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EFFECTIVE CRITERION FOR TRANSFORMATION OF LINEAR FUNCTIONAL-DIFFERENTIAL EQUATIONS OF THE FIRST ORDER INTO CANONICAL FORM WITH CONSTANT COEFFICIENTS AND DEVIATIONS

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ABSTRACT. Effective sufficient and necessary conditions are given that the equation

$$y'(x) + p_0(x)y(x) + p_1(x)y(\xi_1(x)) + \cdots + p_m(x)y(\xi_m(x)) = 0$$

be globally transformable into an equation of form

$$z'(t) + q_0 z(t) + q_1 z(t - r_1) + \cdots + q_m z(t - r_m) = 0$$

on the whole interval of definition.

1. Introduction

Canonical forms for linear functional-differential equations are defined by means of pointwise transformations by F. Neuman [1]. These special forms may serve for example for the investigation of oscillatory behavior of solutions of all equations from certain classes of linear functional-differential equations because each global pointwise transformation preserves distribution of zeros of solutions of a functional-differential equation and its canonical forms.

Oscillatory behavior of solutions of functional-differential equations with constant coefficients and deviations are studied by M. K. Grammatikopoulos, E. A. Grove, ... [2], [3], [4], for example.

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2. Definitions and preliminaries

Consider a linear homogeneous functional-differential equation of the first order of the form

$$y'(x) + \sum_{i=0}^{m} p_i(x) y\big(\xi_i(x)\big) = 0 \tag{1_I}$$

with continuous coefficients $p_i \in C^0(I)$ on a half-open interval $I = [a,b)$ or on an open interval $I = (a,b)$, $-\infty \le a < b \le \infty$, $p_1(x)p_2(x)\dots p_m(x) \not\equiv 0$ on every nonempty subinterval $I_1 \subset I$; with $m \ge 1$ deviating arguments $\xi_i \in C^1(I)$, $\xi_i \colon I \to I$, $\mathrm{d}\xi_i(x)/\mathrm{d}x > 0$ on I, $\xi_i(x) \ne x$ on I, $i = 1,2,\dots,m$, $\xi_0 = \mathrm{id}_I$.

We suppose that

$$\lim_{x \to b^-} \xi_i(x) = b \qquad \text{for} \quad i = 1,2,\dots,m$$

and

$$\lim_{x \to a^+} \xi_i(x) = a \qquad \text{for} \quad i = 1,2,\dots,m$$

in the case that $a \notin I$.

If all ξ_i are of the form

$$\xi_i(x) = x + c_i\,, \qquad c_i \text{ being constants,}$$

the equation (1_I) is said to be with constant deviations and also discrete deviations, see, e.g., [1], [5], [7].

We denote the differential equation (1_I) by $P(y,x,\xi;I)$ to express explicitly the dependent and independent variables and the definition interval of the equation.

Consider two differential equations $P(y,x,\xi;I)$, $Q(z,t,\eta;J)$. We say that $P(y,x,\xi;I)$ is globally transformable into $Q(z,t,\eta;J)$ if there exist a function $f \in C^1(J)$, $f(t) \ne 0$ on J and a C^1 diffeomorphism h of J onto I (i.e., $h \in C^1(J)$, $h(J) = I$, $\mathrm{d}h(t)/\mathrm{d}t \ne 0$ on J) such that

$$z(t) = f(t) y\big(h(t)\big)\,, \tag{2}$$

$$\xi_i \circ h = h \circ \eta_i \tag{3}$$

is a solution of $Q(z,t,\eta;J)$ whenever y is a solution of $P(y,x,\xi;I)$. It follows immediately that for n-tuples $\boldsymbol{y}$ and $\boldsymbol{z}$ of "linearly independent" solutions of the equations $P(y,x,\xi;I)$ and $Q(z,t,\eta;J)$ respectively, there exists a constant nonsingular matrix $\mathbf{A}$ such that

$$\boldsymbol{z}(t) = \mathbf{A} f(t) \boldsymbol{y}\big(h(t)\big)\,, \qquad t \in J\,. \tag{4}$$

The transformation (2), (3) is the most general pointwise transformation of equation (1_I) (see [9], [10]).

