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SOME PROBLEMS OF CLASSIFICATION OF POINTS IN THE DESARGUESIAN HJELMSLEV PLANE

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ABSTRACT. The external, internal, and zero external and two zero internal points of a conic are defined. The number of external and internal points is determined and also the number of zero external and zero internal points of a conic.

We will formulate and prove some results concerning the number of internal and external points (in two meanings) of a regular conic in a Desarguesian Hjelmslev plane over a finite local ring.

1. Concepts and notations

Let $(\mathcal{B}, \mathcal{P}, \mathcal{I})$ be an *incidence structure*; elements of the set $\mathcal{B}$ are the *points*, and elements of the set $\mathcal{P}$ are the *lines* of the incidence structure considered, and $\mathcal{I}$ is the *relation* of incidence.

DEFINITION 1.1. We will call two *points* $A, B \in \mathcal{B}$ *neighbouring* if there exist lines $p_1, p_2 \in \mathcal{P}$, $p_1 \neq p_2$, such that $A, B\,\mathcal{I}\,p_1$ and also $A, B\,\mathcal{I}\,p_2$. Two *lines* $a, b \in \mathcal{P}$ are *neighbouring* if there exist points $A_1, A_2 \in \mathcal{B}$, $A_1 \neq A_2$, such that $A_1\,\mathcal{I}\,a, b$ and also $A_2\,\mathcal{I}\,a, b$.

DEFINITION 1.2. The incidence structure $(\mathcal{B}, \mathcal{P}, \mathcal{I})$ is *homomorphic* with the incidence structure $(\mathcal{B}', \mathcal{P}', \mathcal{I}')$ if there exists a mapping $\Phi\colon \mathcal{B} \cup \mathcal{P} \to \mathcal{B}' \cup \mathcal{P}'$ such that

1. $\Phi(\mathcal{B}) = \mathcal{B}'$, $\Phi(\mathcal{P}) = \mathcal{P}'$,
2. $A\,\mathcal{I}\,a \iff \Phi(A)\,\mathcal{I}'\,\Phi(a)$ for $A \in \mathcal{B}$, $a \in \mathcal{P}$.

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DEFINITION 1.3. The *Hjelmslev plane* is the incidence structure $\mathcal{H} = (\mathcal{B}, \mathcal{P}, \mathcal{I})$ with the properties

1. for all $A, B \in \mathcal{B}$, $A \neq B$, there exists a line $p \in \mathcal{P}$ such that $A, B\, \mathcal{I}\, p$,
2. for all $p_1, p_2 \in \mathcal{P}$, $p_1 \neq p_2$, there exists a point $P \in \mathcal{B}$ such that $p \mathcal{I} p_1, p_2$,
3. there exists a homomorphism Φ from the plane $\mathcal{H}$ onto the projective plane π such that
   a) the points A, B are neighbouring if and only if $\Phi(A) = \Phi(B)$,
   b) the lines a, b are neighbouring if and only if $\Phi(a) = \Phi(b)$.

In this paper, by a *special local ring* we will mean a finite commutative local ring R of which the ideal I of divisors of zero is principal. Let g be the *generator* of the ideal I. The smallest integer $\nu \in N$ such that $g^\nu = 0$ is called the *index of nilpotency* of the ring R. We will suppose that R is not a field, and that the characteristic of the ring R is odd. The symbol $\overline{R}$ will denote the *factor ring* R/I. Let Φ be the *canonical homomorphism* of R onto $\overline{R}$. The symbol R^* will denote the set $R - I$.

