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VECTOR MEASURES AND NUCLEARITY

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It is well known [1, p. 48] that every finite complex-valued measure on the delta ring has finite variation, and hence every vector-valued measure on the delta ring with values in a finite-dimensional locally convex space has finite variation. This is, however, not the case for infinite-dimensional normed spaces as can be exhibited by a counter-example. Nevertheless there is a quite large class of infinite-dimensional locally convex spaces necessarily non-normable that are "well behaved" in this respect. In this paper we give a characterization of a class of the locally convex spaces X with the property that every X-valued vector measure on the delta ring has finite variation. This class contains, e. g., all nuclear spaces and dual-nuclear spaces which once again shows that nuclear locally convex spaces are substantially closer to finite-dimensional spaces than normable ones. We mention some open problems to.

Let X be a locally convex Hausdorff topological vector space — shortly locally convex space — and $P = (p)_{p \in P}$ a family of continuous seminorms defining the locally convex topology on X.

Recall that if we are given a family $(x_i)_{i \in I}$ of elements of the locally convex space X, where I is an arbitrary index set, then $(x_i)_{i \in I}$ is said to be scalarly summable (also called weakly summable [2, 1.2]), if for every x' in X', X' denoting the space of all continuous linear forms on X, the complex family $(x'x_i)_{i \in I}$ is absolutely summable, i. e. $\sum_{i \in I} |x'x_i|$ is finite for every x' in X'. A family $(x_i)_{i \in I}$ is said to be absolutely summable if $\sum_{i \in I} p(x_i)$ is finite for all p in P.

Let $\boldsymbol{D}$ be a delta ring of subsets of a set S and m: $\boldsymbol{D} \to X$ a sigma additive set function, i. e. m is a vector-valued measure on $\boldsymbol{D}$ with values in X. Let p be in P. Recall thet the p-variation of m on $\boldsymbol{D}$ is a non-negative extended-valued function defined by the relation

$$m_p(E) = \sup \sum_{i=1}^{n} p(m(E_i))$$

where the supremum is taken over all finite families of the disjoint sets in $\boldsymbol{D}$ with $\bigcup_{i=1}^{n} E_i = E$. We say that m: $\boldsymbol{D} \to X$ has finite variation if for every p in P the p-variation is finite.

We shall say that a locally convex space X has the property (sas) if every scalarly summable family of the elements of X is absolutely summable.

We shall prove the following result.

Proposition. *Let X be a locally convex space with the property* (sas). *Let $\boldsymbol{S}$ be a sigma algebra of subsets of S. Then every vector measure m: $\boldsymbol{S} \to X$ has bounded variation on S,* i.e. *for every p in P the quantity $m_p(S)$ is a finite number.*

Proof. Let x' be in X'. Then the scalar measure $E \to m_x(E) = x'm(E)$ is bounded on $\boldsymbol{S}$ [1. p. 34], hence for every x' the scalar measure $x'm$ has bounded variation $\mathrm{v}(x'm)$ [1, p. 35], i.e. there exists a non-negative finite constant M_x such that $\mathrm{v}(x'm, S) \leqq M_{x'}$. From this we obtain the inequality

$$(1) \qquad \sum_{i=1}^{k} |x'm(E_i)| \leqq M_{x'} < \infty$$

for every finite family of mutually disjoint sets (E_i) in $\boldsymbol{S}$ forming a decomposition of the space S.

If there is a p in P such that $m_p(S)$ is infinite, then there must exist a sequence of finite families of disjoint sets (E_i^n) in $\boldsymbol{S}$ each forming a decomposition of S such that

$$(2) \qquad \sum_{i=1}^{k_n} p(m(E_i^n)) > 2^n, \quad n = 1, 2, \ldots$$

From the relation (1) there follows, for every x' in X', the relation

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{1}{2^n} |x'm(E_i^n)| \leqq M_{x'} < \infty .$$

Hence the family

$$\{2^{-n} m(E_i^n) : i = 1, \ldots, k_n;\ n = 1, 2, \ldots\}$$

with a countable index set $I = \{(i, n) : i = 1, \ldots, k_n,\ n = 1, 2, \ldots.\}$ is scalarly summable. Since X has the property (sas) by assumption, this family is absolutely summable. So for every p in P there holds

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} 2^{-n} p(m(E_i^n)) < \infty ,$$

which is not possible if the relation (2) should be valid. This contradiction proves our Proposition.

Corollary. *Let X be a nuclear locally convex space. Let $\mathbf{S}$ be a sigma algebra of subsets of S. Then every vector-valued measure m: $\mathbf{S} \to X$ has bounded variation on $\mathbf{S}$.*

Proof. This follows from the fact that every nuclear locally convex space has the property (sas) [2, 4.2.2].

Remark. We have mentioned that every nuclear space has the property (sas). It remains an open problem what properties should a locally convex space with the property (sas) posses in order to be nuclear (cf. [2, 4.2.6]).

Recall that if $\mathbf{D}$ is a delta ring, then for every set E in $\mathbf{D}$ the family of sets from $\mathbf{D}$ which are subsets of E is a sigma algebra. Thus we have proved the following.