We use the stationary groups formed by all the global transformations that transform a given ordinary linear differential equation into itself. Some results about stationary groups of linear differential equations were obtained by F. Neuman [8]. The condition that global transformation (2), (3) transforms an equation $P = P(y, x, \xi; I)$ into itself can be equivalently written in the form of the vector functional equation

$$\boldsymbol{y}(x) = \mathbf{A} f(x) \boldsymbol{y}(h(x)) ,$$

where $\mathbf{A}$ is a nonsingular matrix, $\boldsymbol{y}$ is a "fundamental" solution of P.

From [1], it follows that equation (1_I) is globally transformable into an equation

$$z'(t) + \sum_{i=0}^{m} q_i(t) z(t + c_i) = 0 , \tag{5_J}$$

c_i being constant, defined and satisfying conditions for coefficients and deviations on the interval J, if and only if the following conditions for the transformation (2), (3) are satisfied

$$\eta_i(t) = t + c_i \iff \varphi(\xi_i(x)) = \varphi(x) + c_i \qquad \text{for} \quad i = 1, 2, \dots, m , \tag{6}$$

where

$$x = h(t) \iff t = \varphi(x) ,$$

i.e., $\varphi = h^{-1}$ is the inverse function to h.

The necessary conditions for existence of a common solution $\varphi \in C^1(I)$, $\mathrm{d}\varphi(x)/\mathrm{d}x > 0$ on I, of the system of functional equations (6) are derived by F. Neuman [5]; sufficient conditions are also derived for $m = 1$.

Using this result F. Neuman [1] defined canonical forms for the linear functional-differential equation of the n-th order ($n \geq 1$).

In this paper, we use the same methods as F. Neuman [1], [5] to obtain canonical forms with constant coefficients and constant deviations. The criterion that we give is effective, i.e., it is verifiable for considered any equation.

3. Result

THEOREM. *Suppose that an equation (1_I) is globally transformable into an equation (5_J), and there exist two solutions $y_1, y_2 \in C^1(I)$ of (1_I) with the nonzero Wronskian determinant, $p_k \neq 0$ on I for some $k \in \{1, 2, \dots, m\}$. Then (5_J) is an equation with constant coefficients and constant deviations if and only if for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ there exists a function $L\colon I \to \mathbb{R}$, $L \in C^1(I)$, $L(x) \neq 0$ on I such that the relations*

$$L'(x)/L(x) = p_0(x) - p_0(\xi(x))\xi'(x), p_i(\xi(x))\xi'(x) L(\xi_i(x))/L(x) = p_i(x) , \quad i \in \{1, 2, \dots, m\} , \tag{7}$$

are satisfied on I. *Moreover,* $q_0(t) \equiv 0$ *on* $J = [a_1, \infty)$, $J = (-\infty, \infty)$ *respectively.*

P r o o f. We prove the necessary condition. Consider the equation

$$z'(t) + q_0(t)z(t) + \sum_{i=1}^{m} q_i(t)z(t+c_i) = 0 \tag{8}$$

on J and $q_i(t) \equiv q_i \in \mathbb{R}$. There exists a transformation $z(t) = f(t)y\big(h(t)\big)$ such that $\xi_i(x) = \xi_i\big(h(t)\big) = h(t+c_i)$ if and only if $\varphi\big(\xi_i(x)\big) = \varphi\big(\xi_i\big(h(t)\big)\big) = t + c_i = \varphi(x) + c_i$ for $i \in \{1, 2, \dots, m\}$ according to the assumption that (1_I) is globally transformable into (5_J).

Hence there exist $z_j(t) = f(t)y_j\big(h(t)\big)$, $j = 1, 2$, such that

$$\boldsymbol{z}(t) = f(t)\boldsymbol{y}\big(h(x)\big), \qquad \det\big[\boldsymbol{z}(t), \boldsymbol{z}'(t)\big] = f^2(t)h'(t)\det\big[\boldsymbol{y}(x), \boldsymbol{y}'(x)\big] \neq \mathbf{0}, \tag{9}$$

and $\boldsymbol{z}(t) = \big(z_1(t), z_2(t)\big)^T$ is a solution of a vector differential equation

$$\boldsymbol{z}'(t) + q_0\boldsymbol{z}(t) + \sum_{i=1}^{m} q_i\boldsymbol{z}(t+c_i) = \mathbf{0} \tag{10}$$

on the interval J, $\mathbf{0} = (0,0)^T$, ^{T} is the transpose.