DEFINITION 1.4. The *Desarguesian Hjelmslev plane* over the finite local ring R is the incidence structure $H(R) = (\mathcal{B}, \mathcal{P}, \mathcal{I})$ defined in the following way: the elements of $\mathcal{B}$, that is, the points of the plane $H(R)$, are classes of ordered triples $(\lambda x_1; \lambda x_2; \lambda x_3)$, where $\lambda \in R^*$, $x_1, x_2, x_3 \in R$ and one of the x_i is regular (i.e., $x_i \in R - I$) — the elements $\mathcal{P}$, that is, the lines of the plane $H(R)$, are classes of ordered triples $(\alpha a_1; \alpha a_2; \alpha a_3)$, where $\alpha \in R^*$, $a_1, a_2, a_3 \in R$ and one of the a_i is regular. The point $X = [\lambda x_1; \lambda x_2; \lambda x_3]$ is incident to the line $a = [\alpha a_1; \alpha a_2; \alpha a_3]$ if and only if

$$a_1 x_1 + a_2 x_2 + a_3 x_3 = 0 . \tag{1}$$

We will call the set of all points $[x_1; x_2; x_3] \in H(R)$ whose coordinates satisfy the equation

$$\sum_{i,j=1}^{3} a_{ij} x_i x_j = 0 \tag{2}$$

the *conic* Q in $H(R)$.

We suppose that the conic Q is regular, i.e., $\det[a_{ij}] \notin I$. Observe that in a suitable coordinate system the conic given by (2) satisfies the following equation

$$ax^2 + by^2 + cz^2 = 0 . \tag{3}$$

The Hjelmslev plane $H(R)$ is mapped onto the projective plane $\pi(\overline{R})$ by the canonical homomorphism Φ. It is known that a conic in the projective plane $\pi(\overline{R})$ over the skewfield $\overline{R}$ has exactly $|\overline{R}| + 1$ points. It can be shown that the conic in the Hjelmslev plane has exactly $|R| + |I|$ points ([3]).

Let Q be the conic given by (2), and let $P = [\underline{x}_1; \underline{y}_1; \underline{z}_1] \in H(R)$. Then the polar of the point P to the conic Q is the line $\underline{x}(\underline{a}_{11}\underline{x}_1 + \underline{a}_{12}\underline{y}_1 + \underline{a}_{13}\underline{z}_1) + \underline{y}(\underline{a}_{21}\underline{x}_1 + \underline{a}_{22}\underline{y}_1 + \underline{a}_{23}\underline{z}_1) + \underline{z}(\underline{a}_{31}\underline{x}_1 + \underline{a}_{32}\underline{y}_1 + \underline{a}_{33}\underline{z}_1) = 0$. We will call the line t the *tangent* of the conic Q if t intersects the conic Q at more than two points. We will call the line t the *zero tangent* of the conic Q if t is the polar of a point being in the conic.

Next we will need the statement

THEOREM 1.1. *The line $\underline{Ax} + \underline{By} + \underline{Cz} = 0$ is a tangent of the conic* (2) *if and only if*

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} & A \\ a_{21} & a_{22} & a_{23} & B \\ a_{31} & a_{32} & a_{33} & C \\ A & B & C & 0 \end{bmatrix} = n^2 , \qquad n \in I . \tag{4}$$

Proof. See [3]. $\square$

We will need the following combinatorial result.

LEMMA 1.1. *Let R be a special local ring, and ν be the index of nilpotency of the ring R. Then the number W of singular squares in R is given by the relations*

a) *if $\nu = 2k$, then*

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - |\overline{R}|}{2R+1} , \tag{5}$$

b) *if $\nu = 2k+1$,*

$$W = 1 + \frac{|\overline{R}|^{\nu-1} - 1}{2(|\overline{R}| + 1)} . \tag{6}$$

Proof. See [3]. $\square$

Using the concept that was introduced by V. Chvál we define a *quasiconic* $Q(n)$ in $H(R)$ as the set of all points $[x; y; z] \in H(R)$ satisfying

$$ax^2 + by^2 + cz^2 = r^2 n , \qquad r \in R - I , \ n \in I . \tag{7}$$

2. The classification of points of the plane $H(R)$. The combinatorial point of view

Definition 2.1. Let Q be a conic in the plane $H(R)$. We will call the point $X \in H(R)$ an *external point* of the conic Q if there is a tangent which is incident to this point and $X \notin Q$. We will call the point $X \in H(R)$ an *internal point* of the conic Q if the point X is not incident to a tangent of the conic.

The following auxiliary statement is satisfied for the internal points which are not neighbouring with points of the conic.

