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HADWIGER NUMBERS OF FINITE GRAPHS

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In the present paper we shall study the Hadwiger number of a graph. The concept was defined in [2]. We consider only finite undirected graphs without loops and multiple edges.

We say that a graph G_1 can be contracted onto a graph G_2, if and only if G_2 can be obtained from G_1 by a finite number of the following operations:

(1) identifying two adjacent vertices;

(2) deleting an edge;

(3) deleting an isolated vertex.

To identify two vertices x and y means to delete x and y and all edges incident with them from a graph and to add a new vertex z to it and to join it to all the remaining vertices which were joined to x or y.

The Hadwiger number $\eta(G)$ of a graph G is the maximal number of vertices of a complete graph onto which G can be contracted. Some properties of the Hadwiger number of a graph were shown in [3] and [4]. Here we shall show some further properties of this concept.

At first we shall study Vizing's definition. After performing the operation (1) some multiple edges and loops can occur; this is why the operation (2) in the definition occurs. If some pair of vertices is joined by more than one edge, we may use the operation (2) and delete all of these edges except for one; by this operation we may delete also all the loops. The operation (3) occurs in the definition in order that also disconnected graphs might be considered. The following proposition will make this clear.

Proposition 1. *Let G be a finite undirected graph, let $C_1, \ldots, C_n$ be its connected components. Then*

$$\eta(G) = \max_{1 \leqq i \leqq n} \eta(C_i) .$$

Proof. Let K be the complete graph with $\eta(G)$ vertices onto which G can be contracted. By none of the operations (1), (2), (3) it is possible to make some vertices from different connected components of G to be adjacent or identical. As in K any two distinct vertices are adjacent, all vertices of K are obtained from vertices of one connected component of G and $\eta(G) = \eta(C_i)$ for some i, $1 \leqq i \leqq n$. On the other hand, if $\eta(C_j)$ is the Hadwiger number of some $C_j (1 \leqq j \leqq n)$, then we may contract C_j onto a complete graph with $\eta(C_j)$ vertices and each other connected component of G can be contracted onto one vertex (by using (1) and (2)) and this vertex can be deleted by the operation (3). Then G is contracted onto a complete graph with $\eta(C_j)$ vertices and we obtain $\eta(G) \geqq \eta(C_j)$ for each $j = 1, \ldots, n$. Since we have proved above that $\eta(G) = \eta(C_i)$ for some i, we have proved the assertion.

The proof of Proposition 1 has shown the importance of the operation (3) in the case of a disconnected graph. For connected graphs it is unnecessary, as the following proposition shows.

Proposition 2. *Let G, G' be two non-empty finite connected undirected graphs such that G can be contracted onto G' by using the operations* (1), (2), (3). *Then G can be transformed into G' by using the operations* (1), (2) *only.*

Proof. The graph G is connected, therefore it does not contain isolated vertices. In the procedure of transformation of G into G' an isolated vertex can occur only in two ways; either the whole graph G is contracted by (1) and (2) onto one vertex, or by (2) some separating edge set S in G is deleted and some connected component C of the obtained graph is contracted into one vertex. In the first case we obtain by (3) the empty graph which cannot be contracted onto G', because G' is assumed to be non-empty. In the second case let G'' be the graph obtained by deleting the described isolated vertex. The graph G'' can be obtained without (3) in such a way that not S, but $S - \{e\}$, where $e \in S$, is deleted, C is contracted onto one vertex and then this vertex is identified (operation (1)) with the other end vertex of e. Thus we have proved that we may avoid the operation (3) in constructing G'.

From Proposition 1 we see that it suffices to study connected graphs; Proposition 2 shows that in these studies we need to consider only operations (1) and (2).

In the case of connected graphs we may use the concept of the connected homomorphism introduced by O. Ore [1].

A homomorphism of a graph G onto a graph G' is a surjective mapping τ of the vertex set $V(G)$ of G onto the vertex set $V(G')$ of G' such that two vertices u, v of $V(G')$ are adjacent if and only if there exist vertices $u_0 \in \tau^{-1}(u)$, $v_0 \in \tau^{-1}(v)$, which are adjacent in G, where

$$\tau^{-1}(x) = \{y \in V(G) \mid \tau(y) = x\} \text{ for } x \in V(G') .$$

A homomorphism τ of G onto G' is called a connected homomorphism if and only if the set $\tau^{-1}(x)$ for each $x \in V(G')$ induces a connected subgraph of G.

