mapping

$$T\colon \mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$$

is called a class transformation. Important class transformations are set operations. For some of them we introduce fixed notations:

$$\vee(\mathscr{E}) = \left\{ \bigcup_{i=1}^{n} E_i \,\middle|\, n \in N;\ E_1, \dots, E_n \in \mathscr{E} \right\},$$

$$\vee_d(\mathscr{E}) = \left\{ \bigcup_{i=1}^{n} E_i \,\middle|\, n \in N;\ E_1, \dots, E_n \in \mathscr{E} \text{ and pairw. disj.} \right\},$$

$$\vee_\sigma(\mathscr{E}) = \vee(\mathscr{E}) \cup \left\{ \bigcup_{i=4}^{\infty} E_i \,\middle|\, E_1, E_2, \dots \in \mathscr{E} \right\},$$

$$\wedge(\mathscr{E}) = \left\{ \bigcap_{i=1}^{n} E_i \,\middle|\, n \in N;\ E_1, \dots, E_n \in \mathscr{E} \right\},$$

$$C^*(\mathscr{E}) = \{E^c \mid E \in \mathscr{E}\} \quad \text{with} \quad E^c = X \setminus E,$$

$$C(\mathscr{E}) = \{E, E^c \mid E \in \mathscr{E}\},$$

$$D(\mathscr{E}) = \mathscr{E} \cup \{E \setminus F \mid E, F \in \mathscr{E}\},$$

$$I(\mathscr{E}) = \mathscr{E}$$

with arbitrary $\mathscr{E} \in \mathscr{P}(\mathscr{P}(X))$ in all cases and $\vee$, $\vee_d$, $\vee_\sigma$, $\wedge$, C^*, C, D play the role of T in (1). Obviously there are class transformations defined without set operations for the variable $\mathscr{E}$, e.g. the transformations T_V, T_C with

$$T_V(\mathscr{E}) = \mathscr{E} \cup \left\{ \bigcup_{E \in \mathscr{E}} E \right\},$$

$$T_C(\mathscr{E}) = \mathscr{P}(X) \setminus \mathscr{E}.$$

A class transformation is called isotonic if

$$\mathscr{E}, \mathscr{F} \in \mathscr{P}(\mathscr{P}(X)), \quad \mathscr{E} \subset \mathscr{F} \Rightarrow T(\mathscr{E}) \subset T(\mathscr{F})$$

and expansive if

$$\mathscr{E} \in \mathscr{P}(\mathscr{P}(X)) \Rightarrow \mathscr{E} \subset T(\mathscr{E}).$$

Then $\vee$, $\vee_d$, $\vee_\sigma$, $\wedge$, C, D are isotonic and expansive. C^* is isotonic but not expansive. T_V is expansive but not isotonic (take e.g. $X=\{1, 2, 3\}$, $\mathscr{E}=\{\{1\}, \{2\}\}$, $\mathscr{F}=\{\{1\}, \{2\}, \{3\}\}$, then $\mathscr{E} \subset^- \mathscr{F}$ and $T_V(\mathscr{E})=\{\{1\}, \{2\}, \{1, 2\}\}$, $T_V(\mathscr{F})=\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$). T_C is neither isotonic nor expansive.

Concerning the fixed set X all class transformations T: $\mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$ form a semigroup with composition T_1T_2 defined by $T_1T_2(\mathscr{E}) = T_1(T_2(\mathscr{E}))$. E.g. the special relations

$$\wedge\vee = \vee\wedge, \quad \wedge C^* = C^*\vee, \quad \vee C^* = C^*\wedge, \quad C^*C = CC^* = C \qquad (2)$$
$$\vee_d \subset \vee, \quad C^* \subset C$$

are easily proved.

T^k denotes the k-fold power of T and, more generally, let α be an ordinal, then the α-fold power T^α of T is defined with transfinite induction by

$$T^\alpha(\mathscr{E}) = T\left(\bigcup_{\beta<\alpha} T^\beta(\mathscr{E})\right). \qquad (3)$$

If α is isolated and T is expansive, then (3) reduces to

$$T^\alpha(\mathscr{E}) = T(T^{\alpha-1}(\mathscr{E})).$$

If $T^2 = T$ (i.e. $T(T(\mathscr{E})) = T(\mathscr{E})$ for all $\mathscr{E} \in \mathscr{P}(\mathscr{P}(X))$) is valid, then T is called idempotent. $\vee$, $\vee_d$, $\vee_\sigma$, $\wedge$, C are examples of idempotent class transformations. More generally it may be that $T^{k+1} = T^k$ is fulfilled, even in the case of set operations and $k > 2$. The following theorem gives an example.