Now we consider the transformations (deformations)

$$\boldsymbol{z}(t+c_i) = \mathbf{B}\boldsymbol{z}(t), \qquad i \in \{1, 2, \dots, m\}, \tag{11}$$

where $\mathbf{B}$ is a nonsingular square matrix. Then we have

$$\begin{aligned} &\boldsymbol{z}'(t+c_i) + q_0\boldsymbol{z}(t+c_i) + \sum_{j=1}^{m} q_j\boldsymbol{z}(t+c_i+c_j) \\ &= \mathbf{B}\Big(\boldsymbol{z}'(t) + q_0\boldsymbol{z}(t) + \sum_{j=1}^{m} q_j\boldsymbol{z}(t+c_j)\Big) = \mathbf{B}\cdot\mathbf{0} = \mathbf{0} \end{aligned}$$

on J.

Hence, from (9) and (11), we get that

$$\begin{aligned} \boldsymbol{z}(t+c_j) &= f(t+c_j)\boldsymbol{y}\big(h(t+c_j)\big) = f(t+c_j)\boldsymbol{y}\big(\xi_j(x)\big) \\ &= \mathbf{B}f(t)\boldsymbol{y}\big(h(t)\big) = \mathbf{B}f(t)\boldsymbol{y}(x), \qquad \text{i.e.,} \\ \boldsymbol{y}\big(\xi_j(x)\big) &= \mathbf{B}\frac{f(t)}{f(t+c_j)}\boldsymbol{y}(x) = \mathbf{B}\frac{f\big(\varphi(x)\big)}{f\big(\xi_j(x)\big)}\boldsymbol{y}(x) \end{aligned} \tag{12}$$

for every $j \in \{1, 2, \dots, m\}$ according to (6).

If we define functions

$$L_j(x) = \frac{f(\varphi(x))}{f(\xi_j(x))}, \qquad j \in \{1, 2, \dots, m\}, \tag{13}$$

then for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ there exists a function $L(x) = f(\varphi(x))/f(\xi(x))$ such that $L\colon I \to \mathbb{R}$, $L \in C^1(I)$, $L(x) \neq 0$ on I. In accordance with (12), (13), the vector solution $\boldsymbol{y} = (y_1, y_2)^T$ of the equation

$$\boldsymbol{y}'(x) + p_0(x)\boldsymbol{y}(x) + \sum_{j=1}^{m} p_j(x)\boldsymbol{y}(\xi_j(x)) = \mathbf{0} \tag{14}$$

satisfies

$$\boldsymbol{y}(\xi(x)) = \mathbf{B}L(x)\boldsymbol{y}(x), \qquad L(x) = f(\varphi(x))/f(\xi(x)) \tag{15}$$

on I for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$. If we substitute $\xi(x)$ into equation (14), then we have

$$\boldsymbol{y}'(\xi(x)) + p_0(\xi(x))\xi'(x)\boldsymbol{y}(\xi(x)) + \sum_{j=1}^{m} p_j(\xi(x))\xi'(x)\boldsymbol{y}(x)(\xi_j(\xi(x))) = \mathbf{0} \tag{16}$$

on I ($' = \mathrm{d}/\mathrm{d}x$).

According to the existence of the transformation of equation (1_I) into equation (8) on J, we have $\xi \circ \xi_j = \xi_j \circ \xi$ for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ and every $j \in \{1, 2, \dots, m\}$ (see F. Neuman [5]).

Relation (15) describes the stationary group formed by the global transformations (2), (3), and from (15), (16), we obtain

$$\mathbf{B}\Big(L'\boldsymbol{y} + L\boldsymbol{y}' + p_0(\xi)\xi' L\boldsymbol{y} + \sum_{j=1}^{m} p_j(\xi)\xi' L(\xi_j)\boldsymbol{y}(\xi_j(\xi))\Big) = \mathbf{0}$$

if and only if

$$\boldsymbol{y}'(x) + \Big(p_0(\xi(x))\xi'(x) + \frac{L'(x)}{L(x)}\Big)\boldsymbol{y}(x) + \sum_{j=1}^{m}\Big(p_j(\xi(x))\xi'(x)\frac{L(\xi_j(x))}{L(x)}\Big)\boldsymbol{y}(\xi_j(x)) = \mathbf{0} \tag{17}$$

on I for every $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ and the solution $\boldsymbol{y}$ of (14).