For connected graphs the definition of the Hadwiger number which was written above is evidently equivalent with the following definition.

The Hadwiger number $\eta(G)$ of a finite connected undirected graph G is the maximal number of vertices of a complete graph onto which G can be mapped by a connected homomorphism.

We may define an auxiliary concept ot the H-decomposition of a graph. Let G be a finite connected undirected graph. Then an H-decomposition of G is a decomposition of the vertex set $V(G)$ of G into pairwise disjoint subsets $V_1, \ldots, V_m$ such that $\bigcup_{i-1}^{m} V_i = V(G)$, each V_i induces a connected subgraph G_i of G for $i = 1, \ldots, m$ and for any two positive integers i, j such that $1 \leqq i \leqq m$, $1 \leqq j \leqq m$, $i \neq j$, there exists at least one edge joining a vertex of V_i to a vertex of V_j.

From the definition it is clear that in each graph G with the Hadwiger number $\eta(G)$ there exists at least one H-decomposition of the cardinality $\eta(G)$; it is formed by the system of sets $\tau^{-1}(x)$ for all vertices x of the complete graph onto which G is mapped by a connected homomorphism τ.

Theorem 1. *Let G be a finite connected graph with cutvertices, let $B_1, \ldots, B_k$ be its blocks. Then*

$$\eta(G) = \max_{1 \leqq i \leqq k} \eta(B_i) .$$

Proof. We use the induction according to k and consider an H-decomposition $\mathscr{H}$ of G of the cardinality $\eta(G)$. Let $k = 2$. The graph G has two blocks B_1, B_2; let a be the cutvertex of G. There cannot exist two sets V_i, V_j of an H-decomposition of G such that V_i contains only vertices of B_1, V_j contains only vertices of B_2 and none of these sets contains a; for such sets there would not exists any edge joining a vertex of V_i to a vertex of V_j. As the sets of the H-decomposition are pairwise disjoint, only one contains a. As any of these sets induces a connected subgraph of G, any of such sets which does not contain a, must contain only vertices of one block. This means that there exists a set of $\mathscr{H}$ which contains either all vertices of B_1, or all the vertices of B_2. Without loss of generality let such a set V_i contain all the vertices of B_2. The decomposition of the vertex set of B_1 which consists of $V_i - (V(B_2) - \{a\})$ and all sets of $\mathscr{H}$ different from V_i is evidently an H-decomposition of B_1 of the same cardinality $\eta(G)$ as $\mathscr{H}$. Therefore $\eta(B_1) \geqq \eta(G)$. On the other hand, to any H-decomposition $\mathscr{H}'$ of B_1 we can assign an H-decomposition $\mathscr{H}''$ of G so that to the set of $\mathscr{H}'$ containing a we add all vertices of B_2; thus $\eta(G) \leqq \eta(B_1)$, which means $\eta(B_1) = \eta(G)$. As $\mathscr{H}$ is an H-decomposition of G of the cardinality $\eta(G)$, we see that $\eta(B_1) \geqq \eta(B_2)$; otherwise an H-decomposition of B_2 could be transformed analogously into an H-decomposition of G with more than $\eta(G)$ sets. If $k \geqq 3$, we can make the proof analogously; in this case we denote by B_1 some block of G which contains only one cutvertex (such a block must exist) and instead of B_2 we consider the subgraph of G obtained by deleting all vertices of B_1 except for this cutvertex a.

Proposition 3. *Let G be a finite undirected graph, let G' be a subgraph of G. Then*

$$\eta(G') \leqq \eta(G) .$$

Proof. We shall prove this assertion with the help of the operations (1), (2), (3). By the operation (2) we delete all edges not belonging to G'. Then all vertices not belonging to G' become isolated and we delete them by the operation (3). Thus we obtain G' from G and then we contract it onto a complete graph with $\eta(G')$ vertices. In this way we have obtained a complete graph with $\eta(G')$ vertices from G, which proves the assertion.