Theorem 1. *Every vector-valued measure m on the delta ring $\mathbf{D}$ with values in a locally convex space X with the property* (sas) *in particular with values in a nuclear space, has finite variation on $\mathbf{D}$.*

Recall that a locally convex space is said to be dually-metrizable [2, 0.7.5], if it is sigma quasibarelled and has a countable fundamental system of bounded sets.

We shall show that for metrizable or dually-metrizable locally convex spaces there holds in a certain sense a converse assertion to Theorem 1.

Theorem 2. *Let X be a metrizable or dually-metrizable locally convex space. If every vector-valued measure m: $\mathbf{D} \to X$ has finite variation on $\mathbf{D}$, then X is a nuclear space.*

Proof. It suffices to prove that every summable sequence of elements of X is absolutely summable in this case [2, 4.2.5].

Let $(x_i, i \in N)$ be a summable sequence of elements of X, where N is the set of positve integers. First suppose that X is sequentially complete. Define a set function m: $P(N) \to X$ by $m(E) = \sum_{i \in E} x_i$, for all E from $P(N)$ — the system of all subsets of the set N — especially $m(N) = \sum_{i=1}^{\infty} x_i$. From the summability it follows that m is a vector measure with values in X which has by assumption bounded variation, hence $m_p(N)$ is a finite number. Then

$$\sum_{i=1}^{\infty} p(x_i)| = \sum_{i=1}^{\infty} p(m(\{i\})) \leqq \sum_{i=1}^{\infty} m_p(\{i\}) = m_p(N) < \infty .$$

So $(x_i, i \in N)$ is an absolutely summable sequence. If the space X is not complete, we take its completion and show that this is nuclear, hence X is nuclear as a subspace of a nuclear space [2, 5.1.1]. The proof is complete.

If we adjoint to the assumptions of Theorem 2 else the completeness of the space X, then the space X not only is nuclear but must be even reflexive and in case of a normable space even finite-dimensional as the following theorem shows.

Theorem 3. *Let X be a complete metrizable or quasi-complete dually-metrizable locally convex space. If every vector-valued measure on the delta ring with values in X has finite variation, then the space X is (not only nuclear but also) reflexive.*

Proof. From Theorem 2 it follows that X is nuclear and the completeness implies that X is semireflexive [2, 4.4.11] and since X is metrizable or dually-metrizable, it is reflexive [2, 4.4.12].

A similar theorem as that for nuclear spaces holds also for dual-nuclear spaces. Recall that a locally convex space X is said to be dual-nuclear if its strong dual is nuclear.

Theorem 4. *A vector valued measure on the delta ring with values in a dual-nuclear space has finite variation.*

This follows from Theorem 1 since every dual-nuclear space has the property (sas) [2, 4.2.8].

Conversely we have the following.

Theorem 5. *If X is metrizable or dually-metrizable locally convex space and every vector-valued measure on the delta ring with values in X has finite variation, then X is dual-nuclear.*

Proof. According to Theorem 2 the space X is nuclear and since it is metrizable or dually-metrizable, it is also dual-nuclear [2, 4.3.3].

The last theorem can be else generalized if we consider locally convex spaces with the property (**B**): For every bounded set **B** of absolutely summable sequences of elements of X there exists a closed absolutely convex bounded set B in X such that

$$\sum_{n=1}^{\infty} p_B(x_n) < \infty \quad \text{for all } (x_n, N) \quad \text{from } \mathbf{B},$$

where p_B is the Minkowski functional of the set B. All metrizable and dually-metrizable locally convex spaces have the property (**B**). For such spaces we have the following

Theorem 6. *Let X be a locally convex space with the property* (**B**). *If every vector-valued measure on the delta ring with values in X has finite variation, then X is dual-nuclear.*

Proof. In the same way as in Theorem 2 we show that every summable sequence of elements of X in the given assumptions is absolutely summable, from which it follows that X is dual-nuclear.

Remark. In general a dual-nuclear space need not be nuclear and a nuclear space need not be dual-nuclear [2, 4.3.4].

Corollary 1. *If under the assumptions of Theorem 6 the space X is quasi-*

complete, then X is semireflexive in particular if X is metrizable, then X is reflexive.

Recall that every quasi-complete dual-nuclear locally convex space is semireflexive [2, 4.4.11].

As a corollary of the preceding results we can give the following well-known result.

Corollary 2. *Let X be a normable locally convex space. If every vector-valued measure on the delta ring with the values in X has finite variation, then X is finite-dimensional.*

Proof. Under the assumptions the space X must be nuclear and this is possible only in the case X is finite-dimensional [2, 4.4.14].

From the results of this paper we may conclude the following. In the class of the locally convex spaces with the property (**B**): All vector-valued measures on the delta rings with values in (quasi-complete) space X have finite variation if and only if the space X is dual-nuclear (only if X is semireflexive). In the class of metrizable or dually-metrizable locally convex spaces X: All vector-valued measures on the delta rings with values in (complete) X have finite variation if and only if X is nuclear (only if X is reflexive).

Thus for arbitrary locally convex spaces the similar questions remain still open.

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ВЕКТОРНЫЕ МЕРЫ И ЯДЕРНОСТЬ

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Резюме

В работе характеризован класс локально выпуклих пространств, в которых каждая векторная мера на дельта-кольце со значениями в таком пространстве имеет конечную вариацию. Такими пространствами суть, например, все ядерные пространства.