Compare equations (14) and (17). Then

$$\Big(p_0(\xi)\xi' + \frac{L'}{L} - p_0\Big)\boldsymbol{y} + \sum_{j=1}^{m}\Big(p_j(\xi)\xi'\frac{L(\xi_j)}{L} - p_j\Big)\boldsymbol{y}(\xi_j) = \mathbf{0}, \qquad \boldsymbol{y} \not\equiv \mathbf{0} \tag{18}$$

holds on the interval I. If we now allow that there exist an interval $I_1 \subseteq I$ such that $p_0(\xi)\xi' + L'/L - p_0 \neq 0$ on I_1, then from (18)

$$\boldsymbol{y}(\xi_j(x)) = m(x)\boldsymbol{y}(x),$$

where $m(x)$ is a continuous function, and on I_1, the equation (14) becomes

$$\begin{aligned} &\boldsymbol{y}'(x) + p_0(x)\boldsymbol{y}(x) + \sum_{j=1}^{m} p_j(x)m(x)\boldsymbol{y}(x) \\ &= \boldsymbol{y}'(x) + \left[p_0(x) + \sum_{j=1}^{m} p_j(x)m(x)\right]\boldsymbol{y}(x) = \mathbf{0}\,. \end{aligned}$$

But this contradicts the assumption that the Wronskian determinant of the solutions y_1, y_2 is a nonzero function on I, and we have $p_0(\xi)\xi' + L'/L - p_0 = 0$, and using (18)

$$p_j(\xi)\xi' L(\xi_j)/L - p_j = 0\,, \qquad j = 1, 2, \dots, m\,,$$

for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ on the whole interval I since $\boldsymbol{y} \not\equiv \mathbf{0}$ on I. The necessary condition is proved.

The sufficient condition of the Theorem we prove in another way. We suppose that for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ there exists a function $L\colon I \to \mathbb{R}$, $L \in C^1(I)$, $L(x) \neq 0$ on I, and that (7) is satisfied on I. Then there exist transformations

$$\boldsymbol{y}\big(\xi(x)\big) = \mathbf{B}L(x)\boldsymbol{y}(x)$$

globally converting any equation (1_I) into itself on the interval I, and $\xi \circ \xi_i = \xi_i \circ \xi$ for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$, $i \in \{1, 2, \dots, m\}$. Consider the transformation

$$y(x) = f(x)v(x)\,, \tag{19}$$

where $f \in C^1(I)$, $f(x) \neq 0$ on I. This transformation converts any equation (1_I) into an equation

$$v'(x) + \left(p_0(x) + \frac{f'(x)}{f(x)}\right)v(x) + \sum_{i=1}^{m} p_i(x)\frac{f\big(\xi_i(x)\big)}{f(x)}v\big(\xi_i(x)\big) = 0\,, \tag{20}$$

and we define

$$f'(x)/f(x) = -p_0(x) \tag{21}$$

on I. Then $L'(x)/L(x) = p_0(x) - p_0\big(\xi(x)\big)\xi'(x) = f'\big(\xi(x)\big)/f\big(\xi(x)\big) - f'(x)/f(x)$, i.e.,

$$L(x) = cf\big(\xi(x)\big)/f(x)\,, \qquad c \in \mathbb{R} - \{0\}\,, \tag{22}$$

and we can suppose that

$$L(x) = f\big(\xi(x)\big)/f(x) > 0\,, \qquad L \in C^1(I)\,, \tag{23}$$

on the whole interval I. Moreover, from (7), we get

$$p_i(\xi)\xi'\frac{f\big(\xi_i(\xi)\big)}{f(\xi)} = p_i\frac{f(\xi_i)}{f} \tag{24}$$

for every function $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ on I, $i = 1, 2, \dots, m$. Equation (20) is then of the form

$$v'(x) + \sum_{i=1}^{m} p_i(x) \frac{f(\xi_i(x))}{f(x)} v(\xi_i(x)) = 0. \tag{25}$$