By $K_{m,n}$ we shall denote a complete bipartite graph, i. e. a graph whose vertex set is the union of two disjoint sets A, B such that $|A| = m$, $|B| = n$, each vertex of A is adjacent to each vertex of B, no two vertices of A and no two vertices of B are adjacent.

Theorem 2. *Let $K_{m,n}$ be a complete bipartite graph. Then*

$$\eta(K_{m,n}) = \min(m, n) + 1 .$$

Proof. Without loss of generality let $m \leqq n$. As it is well known, $K_{m,n}$ contains an set F consisting of m idnependent edges. Choose an edge $e \in F$. If we contract each edge of $F - \{e\}$ onto one vertex, these $m - 1$ vertices will induce a complete graph K_{m-1} with $m - 1$ vertices. Each of the end vertices of e is adjacent to all of these $m - 1$ and they are also adjacent to each other (by e), therefore we have a graph containing a complete graph K_{m+1} with $m + 1$ vertices as a subgraph. As $\eta(K_{m+1}) = m + 1$, we have $\eta(G) \geqq m + 1$. Now assume that $\eta(K_{m,n}) \geqq m + 2$. There must exist an H-decomposition of $K_{m,n}$ with at least $m + 2$ sets. As these sets are pairwise disjoint and $|A| = m$, at least two of them do not contain any vertex of A and thus they are subsets of B. As any of these sets induces a connected sugraph of $K_{m,n}$, the sets must contain exactly one element each, because B is an independent subset and thus only one-element subsets of B induce connected subgraphs. But then there exists no edge joining a vertex of one of these sets to a vertex of another, which is a contradiction.

Theorem 3. *Let G be a finite connected undirected graph. Let G contain a clique C with k vertices, let G' be the subgraph of G induced by the set $V(G) - V(C)$. Then*

$$\eta(G) \leqq k + \eta(G').$$

Proof. Suppose that there exists an H-decomposition $\mathscr{H}$ of G with at least $k + \eta(G') + 1$ sets. As these sets are disjoint, there are at least $\eta(G') + 1$ sets of $\mathscr{H}$ which do not contain vertices of C. But these sets form an H-decomposition of some subgraph G'' of G' with at least $\eta(G') + 1$ sets and thus $\eta(G'') \geqq \eta(G') + 1$, which is impossible, because G'' is a subgraph of G'.

Proposition 4. *Let G be a graph obtained from two finite undirected graph G' and G'' by joining each vertex of G' with each vertex of G'' by an edge. Then*

$$\eta(G) \geqq \eta(G') + \eta(G'').$$

Proof. Evidently by contracting G' onto a complete graph with $\eta(G')$ vertices and simultaneously G'' onto a complete graph with $\eta(G'')$ vertices the graph G is contracted onto a complete graph with $\eta(G') + \eta(G'')$ vertices. Thus

$$\eta(G) \geqq \eta(G') + \eta(G'').$$

The equality sign in this relation need not occur, as Theorem 2 shows. The graph $K_{m,n}$ is such a graph, where G' and G'' are graphs consisting both only of isolated vertices, therefore $\eta(G') = \eta(G'') = 1$, but $\eta(K_{m,n})$ can be greater than two.

Corollary. *Let G be a graph obtained from a clique C with k vertices and of some finite undirected graph G' by joining each vertex of C with each vertex of G' by an edge. Then*

$$\eta(G) = k + \eta(G').$$

Now we shall prove a theorem concerning the Cartesian products of graphs. If G_1, G_2 are two undirected graphs, then their Cartesian product $G_1 \times G_2$ is the graph whose vertices are all ordered pairs $[x_1, x_2]$, where $x_1 \in V(G_1)$, $x_2 \in V(G_2)$ and two vertices $[x_1, x_2]$, $[y_1, y_2]$ are joined by an edge in $G_1 \times G_2$ if and only if either $x_1 = y_1$ and x_2, y_2 are adjacent in G_2, or x_1, y_1 are adjacent in G_1 and $x_2 = y_2$.