1.1. Theorem. *The class transformation $T = \vee C$ fulfils $T^3 = T^2$.*

Proof. First of all we show

$$(\vee C)^2 = \vee\wedge C. \qquad (4)$$

Replacing $\mathscr{E}$ by $\vee(\mathscr{E})$ in $C(\mathscr{E}) = \mathscr{E} \cup C^*(\mathscr{E})$ we have from (2) the equation

$$C\vee(\mathscr{E}) = \vee(\mathscr{E}) \cup C^*\vee(\mathscr{E}) = \vee(\mathscr{E}) \cup \wedge C^*(\mathscr{E})$$

and then taking $C(\mathscr{E})$ instead of $\mathscr{E}$ we obtain with

$$\vee C \vee C(\mathscr{E}) = \vee(\vee C(\mathscr{E}) \cup \wedge C^*C(\mathscr{E})) =$$
$$= \vee(C(\mathscr{E}) \cup \wedge V(\mathscr{E})) = \vee\wedge C(\mathscr{E})$$

equation (4). From (2) we get

$$C\vee\wedge C(\mathscr{E}) = \vee\wedge C(\mathscr{E}) \cup C^*\vee\wedge C(\mathscr{E}),$$
$$C^*\vee\wedge C(\mathscr{E}) = \wedge\vee C^*C(\mathscr{E}) = \vee\wedge C(\mathscr{E})$$

and therefore

$$C\vee\wedge C(\mathscr{C})=\vee\wedge C(\mathscr{C}). \qquad (5)$$

Applying (4), (5) and (2) we conclude

$$\vee C(\vee C)^2=\vee(C\vee\wedge C)=\vee(\vee\wedge C)=\vee\wedge C=(\vee C)^2.$$

This proves the theorem.

The semigroup of all class transformations allows the ordering

$$T_1\leqslant T_2\Leftrightarrow(\mathscr{C}\in\mathscr{P}(\mathscr{P}(X))\Rightarrow T_1(\mathscr{C})\subset T_2(\mathscr{C})).$$

Concerning this ordering there exists an upper bound class transformation T^e for every class transformation T that is defined by

$$T^e(\mathscr{C})=T(\mathscr{C})\cup\mathscr{C},\quad(\mathscr{C}\subset\mathscr{P}(X)).$$

Obviously T^e has the properties

$$T\leqslant T^e,$$
$$T\leqslant T',\ T'\text{ expansive}\Rightarrow T^e\leqslant T',$$
$$T\text{ isotonic}\Rightarrow T^e\text{ isotonic}.$$

2. Invariant classes of sets

2.1. Definition. *Let $\mathscr{C}\in\mathscr{P}(\mathscr{P}(X))$ and T be a class transformation on X. $\mathscr{C}$ is said to be T-invariant if*

$$T(\mathscr{C})\subset\mathscr{C}. \qquad (6)$$

The set of all T-invariant $\mathscr{C}$ in $\mathscr{P}(\mathscr{P}(X))$ will be denoted by $\mathscr{I}_X(T)$ or $\mathscr{I}(T)$.

We mention some elementary facts concerning this definition. For any T the set $\mathscr{I}(T)$ is nonvoid since $T(\mathscr{P}(X))\subset\mathscr{P}(X)$. If T is expansive, then (6) is equivalent to $T(\mathscr{C})=\mathscr{C}$. Since $T(\mathscr{C})\subset\mathscr{C}$ iff $T(\mathscr{C})\cup\mathscr{C}\subset\mathscr{C}$, we have equality of T- and T^e-invariance: $\mathscr{I}(T)=\mathscr{I}(T^e)$. Every idempotent T gives rise to T-invariant classes: $\mathscr{F}=T(\mathscr{C})$ fulfils $T(\mathscr{F})=\mathscr{F}$ for all $\mathscr{C}\in\mathscr{P}(\mathscr{P}(X))$. For many cases it is important to have invariance of a class $\mathscr{C}$ under several class transformations $T_1,\ldots,T_n$. In this case $T_1\ldots T_n(\mathscr{C})\subset\mathscr{C}$ is obvious for isotonic $T_1,\ldots,T_{n-1}$ and therefore