Now we define a transformation

$$x = h(t) \iff t = \varphi(x) = \int_{x_0}^{x} \left| p_k(s) \frac{f(\xi_k(s))}{f(s)} \right| \mathrm{d}s + a_1, \tag{26}$$

where $a_1 \in \mathbb{R}$, $x_0 \in I$ and $k \in \{1, 2, \dots, m\}$ is fixed, $p_k \neq 0$ on I. Such transformation always exists according to the assumptions of the Theorem. Then

$$\varphi'(x) = \left| p_x(x) f(\xi_k(x)) / f(x) \right| > 0$$

and

$$\begin{aligned} &\big(\varphi(\xi(x)) - \varphi(x)\big)' \\ &= \left| p_k(\xi(x)) f(\xi_k(\xi(x))) \xi'(x) / f(\xi(x)) \right| - \left| p_x(x) f(\xi_k(x)) / f(x) \right| = 0 \end{aligned}$$

by means of (24). Hence $\varphi(\xi(x)) = \varphi(x) + c$, $c \in \mathbb{R}$, and (6) gives

$$\varphi(\xi_i(x)) = \varphi(x) + c_i \iff \eta_i(t) = t + c_i, \tag{27}$$

and

$$\xi_i(x) = \xi_i(h(t)) = \varphi^{-1}(\varphi(x) + c_i) = \varphi^{-1}(t + c_i) = h(t + c_i) \tag{28}$$

for all $i \in \{1, 2, \dots, m\}$.

If we define a transformation

$$v(x) = v(h(t)) = z(t), \tag{29}$$

we obtain

$$\begin{aligned} v(\xi_i(x)) = v(\xi_i(h(t))) = v(h(t + c_i)) = z(t + c_i), \qquad i = 1, 2, \dots, m, \\ v'(x) = \big(z(\varphi(x))\big)' = z'(\varphi(x))\varphi'(x) = z'(t)\varphi'(x), \end{aligned}$$

where

$$\varphi'(x) = \left| p_k(x) f(\xi_k(x)) / f(x) \right| = \big(p_k(x) f(\xi_k(x)) / f(x)\big) \operatorname{sign} p_k(x)$$

since (23) implies $f(\xi_k(x)) / f(x) > 0$ on I.

The transformation (26), (29) globally transforms equation (25) into

$$z'(t) + \sum_{i=1}^{m} \frac{p_i(x) f(\xi_i(x))}{p_k(x) f(\xi_k(x))} \operatorname{sign} p_k(x) z(t + c_i) = 0, \tag{30}$$

$q_i(t) = q_i(\varphi(x)) = N_i(x) := \dfrac{p_i(x)f(\xi_i(x))}{p_k(x)f(\xi_k(x))}\operatorname{sign} p_k(x)$, $i = 1,2,\dots,m$, are continuous coefficients of equation (30),

$$q_k(t) = N_k(x) = \operatorname{sign} p_k(x) = \varepsilon = \pm 1\,,$$

and using (24)

$$\begin{aligned} N_i(\xi(x)) &= \frac{p_i(\xi(x))\xi'(x)f(\xi_i(\xi(x)))}{f(\xi(x))} \cdot \frac{f(\xi(x))}{p_k(\xi_k(x))\xi'(x)f(\xi_k(\xi(x)))} \cdot \operatorname{sign} p_k(x) \\ &= \frac{p_i(x)f(\xi_i(x))}{f(x)} \cdot \frac{f(x)}{p_k(x)f(\xi_k(x))} \operatorname{sign} p_k(x) = N_i(x) \end{aligned}$$

holds on I for $i = 1,2,\dots,m$. Moreover,

$$\xi'(x) > 0\,, \qquad \xi(x) \neq x\,,$$

for all $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ on I in accordance with the assumptions for the equation (1_I).