Lemma 2.1. *Let $X \in H(R)$ be a point not neighbouring with points of the conic Q. The point $X \in H(R)$ is an internal point of the conic Q if and only if the point $\overline{X} \in \pi(\overline{R})$ is an internal point of the conic $\overline{Q}$.*

Proof.

a) Let $\overline{X} \in \pi(\overline{R})$ be an internal point of the conic $\overline{Q}$. From Definition 2.1, it then follows that there exists no tangent through the point $\overline{X} \in \pi(\overline{R})$.

b) Let the point $X \in H(R)$ be an internal point of the conic Q, and let $\overline{X} \in \pi(\overline{R})$ be a non-internal point of the conic $\overline{Q}$. From the definition, it follows that there is a tangent through the point X to the conic. We denote by $\overline{T}$ the point in which the tangent intersects the conic. Then the line $XT \in H(R)$ (where T is the model of the point $\overline{T}$ on the conic Q) is a tangent of the conic Q in $H(R)$, i.e., the point X is external. This contradicts the assumption. Consequently, $\overline{X}$ is an internal point of the conic $\overline{Q}$. □

We now introduce two lemmas.

Lemma 2.2. *Let $f(x) = \alpha x^2 + \beta x + \gamma$, let $\alpha, \gamma \in I$, $\beta \notin I$. Then f attains all values in R.*

Proof. Let $f(x) = f(y)$, then $\alpha x^2 + \beta x + \gamma = \alpha y^2 + \beta y + \gamma$, and after a reordering, we have

$$\alpha(x^2 - y^2) + \beta(x - y) = 0,$$

then

$$(x - y)\left[\beta + \alpha(x - y)\right] = 0.$$

□

From the preceding relation, for $\beta \notin I$, it follows that $x = y$.

Lemma 2.3. *Let $\alpha x^2 + \beta y^2 + \gamma xy = n$ for $\alpha, \beta \in I$, and $\gamma \notin I$, $n \in I$. Then this equation has a regular solution.*

Proof. Let x, y satisfy

$$\alpha x^2 + \beta y^2 + \gamma xy = n, \qquad \alpha, \beta \in I, \ \gamma \notin I, \ n \in I.$$

We assume that $y = 1$, which implies that $\alpha x^2 + \beta + \gamma x = n$. We write $f(x) = \alpha x^2 + \beta + \gamma x$. From the preceding lemma, the existence of x_0 follows such that $f(x_0) = n$; then $\alpha x_0^2 + \beta + \gamma x_0 = n$. Then $(x_0; 1)$ is the required regular solution. □

Now we introduce the concept of zero external and zero internal points of the conic.

DEFINITION 2.2. The point $X \in H(R)$ is called a *zero external point* of the conic in $H(R)$ if there is a zero tangent incident to this point, which is not on the conic. The *zero internal points* of the conic are the points which are not on a zero tangent nor on the conic.

We wish to find the numbers of zero external and zero internal points of the conic. We will need the following statement.

THEOREM 2.1. *Let P be a point that is not neighbouring with points of the conic, and let t be a tangent to the conic Q incident to the point P. Then zero tangents are also incident to the point P.*

Proof. We consider the equation of the conic Q in the form

$$ax^2 + by^2 + cz^2 = 0. \tag{8}$$

Let $P = [1; y_0; z_0]$ be the point from which there is a tangent to the conic, and let the line $Ax + By + Cz = 0$ be a line incident to the point p, so that

$$A + By_0 + Cz_0 = 0. \tag{9}$$

The line (9) is a tangent to the conic (8) if and only if

$$A^2bc + B^2ac + C^2ab = -n^2.$$

From (9) it follows that $A = -By_0 - Cz_0$. After substitution into the preceding equation, we obtain

$$B^2(bcy_0^2 + ac) + C^2(bcz_0^2 + ab) + 2BCbcy_0z_0 = -n^2. \tag{10}$$

We may distinguish two cases:
1. At least one of the coefficients in the equation (10) at B^2, C^2 is regular. Then the equation