$$\mathscr{I}(T_1)\cap\ldots\cap\mathscr{I}(T_n)\subset\mathscr{I}(T_1\ldots T_n)$$

holds. For the converse of this implication we have

2.2. Theorem. *If $T_1,\ldots,T_n$ are expansive and $T_1,\ldots,T_{n-1}$ are isotonic, then*

$$\mathscr{I}(T_1\ldots T_n)\subset\mathscr{I}(T_1)\cap\ldots\cap\mathscr{I}(T_n).$$

Proof. For fixed i it follows with expansiveness for $T_1, \ldots, T_{i-1}$ that

$$T_i(\mathscr{E}) \subset T_{i-1}T_i(\mathscr{E}) \subset \ldots \subset T_1 \ldots T_i(\mathscr{E})$$

and with isotony for $T_1, \ldots, T_{n-1}$ and therefore also for the products $T_1 \ldots T_{i+k}$ $(k=0, \ldots, n-i-1)$ we have

$$T_1 \ldots T_i(\mathscr{E}) \subset T_1 \ldots T_i T_{i+1}(\mathscr{E}) \subset \ldots \subset T_1 \ldots T_n(\mathscr{E}).$$

If $\mathscr{E} \in \mathscr{I}(T_1 \ldots T_n)$, the theorem follows from $T_i(\mathscr{E}) \subset T_1 \ldots T_n(\mathscr{E}) \subset \mathscr{E}$ and i being arbitrary in $\{1, \ldots, n\}$.

It should be remarked that theorem 2.2 is only sufficient and does not cover all important cases. In this respect we prove only

2.3. Theorem. *Let $T_1, T_2 : \mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$; then either of the conditions*
(a) T_1 is isotonic and $T_1^n = I$ and T_2 is expansive,
(b) T_1 is idempotent and expansive and T_2 is isotonic and $T_2^n = I$ is sufficient for

$$\mathscr{E} \in \mathscr{I}(T_1 T_2) \Rightarrow \mathscr{E} \in \mathscr{I}(T_1) \cap \mathscr{I}(T_2).$$

Proof. (a) is sufficient since

$$T_1(\mathscr{E}) \subset T_1 T_2(\mathscr{E}) \subset \mathscr{E}$$
$$T_2(\mathscr{E}) = T_1^n T_2(\mathscr{E}) = T_1^{n-1}(\mathscr{E}) \subset \mathscr{E}$$

and in the case of (b) we have

$$T_2(\mathscr{E}) \subset T_1 T_2(\mathscr{E}) \subset \mathscr{E}$$

and therefore

$$T_1(\mathscr{E}) = T_1 T_2(\mathscr{E}) \subset \mathscr{E}.$$

2.4. Remark. As an application of theorem 2.3 it is possible to characterize an algebra $\mathscr{E}$ of subsets of X not only by "$\vee(\mathscr{E}) = \mathscr{E}$ and $C^*(\mathscr{E}) = \mathscr{E}$" but also by "$C^* \vee(\mathscr{E}) = \mathscr{E}$" or by "$\vee C^*(\mathscr{E}) = \mathscr{E}$". Similar results hold for σ-algebras: "$\vee_\sigma C^*(\mathscr{E}) = \mathscr{E}$". Of course theorem 2.2 is also applicable to the description of algebras by only one equation. But then the nonexpansive C^* must be extended to the expansive class transformation C and one has then $\vee C(\mathscr{E}) = \mathscr{E}$ and for σ-algebras $\vee_\sigma C(\mathscr{E}) = \mathscr{E}$ respectively.

3. Generation of T-invariant classes

If $\mathscr{E}$ is not T-invariant, then it is important to construct a (minimal) T-invariant class $\mathscr{F}$ containing $\mathscr{E}$. It is well known in measure theory this problem frequently appears if suitable classes of sets are desired for defining measures on them.