Theorem 4. *Let G_1, G_2 be two finite connected undirected graphs, let $G_1 \times G_2$ be their Cartesian product. Then*

$$\eta(G_1 \times G_2) \geqq \eta(G_2) + \eta(G_2) - 1.$$

Proof. Let τ_1 be a connected homomorphism of G_1 onto a complete graph C_1 with $\eta(G_1)$ vertices, let τ_2 be a connected homomorphism of G_2 onto a complete graph C_2 with $\eta(G_2)$ vertices. Then we consider the mapping τ such that for each vertex $[x_1, x_2]$ of $G_1 \times G_2$ we have

$$\tau([x_1, x_2]) = [\tau_1(x_1), \tau_2(x_2)].$$

The mapping τ is evidently a connected homomorphism of $G_1 \times G_2$ onto $C_1 \times C_2$. It remains to prove that

$$\eta(C_1 \times C_2) \geqq \eta(G_1) + \eta(G_2) - 1.$$

The vertices of C_1 will be denoted by $u_1, \ldots, u_{\eta(G_1)}$, the vertices of C_2 will be denoted by $v_1, \ldots, v_{\eta(G_2)}$. Let $A_i = \{[u_i, v_j] \mid j = 1, \ldots, \eta(G_2)\}$ for $i = 1, \ldots, \eta(G_1)\}$. The system of sets $\mathscr{H} = \{\{[u_1, v_1]\}, \{[u_1, v_2]\}, \ldots, \{[u_1, v_{\eta(G_2)}]\}, A_2, \ldots, A_{\eta(G_1)}\}$ is evidently an H-decomposition of $C_1 \times C_2$ of the cardinality $\eta(G_1) + \eta(G_2) - 1$. Therefore

$$\eta(G_1 \times G_2) \geqq \eta(C_1 \times C_2) \geqq \eta(G_1) + \eta(G_2) - 1.$$

This inequality cannot be improved in general. If G_1 is a graph consisting of one edge and its end vertices and $G_2 \cong G_1$, then $G_1 \times G_2$ is a circuit of the length four. We have

$$\eta(G_1) = \eta(G_2) = 2, \; \eta(G_1 \times G_2) = 3.$$

Theorem 5. *Let G be a finite undirected graph, let u be its vertex. Let G' be the graph obtained from G by deleting u and all edges incident with u. Then*

$$\eta(G') \geqq \eta(G) - 1.$$

Proof. Let $\mathscr{H}$ be an H-decomposition of G with $\eta(G)$ vertices. Let H_0 be the set of $\mathscr{H}$ containing u. The system of sets $\mathscr{H} - \{H_0\}$ forms an H-decomposition of some subgraph G'' of G' of the cardinality $\eta(G) - 1$. Therefore

$$\eta(G'') \geqq \eta(G) - 1$$

and, as G'' is a subgraph of G',

$$\eta(G') \geqq \eta(G) - 1.$$

Theorem 6. *Let G be a finite undirected graph, let e be its edge. Let G' be the graph obtained from G by deleting e. Then*

$$\eta(G') \geqq \eta(G) - 1.$$

The proof is analogous to the proof of Theorem 5.

Now we shall study graphs which are critical with respect to the Hadwiger number. A graph G is said to be vertex-critical (or edge-critical) with respect to the Hadwiger number, if each graph obtained from G by deleting a vertex (or an edge respectively) has the Hadwiger number less than $\eta(G)$.

Theorem 6. *Let G be a finite undirected graph which is vertex-critical with respect to the Hadwiger number. Then G is connected and any H-decomposition*

$\mathscr{H} - \{V_1, \ldots, V_{\eta(G)}\}$ of G with $\eta(G)$ sets has the following property: for each i $1, \ldots, \eta(G)$, if $u \in V_i$, then either for some $j \neq i$ all edges joining a vertex of V_i with a vertex of V_j are incident with u, or u is a cut-vertex of the subgraph G_i of G induced by V_i.