$$\overline{B}^2\left(\overline{b}\overline{c}\overline{y}_0^2 + \overline{a}\overline{c}\right) + \overline{C}^2\left(\overline{b}\overline{c}\overline{z}_0^2 + \overline{a}\overline{b}\right) + 2\overline{B}\overline{C}\overline{b}\overline{c}\overline{y}_0\overline{z}_0 = 0 \tag{11}$$

has exactly two solutions because its discriminant

$$\overline{D} = -4\overline{a}\overline{b}\overline{c}\left(\overline{b}\overline{y}_0^2 + \overline{c}\overline{z}_0 + \overline{a}\right)$$

is regular since there are exactly two tangents from the point $[1;y_0;z_0]$ to the conic. It was shown in Theorem 1.1. of [3] that the equation

$$B^2(bcy_0^2+ac)+C^2(bcz_0^2+ab)+2BCbcy_0z_0=0$$

has the solutions B_1, C_1, and then the line

$$(-B_1y_0-C_1z_0)x+B_1y+C_1z=0$$

is a zero tangent of the conic Q incident to the point P.
2. Both coefficients at B^2, C^2 in the equation (10) are singular. It follows from Lemma 2.3 that the equation

$$B^2(bcy_0^2+ac)+C^2(bcz_0^2+ab)+2BCbcy_0z_0=-n^2$$

has a regular solution, consequently a zero tangent is incident to the point $P=[1;y_0;z_0]$. □

To prove the next theorem, we need a lemma:

LEMMA 2.4. *A zero tangent is incident to the point $P=[x_0;y_0;z_0]$ neighbouring with a point of a conic if and only if*

$$-4abc(ax_0^2+by_0^2+cz_0^2)=n^2\,,\qquad n\in I\,,$$

where $ax^2+by^2+cz^2=0$ is the equation of the conic considered.

Proof.
a) We consider the equation of the tangent in the form

$$Ax+By+Cz=0\,. \tag{12}$$

Let the equation of the conic be

$$ax^2+by^2+cz^2=0\,. \tag{13}$$

The line (12) is a zero tangent of the conic (13) if and only if

$$A^2bc+B^2ac+C^2ab=0\,. \tag{14}$$

Let us assume that z_0 is regular. Then

$$C=\frac{-Ax_0-By_0}{z_0}\,.$$

After substitution into (14) and reordering of terms, we get

$$A^2(bcz_0^2+abx_0^2)+B^2(acz_0^2+aby_0^2)+2abABy_0x_0=0\,.$$

If the point $[x_0; y_0; z_0]$ is on a tangent to a conic, then the discriminant of the last equation must be a singular square. Let us consider still the assumption that the point $[x_0; y_0; z_0]$ is neighbouring with points of the conic, then

$$ax_0^2 + by_0^2 + cz_0^2 \in I\,.$$

Then

$$D = -4abcz_0^2(ax_0^2 + by_0^2 + cz_0^2) = n^2\,, \qquad n \in I\,.$$

b) Conversely, let $[x_0; y_0; z_0]$ satisfy

$$-4abc(ax_0^2 + by_0^2 + cz_0^2) = n^2\,.$$

It is necessary to show that there are coëfficients A, B, C such that

$$A^2bc + B^2ac + C^2ab = 0\,, \tag{15}$$

$$Ax_0 + By_0 + Cz_0 = 0\,. \tag{16}$$

We consider the equation (15) as the equation of a conic in variables A, B, C, and the equation (16) as the line with coefficients x_0, y_0, z_0. Let us consider the determinant

$$\det \begin{bmatrix} bc & 0 & 0 & x_0 \\ 0 & ac & 0 & y_0 \\ 0 & 0 & ab & z_0 \\ x_0 & y_0 & z_0 & 0 \end{bmatrix} = -abc(ax_0^2 + by_0^2 + cz_0^2) = \frac{n^2}{4}\,, \tag{17}$$

from the above mentioned follows the assumption. It follows from the relation (17) that the line (16) is a tangent of the conic (15). □