3.1. Lemma. *If* $(\mathcal{F}_\alpha)_{\alpha \in I}$ *is a family of classes* $\mathcal{F}_\alpha \in \mathcal{P}(\mathcal{P}(X))$ *and* T *is isotonic, then*

$$T(\mathcal{F}_\alpha) \subset \mathcal{F}_\alpha, \ (\alpha \in I) \Rightarrow T\left(\bigcap_{\alpha \in I} \mathcal{F}_\alpha\right) \subset \bigcap_{\alpha \in I} \mathcal{F}_\alpha.$$

Proof. With the isotony of T we have

$$T\left(\bigcap_{\alpha \in I} \mathcal{F}_\alpha\right) \subset T(\mathcal{F}_{\alpha_0})$$

for each $\alpha_0 \in I$, therefore $T\left(\bigcap_{\alpha \in I} \mathcal{F}_\alpha\right) \subset \bigcap_{\alpha \in I} T(\mathcal{F}_\alpha)$ and from the assumption the lemma results.

This lemma and the above mentioned fact of $T(\mathcal{P}(X)) \subset \mathcal{P}(X)$ for all class transformations T yield

3.2. Theorem. *Let* $\mathcal{E} \in \mathcal{P}(\mathcal{P}(X))$ *and* T *an isotonic class transformation. Then there exists a unique* $\mathcal{M} \in \mathcal{P}(\mathcal{P}(X))$ *with*

$$\mathcal{E} \subset \mathcal{M}, \tag{7}$$

$$T(\mathcal{M}) \subset \mathcal{M}, \tag{8}$$

$$\mathcal{E} \subset \mathcal{G}, \ T(\mathcal{G}) \subset \mathcal{G} \Rightarrow \mathcal{M} \subset \mathcal{G}. \tag{9}$$

Proof. For the uniqueness of this $\mathcal{M}$ it is sufficient to show (7), (8) for $\mathcal{M}_1 \cap \mathcal{M}_2$ if the properties (7), (8), (9) are valid for $\mathcal{M}_1$, $\mathcal{M}_2$. Then apply (9) with $\mathcal{G} \to \mathcal{M}_1$ resp. $\mathcal{G} \to \mathcal{M}_2$ and conclude $\mathcal{M}_1 \cup \mathcal{M}_2 \subset \mathcal{M}_1 \cap \mathcal{M}_2$ and therefore $\mathcal{M}_1 = \mathcal{M}_2$. But (7) is obvious for $\mathcal{M}_1 \cap \mathcal{M}_2$ and (8) follows from the isotony of T. For the existence of such an $\mathcal{M}$ we define

$$\mathcal{M} = \cap \{\mathcal{F} \mid \mathcal{E} \subset \mathcal{F}, \ T(\mathcal{F}) \subset \mathcal{F}\}.$$

Then of course (7) holds and by lemma 3.1 and

$$T(\cap \{\mathcal{F} \mid \mathcal{E} \subset \mathcal{F}, \ T(\mathcal{F}) \subset \mathcal{F}) \subset \cap \{\mathcal{F} \mid \mathcal{E} \subset \mathcal{F}, \ T(\mathcal{F}) \subset \mathcal{F}\}$$

(8) is true. (9) is a direct implication of the definition of $\mathcal{M}$.

3.3. Definition. *For every* $\mathcal{E} \in \mathcal{P}(\mathcal{P}(X))$ *and every isotonic class transformation* T *the class* $\mathcal{M} \in \mathcal{P}(\mathcal{P}(X))$ *in the theorem 3.2 is said to be the T-invariant class generated by* $\mathcal{E}$. *Notation:* $\mathcal{M} = \mathcal{M}_T(\mathcal{E})$.

It is interesting to know whether there is a difference between $\mathcal{M}_T(\mathcal{E})$ and $\mathcal{M}_{T^e}(\mathcal{E})$. The last class is defined very well since T^e is isotonic if T is.