Proof. The connectivity of G follows from Proposition 1. As the graph G' obtained from G by deleting u and all edges incident with u has the Hadwiger number less than $\eta(G)$, the graph G' cannot contain an H-decomposition of the cardinality $\eta(G)$. Therefore $(\mathscr{H} - \{V_i\}) \cup \{V_i - \{u\}\}$ cannot be an H-decomposition of G'. As all V_j for $j \neq i$ remain unchanged, the set $V_i - \{u\}$ either does not induce a connected subgraph of G', or there exists some $j \neq i$ such that no vertex of $V_i - \{u\}$ is joined by an edge with a vertex of V_j; this means that all edges joining a vertex of V_i with a vertex of V_j are incident with u.

Theorem 8. *Let G be c finite undirected graph which is edge-critical with respect to the Hadwiger number. Then any H-decomposition $\mathscr{H} = \{V_1, \ldots, V_{\eta(G)}\}$ with $\eta(G)$ sets has the following properties:*

(a) *For each $i = 1, \ldots, \eta(G)$ the subgraph G_i of G induced by V_i is a tree.*

(b) *For $i, j = 1, \ldots, \eta(G)$, $i \neq j$, there exists exactly one edge joining a vertex of V_i to a vertex of V_j.*

(c) *Each terminal vertex of G_i is incident with at least one edge not belonging to G_i (for $i = 1, \ldots, \eta(G)$). Moreover, G is connected.*

Proof. Let G' be the graph obtained from G by deleting e. Let $\mathscr{H}$ be an H-decomposition of G of the cardinality $\eta(G)$. As $\eta(G') < \eta(G)$, the decomposition $\mathscr{H}$ cannot be an H-decomposition of G'. Therefore either some $V_i \in H$ does not induce a connected subgraph of G', or for some i, j no vertex of V_i is joined to a vertex of V_j. This means that e is a bridge in the subgraph G_i of G induced by V_i for some i, or e is the unique edge joining a vertex of V_i with a vertex of V_j for some $i, j, i \neq j$. As e was arbitrarily chosen, this holds for each edge of G. Thus each edge of some G_i must be a bridge in it; a connected graph in which each edge is a bridge is a tree — (a) is proved. An edge not belonging to any G_i must be the unique edge joining a vertex of V_i with a vertex of V_j for some $i, j, i \neq j$ — (b) is proved. It remains to prove the property (c). If some terminal vertex u of G_i for some i were not be joined with a vertex of any V_j for $j \neq i$, then by deleting the (unique) edge incident with it this vertex would become isolated and $(H - \{V_i\}) \cup \{V_i - \{u\}\}$ would be an H-decomposition of the graph obtained from G by deleting e and u and this graph would have the Hadwiger number $\eta(G)$ which would be a contradiction to the assumption that G is edge-critical. The connectivity of G follows from Proposition 1.

Theorem 9. *Let G be a finite undirected graph. Then G contains a subgraph G_0 such that*

$$\eta(G_0) = \eta(G)$$

and G_0 is vertex-critical with respect to the Hadwiger number.

Proof. If G is vertex-critical, then $G_0 = G$. If not, we choose a vertex u_1 such that the graph obtained from G by deleting u_1 and all edges incident with u_1 has the same Hadwiger number as G. If this graph is vertex-critical, it is G_0. If not, we choose a vertex u_2 such that after deleting it we obtain a graph with the Hadwiger number $\eta(G)$. Thus we proceed further and after a finite number of steps we must obtain G_0.

Theorem 10. *Let G be a finite undirected graph. Then G contains a subgraph G_0 such that*

$$\eta(G_0) = \eta(G)$$

and G_0 is edge-critical with respect to the Hadwiger number.

Proof is analogous to the proof of Theorem 9. Instead of vertices we delete edges; if some isolated vertex occurs, we delete it, too.

Concluding we shall express two conjectures.

Conjecture 1. *For any two finite connected undirected graphs G_1 and G_2 we have*

$$\eta(G_1 \times G_2) = \eta(G_1) + \eta(G_2) - 1 .$$

Conjecture 2. *Let G be a finite undirected graph with n vertices, let $\bar{G}$ be its complement. Then*

$$\eta(G) + \eta(\bar{G}) \leqq n + 1 .$$

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