Due to the condition $\lim \xi(x) = b$ for $x \to b^-$, the nth iterate $\xi^{[n]}$ of $\xi \in \{\xi_1, \xi_2, \dots, \xi_m\}$ exists for all positive or negative integers n depending on whether $\xi(x) > x$ or $\xi(x) < x$ on $[a,b)$ and

$$\lim_{n\to\infty} \xi^{[n]}(x) = b \quad \text{for } \xi(x) > x\,, \qquad \lim_{n\to -\infty} \xi^{[n]}(x) = b \quad \text{if } \xi(x) < x\,.$$

Hence

$$N_i(\xi^{[n]}(x)) = N_i(\xi^{[n-1]}(x)) = \dots = N_i(x)\,, \qquad x \in I\,,$$

gives

$$N_i(x) = N_i(b^-) \in \mathbb{R}\,,$$

i.e., $q_i(t) = N_i(x)$, $i = 1,2,\dots,m$, are constant functions.

Repeating arguments given by F. Neuman [1] we can prove that $\varphi(I) = [a_1, \infty)$ in the case $I = [a,b)$, and $\varphi(I) = (-\infty,\infty)$ in the case $I = (a,b)$ according to the assumptions

$$\lim_{x\to a^+} \xi_i(x) = a \qquad \text{for} \quad i = 1,2,\dots,m\,.$$

The Theorem is proved. □

EXAMPLE. The equation

$$y'(x) + \frac{a}{x}y(x) + \frac{b\sqrt{x}}{x^3 \ln x}y(\sqrt{x}) + \frac{cx^5}{\ln x}y(x^3) = 0\,,$$

$x \in I = (1,\infty)$; $a,b,c \in \mathbb{R}$, $bc \neq 0$; is globally transformable into an equation with discrete deviations (see [5]). Then

$$L'(x)/L(x) = a\big(1/x - \xi'(x)/\xi(x)\big) \iff L(x) = k\big(x/\xi(x)\big)^a, \quad k \in \mathbb{R} - \{0\},$$

and the conditions (7) are equivalent to

$$\frac{bx^{1/4}x^{-1/2}2^{-1}}{x^{3/2}\ln x^{1/2}} \cdot \frac{\big(x^{1/2}x^{-1/4}\big)^a}{\big(xx^{-1/2}\big)^a} = \frac{bx^{1/2}}{x^3 \ln x}, \qquad \frac{cx^{5/2}x^{-1/2}2^{-1}}{\ln x^{1/2}} \cdot \frac{\big(x^3x^{-3/2}\big)^a}{\big(xx^{-1/2}\big)^a} = \frac{cx^5}{\ln x},$$

$$\frac{bx^{3/2}3x^2}{x^9 \ln x^3} \cdot \frac{\big(x^{1/2}x^{-3/2}\big)^a}{\big(xx^{-3}\big)^a} = \frac{bx^{1/2}}{x^3 \ln x}, \qquad \frac{cx^{15}3x^2}{\ln x^3} \cdot \frac{\big(x^3x^{-9}\big)^a}{\big(xx^{-3}\big)^a} = \frac{cx^5}{\ln x},$$

$x \in I$. The given equation is globally transformable into an equation with constant coefficients and discrete deviations if and only if $a = 3$; $b,c \in \mathbb{R} - \{0\}$. We have the corresponding transformations

$$y(x) = f(x)v(x), \qquad v(x) = v\big(h(t)\big) = z(t),$$

where

$$f'(x)/f(x) = -p_0(x) = -3/x \iff f(x) = M/x^3, \quad M \in \mathbb{R} - \{0\};$$

$$x = h(t) \iff t = \varphi(x) = \int_{x_0}^{x} \big|p_k(s) f\big(\xi_k(s)\big)/f(s)\big| \, \mathrm{d}s + a_1,$$

$$x_0 \in I, \quad a_1 \in \mathbb{R}, \quad k \in \{1,2\},$$

and

$$\varphi(x) = \int_{x_0}^{x} \left| p_1(s) \frac{Ms^3}{M \cdot \big(\xi_1(s)\big)^3} \right| \mathrm{d}s + a_1 = \int_{x_0}^{x} \left| \frac{bs^{1/2}}{s^3 \ln s} \cdot \frac{x^3}{s^{3/2}} \right| \mathrm{d}s + a_1$$

$$= \int_{x_0}^{x} \left| \frac{b}{s \ln s} \right| \mathrm{d}s + a_1 = |b| \ln\ln x + a_2, \qquad a_2 \in \mathbb{R},$$

for example.

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