3.4. Theorem. *For every* $\mathcal{E} \in \mathcal{P}(\mathcal{P}(X))$ *and every isotonic class transformation* T *one has*

$$\mathcal{M}_T(\mathcal{E}) = \mathcal{M}_{T^e}(\mathcal{E}).$$

Proof. Note the equality of the sets

$$\{\mathscr{F} \in \mathscr{P}(\mathscr{P}(X)) \mid \mathscr{E} \subset \mathscr{F} \text{ and } T(\mathscr{F}) \subset \mathscr{F}\}$$

and

$$\{\mathscr{F} \in \mathscr{P}(\mathscr{P}(X)) \mid \mathscr{E} \subset \mathscr{F} \text{ and } T^e(\mathscr{F}) \subset \mathscr{F}\}$$

being essential in the definition of $\mathscr{M}_T(\mathscr{E})$ and $\mathscr{M}_{T^e}(\mathscr{E})$.

3.5. Remark. This theorem enables us to replace the isotonic T by the isotonic and expansive T^e whenever $\mathscr{M}_T(\mathscr{E})$ is required. This is of importance because expansive class transformations are better to handle as we shall see below.

4. Generation of T-invariant classes with m-bounded T

First we recall some important facts of the set theory. If X is a well-ordered set, its order type will be denoted by $\bar{X}$. Every ordinal $\alpha, \beta, \ldots$ is the order type of a well-ordered set. ω is the order type of N with the natural order. Ω is the smallest ordinal greater than every countable ordinal. The power $\bar{\alpha}$ of an ordinal α is the power of one of the well-ordered sets X with order type α. $\bar{X}$ denotes the power of the set X.

An infinite ordinal α is said to be initial iff

$$\beta < \alpha \Rightarrow \bar{\beta} < \bar{\alpha}$$

holds. In this sense ω and Ω are initial. With

$$P(\alpha) = \{\beta \mid \beta \text{ initial}, \beta < \alpha\}$$

the ordinal $\iota(\alpha) = \overline{P(\alpha)}$ is said to be the index of α. It is easy to prove $\iota(\omega) = 0$, $\iota(\Omega) = 1$ and

$$\alpha_1 < \alpha_2 \Rightarrow \iota(\alpha_1) < \iota(\alpha_2).$$

For every ordinal β there exists an initial ordinal α with $\beta = \iota(\alpha)$. Every initial ordinal is denoted by ω_α where $\iota(\omega_\alpha) = \alpha$. Then $\omega = \omega_0$, $\Omega = \omega_1$. The powers of initial ordinals are the alephs:

$$\aleph_\alpha = \bar{\omega}_\alpha, \quad (\alpha \text{ ordinal}).$$

For these powers the important theorem of Hessenberg ([4], p. 593)

$$\aleph_\alpha^2 = \aleph_\alpha, \quad (\alpha \text{ ordinal})$$

holds.

As it is well known T-invariant classes with expansive T containing a given $\mathscr{E}$ may be constructed by forming the possibly transfinite sequence

$$\mathscr{E} \subset T(\mathscr{E}) \subset T^2(\mathscr{E}) \subset \ldots \subset T^{\omega_0}(\mathscr{E}) \subset \ldots$$

$$T\left(\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E})\right)=\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E}) \tag{10}$$

with ω_α being the initial ordinal of $\aleph_\alpha$.

Proof. Note first that $\aleph_\alpha$-boundednes implies

$$T\left(\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E})\right)=\cup\left\{T(\mathscr{E}_0)\,|\,\mathscr{E}_0\in\mathscr{P}_{\aleph_\alpha}\left(\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E})\right)\right\}.$$

Since by hypothesis T is expansive, (10) is verified by checking

$$\cup\left\{T(\mathscr{E}_0)\,|\,\mathscr{E}_0\in\mathscr{P}_{\aleph_\alpha}\left(\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E})\right)\right\}\subset\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E}). \tag{11}$$

For $\mathscr{E}_0\in\mathscr{P}_{\aleph_\alpha}\left(\bigcup_{\gamma<\omega_\alpha} T^\gamma(\mathscr{E})\right)$ there exists an index set I with $\overline{\overline{I}}<\aleph_\alpha$ and a representation

$$\mathscr{E}_0=\{E_\iota\,|\,\iota\in I\}\quad\text{with}\quad E_\iota\in T^{\gamma_\iota}(\mathscr{E}),\quad \gamma_\iota<\omega_\alpha\,;\quad (\iota\in I).$$