THEOREM 2.2. *Let Q be a conic in $H(R)$. Then the number of neighbouring points on the conic which are on zero tangents is given by the relation*

$$(|R| + |I|)W\,.$$

P r o o f. It follows from the preceding lemma that a point belongs to the required set if and only if its coordinates satisfy the equation (we suppose z_0 regular)

$$-4abc(ax_0^2 + by_0^2 + cz_0^2) = n^2\,,$$

i.e., if the point $[x_0; y_0; z_0]$ is on $Q(n^2)$, where

$$Q(n^2) : -4abc(ax^2 + by^2 + cz^2) = n^2\,. \tag{18}$$

The quasiconic (18) includes exactly $|Q| \cdot [n^2]$ points. If n runs through all singular elements of the local ring R, then for the numbers of points on all quasiconics

$$\sum |[n^2]| \cdot |Q| = W(|R| + |I|)\,.$$

□

The following theorem introduces numbers of zero external and zero internal points.

THEOREM 2.3. *Let Q be a conic in $H(R)$. Then*

1.
$$(|\overline{R}|+1)|I|\left(\frac{|\overline{R}|}{2}\cdot|I|+W-1\right)$$
is the number of zero external points,

and

2.
$$\frac{|\overline{R}|(|\overline{R}|-1)}{2}\cdot|I|^2+(|\overline{R}|+1)|I|(|I|-W)$$
is the number of zero internal points of the conic.

Proof.

1. Let us consider first the situation in the projective plane $\pi(\overline{R})$. The number of external points is
$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}.$$
It maps exactly $|I|^2$ points of the plane $H(R)$ on each point of the plane $\pi(\overline{R})$. Then the number of external points not neighbouring with points of the conic is exactly
$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2.$$
This follows from Theorem 2.1. Let us now consider the points on the zero tangents neighbouring with points of the conic. The number of these points is, according to Theorem 2.2, $(|\overline{R}|+1)\cdot|I|W$. The external points are only those that are not on the conic. Then the number of all zero external points is
$$\begin{aligned}&\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2+(|\overline{R}|+1)|I|W-(|\overline{R}|+1)|I|\\&=(|\overline{R}|+1)\cdot|I|\cdot\left(\frac{|\overline{R}|}{2}\cdot|I|+W-1\right).\end{aligned}$$

2. Similarly, we will find the number of zero internal points. Zero internal points are points that map onto internal points in the plane $\pi(\overline{R})$ and also those points neighbouring with the points on the conic that are not on zero tangents. The number of these points is (according to Theorem 2.1)
$$\begin{aligned}&\frac{|\overline{R}|(|\overline{R}|-1)}{2}\cdot|I|^2+(|\overline{R}|+1)\cdot|I|^2-(|\overline{R}|+1)\cdot|I|W\\&=\frac{|\overline{R}|(|\overline{R}|-1)}{2}\cdot|I|^2+(|\overline{R}|+1)\cdot|I|(|I|-W).\end{aligned}$$
□

Next we have:

THEOREM 2.4. *There are exactly*

$$s = 1 + |I|\left(|R| - |I| \cdot \frac{1 - |\overline{R}|^{2\nu-2}}{|\overline{R}|^{2\nu-2}(1 - |\overline{R}|^2)}\right) \tag{19}$$

neighbouring points with a given point of the conic that are on tangents.

Proof. Let us take the equation of the conic

$$xy + bxz + cyz = 0 \tag{20}$$

and choose the point $[1;0;0]$ on this conic, (the choice of the point is arbitrary because it is possible to choose the coordinate system so that every point is transformed in $[1,0,0]$). Let us consider points neighbouring with the point $[1;0;0]$; their number is $|I|^2$. We will consider only those neighbouring points that are incident to a tangent to the conic. Let us consider the line incident to points $[1;0;0]$, $[1;j;k]$, $j,k \in I$. It is the line expressed by the equation

$$ex + fz = 0\,,$$

consequently,

$$ej + fk = 0\,.$$

Suppose, e.g., that $k = jj'$, then $ej + fjj' = 0$, then $j(e + fj') = 0$. Then $e + fj' \in Anj$. We write $e + fj' = j^*$, and we assume that $f = 1$. Then $j' = j^* - e$. The equation of the conic considered will have the form