Further let $C=(\gamma_\iota)_{\iota\in I}$ be the family of all indices γ_ι occurring in the above representation of $\mathscr{E}_0$. If C^* denotes the set built by wellordering the family C, we have further on

$$\gamma\in C^*\Rightarrow\gamma<\omega_\alpha$$

and $\overline{\overline{C^*}}\leqslant\overline{\overline{I}}<\aleph_\alpha$ and therefore, since α is isolated,

$$\overline{\overline{C^*}}\leqslant\aleph_{\alpha-1}. \tag{12}$$

Now let γ_0 be the smallest ordinal above C^* ([6], p. 234) and put

$$\gamma^*=\begin{cases}\gamma_0-1 & \text{if } \gamma \text{ is isolated,}\\ \gamma_0 & \text{otherwise.}\end{cases}$$

In the first case γ^* belongs to C^* and we have $\overline{\gamma^*}=\overline{\gamma_0-1}$ and in the second case with $W(\gamma^*)=\bigcup_{\gamma\in C^*}W(\gamma)$; $\overline{\overline{W(\gamma)}}\leqslant\aleph_{\alpha-1}$ for all $\gamma\in C^*$ and (11) we have

$$\overline{\overline{W(\gamma^*)}}\leqslant\overline{\overline{C^*}}\cdot\aleph_{\alpha-1}=\aleph_{\alpha-1}\cdot\aleph_{\alpha-1}=\aleph_{\alpha-1},$$

the latter equality by Hessenberg's theorem, and again $\overline{\gamma^*}<\aleph_\alpha$ is valid. Therefore we have

$$\overline{\gamma^*}<\aleph_\alpha$$

in both cases. From this it follows that $\gamma^* < \omega_\alpha$ and

$$\gamma^* + 1 < \omega_\alpha, \tag{13}$$

since ω_α is a limit ordinal like every initial ordinal is. For $\mathscr{E}_0$ we check the implications

$$\mathscr{E}_0 = \{E_\iota \mid \iota \in I\} \subset \bigcup_{\iota \in I} T^{\gamma_\iota}(\mathscr{E}) = \bigcup_{\gamma \in C^*} T^\gamma(\mathscr{E}) \subset \bigcup_{\gamma < \gamma^*} T^\gamma(\mathscr{E})$$

and

$$T(\mathscr{E}_0) \subset T\left(\bigcup_{\gamma \leq \gamma^*} T^\gamma(\mathscr{E})\right) \subset T\left(\bigcup_{\gamma < \gamma^* + 1} T^\gamma(\mathscr{E})\right) = T^{\gamma^{*}+1}(\mathscr{E}),$$

where the sign of equality comes from (3). But considering (13) this is equivalent to (11) and the theorem is proved.

4.5. Corollary. *If α is a limit ordinal and T: $\mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$ is expansive and $\aleph_\alpha$-bounded, then (10) holds with $\omega_{\alpha+1}$ instead of ω_α.*

Proof. $\alpha + 1$ is isolated and T especially $\aleph_{\alpha+1}$-bounded. The rest follows from 4.4.

4.6. Notation. To simplify the following we write

$$[\alpha] = \begin{cases} \alpha & \text{if } \alpha \text{ is isolated,} \\ \alpha + 1 & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Then summarizing our considerations we conclude

4.7. Theorem. *Let X be a set and T: $\mathscr{P}(\mathscr{P}(X)) \to \mathscr{P}(\mathscr{P}(X))$ isotonic, expansive and $\aleph_\alpha$-bounded. Then with*

$$\xi = \min\{\eta \mid 2^{2^{\bar{\bar{X}}}} \leq \aleph_\eta\} \quad \textit{and} \quad \beta = \omega_{[\min(\alpha, \xi)]}$$

for every $\mathscr{E} \subset \mathscr{P}(X)$ the T-invariant class $\mathscr{M}_T(\mathscr{E})$ as defined in 3.3 is

$$\mathscr{M}_T(\mathscr{E}) = T^\beta(\mathscr{E}). \tag{14}$$

Proof. Isotony of T shows $\mathscr{M}_T(\mathscr{E})$ to be correct and expansiveness of T makes lemma 4.4 applicable. To prove (14) we show (8) and (9) with $\mathscr{M} = T^\beta(\mathscr{E})$, since (7) is obvious by expansiveness of T. The hypothesis and 4.2 imply $\aleph_{\min(\alpha, \xi)}$-boundedness and with $\delta = \min(\alpha, \xi)$, lemma 4.4 and corollary 4.5 it follows that

$$T\left(\bigcup_{\gamma < \omega_{[\ ]}} T^\gamma(\mathscr{E})\right) = \bigcup_{\gamma < \omega_{[\delta]}} T^\gamma(\mathscr{E}). \tag{15}$$

From (15) we get

$$T(T^{\omega_{[\delta}}(\mathscr{E})) = T\left(T\left(\bigcup_{\gamma < \omega_{[\delta]}} T^\gamma(\mathscr{E})\right)\right) = T\left(\bigcup_{\gamma < \omega_{[\delta]}} T^\gamma(\mathscr{E})\right) - T^{\omega_{[\delta}}(\mathscr{E}),$$

such that (8) is valid. To prove (9) we check

$$\mathscr{E} \subset \mathscr{G}, \quad T(\mathscr{G}) \subset \mathscr{G} \Rightarrow T^\alpha(\mathscr{E}) \subset \mathscr{G}$$

for every ordinal α by transfinite induction: $T^0(\mathscr{E}) = \mathscr{E} \subset \mathscr{G}$ is assumed; if $T^\beta(\mathscr{E}) \subset \mathscr{G}$ is proved for all $\beta < \alpha$, we get

$$T^\alpha(\mathscr{E}) = T\left(\bigcup_{\beta<\alpha} T^\beta(\mathscr{E})\right) \subset T(\mathscr{G}) \subset \mathscr{G}.$$

Then we put $\alpha = \omega_{[\delta]}$ and (9) is obvious in our case.

4.8. Remarks. 1. The proof of 4.6 shows that even

$$\mathscr{M}_T(\mathscr{E}) = \bigcup_{\gamma<\beta} T^\gamma(\mathscr{E})$$

is true and this is somewhat stronger than $\mathscr{M}_T(\mathscr{E}) = T^\beta(\mathscr{E})$.

2. As remarked after theorem 2.3 a σ-algebra $\mathscr{E}$ of subsets of X is characterized by $\vee_\sigma C(\mathscr{E}) = \mathscr{E}$. The class transformation $T = \vee_\sigma C$ is $\aleph_1$-bounded as it is easily seen with 4.1, therefore we have with theorem 4.7.

$$\mathscr{M}_{\vee_\sigma C}(\mathscr{E}) = (\vee C)^{\omega_1}(\mathscr{E}).$$

3. For an algebra we get with 4.7.

$$\mathscr{M}_{\vee C}(\mathscr{E}) = (\vee C)^{\omega_0}(\mathscr{E}).$$

But here theorem 1.1 gives an essentially sharper result $\mathscr{M}_{\vee C}(\mathscr{E}) = (\vee C)^2(\mathscr{E})$. We conclude that theorem 4.7 does not give the strongest result even in the case of arbitrary $\mathscr{E} \subset \mathscr{P}(X)$. The reason is rather obvious: our method of proving lemma 4.4 (and therefore theorem 4.7) was essentially based on Hessenberg's theorem, being not valid for finite cardinals. Nevertheless it would be of interest to investigate, whether there is a modification of the notion of $\boldsymbol{m}$-boundedness, covering those cases also.

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КОНСТРУКЦИЯ СИСТЕМЫ МНОЖЕСТВ ИНВАРИАНТНЫХ ОТНОСИТЕЛЬНО ОПЕРАЦИЙ

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Резюме

Статья посвящена построению систем множеств инвариантных относительно некоторых (множественных) операций $T_1, \ldots, T_n$ и содержащих заданную систему множеств $\mathscr{E}$. Случай нескольких операций связаны со случаем одной операции (теор. 2.2 и 2.3). С помощью определяемого условия $\mathfrak{m}$-ограничности операции T (опр. 4.1) получаем оценку длины (часто трансфинитно) последовательности $\mathscr{E}$, $T(\mathscr{E})$, $T^2(\mathscr{E}), \ldots$, которая необходима для T-инвариантности.