$$(j^* - j')y + z = 0 \implies z = (j' - j^*)y\,.$$

Substituting this into the equation of the conic (20) we get, after a reordering of terms,

$$c(j' - j)y^2 + xy\big[1 + b(j' - j^*)\big] = 0\,.$$

We determine the discriminant of the equation

$$D = \big[1 + b(j' - j)\big]^2\,. \tag{21}$$

Consider a tangent, then $D = i^2$, $i \in I$, therefore $1 + b(j' - j^*) = i$, and then

$$j' = \frac{i - 1 + bj^*}{b}\,, \tag{22}$$

and the number $j' \in I$ for which (22) holds is $|I|$. We now have to find the number of distinct points of the form $[1;j;jj']$. It is necessary to count the number of distinct possibilities for the validity of the relation

$$j'j = j''j\,,$$

which implies that $(j'-j'')j = 0 \iff j'-j'' \in Anj$. For the cardinality of the annihilator, we have

$$|Anj| = |Anj_1 g^\alpha| = |Ang^\alpha| = |Ang|^\alpha = |\overline{R}|^\alpha .$$

We now have to find the number of singular elements of the ring R which have an annihilator with the same number of elements. They are elements that differ only by a regular multiple, so

$$rg^\alpha = r'g^\alpha, \qquad r, r' \in R - I .$$

Certainly, $(r - r')g^\alpha = 0$, then $r - r' \in Ang^\alpha$. The number of elements rg^α, $r \in R-I$, in R that have annihilator with the same number of elements is given by

$$\frac{|R| - |I|}{|Ang|^\alpha} = \frac{|R| - |I|}{|\overline{R}|^\alpha} .$$

Then the number of points which are neighbouring with the given point on the conic from which there are tangents to the conic is

$$1 + \sum \frac{|I|}{|\overline{R}|^\alpha} \cdot \frac{|R| - |I|}{|\overline{R}|^\alpha}, \tag{23}$$

where the summand I represents the case $j = 0$. We now obtain from (23)

$$1 + \sum_{\alpha=1}^{\nu-1} \frac{|I|(|R| - |I|)}{|\overline{R}|^2} = 1 + (|R| - |I|)|I| \cdot \frac{1 - |R|^{2\nu-2}}{|\overline{R}|^{2\nu-2}(1 - |\overline{R}|^2)} .$$

□

With reference Theorem 2.4, we will formulate the following statement:

THEOREM 2.5. *For the given conic there are exactly*

1.
$$\frac{|\overline{R}|(|\overline{R}| + 1)}{2} \cdot |I|^2 + (|\overline{R}| + 1) \cdot (s - |I|)$$

external points

and

2.
$$\frac{|\overline{R}|(|\overline{R}| - 1)}{2} \cdot |I|^2 + (|I|^2 - s)(|\overline{R}| + 1)$$

internal points in $H(R)$.

Proof.

1. It follows immediately from Lemma 2.1 that the number of external points not neighbouring with points of the conic is given by the relation

$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2 .$$

Let us consider external points neighbouring with points of the conic. It follows immediately from Theorem 2.4 that there are

$$(|\overline{R}|+1)s-(|\overline{R}|+1)\cdot|I|$$

neighbouring points with points of the conic (besides points of the conic), and consequently, the number of external points is

$$\frac{|\overline{R}|(|\overline{R}|+1)}{2}\cdot|I|^2+(|\overline{R}|+1)s-(|\overline{R}|+1)\cdot|I| .$$

This proves the first part of the theorem.

2. It follows from Lemma 2.1 that the number of internal points not neighbouring with points of the conic is given by the relation

$$\frac{|\overline{R}|(|\overline{R}|-1)}{2}\cdot|I|^2 .$$

Let us now consider those points neighbouring with the conic that are not incident to the tangent. The number of such points is

$$(|I|^2-s)(|\overline{R}|+1)$$

as shown by Theorem 2.4 and the preceding part of the proof. Then the number of internal points is

$$\frac{|\overline{R}|(|\overline{R}|-1)}{2}\cdot|I|^2+(|I|^2-s)(|\overline{R}|+1) ,$$

as claimed. □

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