and assuming $T^{\alpha}(\mathscr{E})=T^{\alpha_0}(\mathscr{E})$ for $\alpha \geqslant \alpha_0$ and a certain ordinal α_0 (for the definition of $T^{\alpha}(\mathscr{E})$ see sect. 1 above). Of course α_0 will depend essentially on T. In case of an algebra according to theorem 1.1 and the remark following theorem 2.3 $\alpha_0=2$ is possible and in case of a ring P. R. Halmos ([2], p. 23) proved $\alpha_0=\omega_0$ if T is the operation of finite unions of differences of sets, i.e. $T=\vee D$. For σ-algebras with the operator $T=\vee C$ one can prove $\alpha=\omega_1$ ([7]; p. 32 problem 4.d). Of course such properties were known much earlier. For σ-algebras (to be exact: Borel sets in metric spaces) see e.g. [3], p. 305 and for generated algebras $\mathscr{A}(\mathscr{E})$ K. Jacobs ([5], p. I.1.3) proved $\mathscr{A}(\mathscr{E})=\vee \wedge C(\mathscr{E})$. In the following we consider the reasons for the different lengths of the transfinite sequences. We examine especially their dependence on the power of X and on the properties of T.

By $\mathscr{P}_{\boldsymbol{m}}(\mathscr{E})$ we denote the system of all subclasses $\mathscr{E}_0 \subset \mathscr{E}$ with a power less than $\boldsymbol{m}$.

4.1. Definition. *If T is a class transformation, $\boldsymbol{m}$ a cardinal and*

$$\mathscr{P}_{\boldsymbol{m}}(\mathscr{E})=\{\mathscr{E}_0 \subset \mathscr{E} \mid \bar{\bar{\mathscr{E}}}_0<\boldsymbol{m}\}$$

the set of all subclasses $\mathscr{E}_0 \subset \mathscr{E}$ with a power less than $\boldsymbol{m}$, then T is said to be $\boldsymbol{m}$-bounded if

$$T(\mathscr{E})=\cup\{T(\mathscr{E}_0) \mid \mathscr{E}_0 \subset \mathscr{P}_{\boldsymbol{m}}(\mathscr{E})\} \quad \text{for all} \quad \mathscr{E} \in \mathscr{P}(\mathscr{P}(X)).$$

At a first glance $\boldsymbol{m}$-boundedness seems to be a very special property, but we have the

4.2. Theorem. *Every expansive T: $\mathscr{P}(\mathscr{P}(X)) \rightarrow \mathscr{P}(\mathscr{P}(X))$ is $2^{2^{\bar{\bar{X}}}}$-bounded.*

Proof. Independently of the special T every class $\mathscr{E}_0 \subset \mathscr{E} \subset \mathscr{P}(X)$ fulfils $\bar{\bar{\mathscr{E}}}_0 \leqslant \overline{\overline{\mathscr{P}(X)}}=2^{\bar{\bar{X}}}<2^{2^{\bar{\bar{X}}}}$ and therefore with $\boldsymbol{m}=2^{2^{\bar{\bar{X}}}}$ $\mathscr{E}$ itself is found in

$$\cup\{T(\mathscr{E}_0) \mid \mathscr{E}_0 \in \mathscr{P}_{\boldsymbol{m}}(\mathscr{E})\}$$

and the boundedness is obvious.

4.3. Examples. (a) Especially for $T=\vee$ we have at least $\aleph_0$-boundedness: an arbitrary $E \in \vee(\mathscr{E})$ is of the form $E=\bigcup_{i=1}^{n} E_i$, $(E_i \in \mathscr{E})$ and therefore $E \in \vee(\mathscr{E}_0)$ with some $\mathscr{E}_0 \in \mathscr{P}_{\aleph_0}(\mathscr{E})$. For finite X we even have boundedness from theorem 4.2.

(b) Similarly for $T=\vee_\sigma$ and infinite $\mathscr{E}_0$ we have $\bar{\bar{\mathscr{E}}}_0<\aleph_1$ and therefore only $\aleph_1$-boundedness is attainable if X is infinite.

The boundedness-property in theorem 4.2 is derived from $\bar{\bar{X}}$ only, but the examples show that a more restrictive boundedness is possible. In any case the following theorem can be formulated.

4.4. Lemma. *Let α be an isolated ordinal and T: $\mathscr{P}(\mathscr{P}(X)) \rightarrow \mathscr{P}(\mathscr{P}(X))$ expansive and $\aleph_\alpha$-bounded, then*