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CENTERS AND CENTROIDS OF UNICYCLIC GRAPHS

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The terminology used in this note is standard and follows that of Harary [2]. We consider only simple graphs. The *distance* between vertices u and v of a graph G is the smallest number of edges in a u-v path in G and is denoted $\mathrm{dist}_G(u, v)$. The *eccentricity* of a vertex u in G, denoted $\mathrm{e}_G(u)$, is the distance between u and a vertex in G, farthest from u. The subgraph of G induced by the set of vertices with minimum eccentricity is called the *center* of G and is denoted by $C(G)$. The *distance* of a vertex u in G, denoted $\mathrm{m}_G(u)$, is the sum of distances between u and all vertices of G. The subgraph of G induced by the vertices with minimum distance is called the *centroid* of G, and is denoted by $M(G)$. For every vertex u of G and every set S of vertices of G we also define

$$\mathrm{m}_G(u\,;S)=\sum_{v\in S}\mathrm{dist}_G(u, v).$$

The well-known Jordan's theorem identifies centers and centroids of trees.

Theorem 1. (Jordan [4]) *If T is a tree then $C(T)=K_1$ or K_2 and $M(T)=K_1$ or K_2.* ∎

Next results of this type, i.e. characterizing centers or centroids of graphs from a specified class appeared only one hundred years after the theorem of Jordan. Proskurowski [5], [6] characterized centers of maximal outerplanar graphs and 2-trees and Hedetniemi et al. [3] determined centers and centroids of $C_{(n)}$-trees. It is the aim of this paper to find all centers and centroids of unicyclic graphs (a graph is *unicyclic* if it is connected and has exactly one cycle).

Let C be a cycle. By $\mathcal{U}(C)$ we shall denote the class of all unicyclic graphs having C as their cycle, and by $\mathcal{U}\mathcal{I}(C)$ the subclass of $\mathcal{U}(C)$ containing graphs whose every vertex not in C is *pendant*, i.e. has its vertex degree equal to 1.

First we shall state two general results.

Theorem 2. (Harary and Norman [1]) *The center of a connected graph G is contained in a block of G.* ∎

Theorem 3. *The centroid of a connected graph G is contained in a block of G.*

Proof. Suppose that the theorem fails and that G is a counterexample to it. Then its centroid $M(G)$ contains two vertices v_1 and v_2 belonging to different blocks of G, say B_1 and B_2, respectively. Let u be the cutvertex of G which belongs to B_1 and separates v_1 and v_2. Define V_1 to be the set vertices containing u and the vertices of the connected component of G-u which contains v_1 and put $V_2=(V(G)\setminus V_1)\cup\{u\}$. Finally, put $k_i=\mathrm{dist}_G(v_i, u)$, $i=1,2$. It is easy to see that $\mathrm{m}_G(v_i; V_i)>\mathrm{m}_g(u; V_i)-k_i|V_i|$, $i=1,2$. Hence

$$\begin{aligned}\mathrm{m}_G(v_1)&=\mathrm{m}_G(v_1; V_1)+k_1|V_2|+\mathrm{m}_G(u; V_2)\\&>\mathrm{m}_G(u; V_1)-k_1|V_1|+k_1|V_2|+\mathrm{m}_G(u; V_2)\\&=\mathrm{m}_G(u)-k_1(|V_1|-|V_2|),\end{aligned}\tag{1}$$

and analogously

$$\mathrm{m}_G(v_2)>\mathrm{m}_G(u)-k_2(|V_2|-|V_1|).\tag{2}$$

Since $v_i\in M(G)$, $\mathrm{m}_G(u)\geqslant\mathrm{m}_G(v_i)$, $i=1,2$, and consequently (1) and (2) imply $|V_1|>|V_2|$ and $|V_2|>|V_1|$, respectively. This contradiction completes the proof. ∎

For unicyclic graphs Theorems 2 and 3 imply the following corollaries.

Corollary 4. *If $G\in\mathscr{U}(C)$ then $C(G)=K_1$ or K_2, or $C(G)\subseteq C$.* ∎

Corollary 5. *If $G\in\mathscr{U}(C)$ then $M(G)=K_1$ or K_2, or $M(G)\subseteq C$.* ∎

Corollary 6. *If $G\in\mathscr{U}\mathscr{I}(C)$ then $M(G)\subseteq C$.* ∎

We shall now determine all induced subgrahps of a cycle C which are the centers of unicyclic graphs from $\mathscr{U}(C)$. The collection of all such subgraphs will be denoted by $\mathscr{C}(C)$.

Theorem 7. *(a) If $|C|$ is even then $\mathscr{C}(C)$ consists of all induced subgraphs of C.*

(b) If $|C|$ is odd and $H\subseteq C$ then $H\in\mathscr{C}(C)$ if and only if for every path uvw in C, v is in H whenever u and w are in H.

Proof. (a) Let H be an arbitrary induced subgraph of C. Put $A=\{x\in C$: the farthest vertex from x in C is not in $H\}$ and define a unicyclic graph G by adding $|A|$ new vertices x_a, $a\in A$, to C and joining each x_a to a. It is readily verified that $H=C(G)$ (see Figure 1).

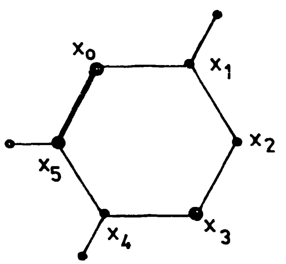

Figure 1. $V(H)=\{x_0, x_3, x_5\}$, $A=\{x_1, x_4, x_5\}$; the center is shown in dark vertices and bold edges.

(b) Suppose first that $H \in \mathscr{C}(C)$ and let G be a unicyclic graph with a cycle C such that $H = C(G)$. Let uvw be a path of C and let u and w be in H. We shall prove that v is in H, too. Let z be a vertex in G, farthest from v and let P be a shortest path between v and z. There are three possibilities.

1. $u \notin P$, $w \notin P$. In this case $\mathrm{dist}_G(u, z) = \mathrm{dist}_G(v, z) + 1$ and, consequently, $\mathrm{e}_G(u) \geq \mathrm{e}_G(v) + 1$. Hence $u \notin C(G) = H$ contrary to the assumption.
2. $u \in P$, $w \notin P$. In this case $\mathrm{dist}_G(w, z) \geq \mathrm{dist}_G(v, z) = \mathrm{e}_G(v)$, which in turn implies that $\mathrm{e}_G(w) \geq \mathrm{e}_G(v)$. Since $w \in C(G)$, it implies that $\mathrm{e}_G(w) = \mathrm{e}_G(v)$ and consequently $v \in C(G)$.
3. $u \notin P$, $w \in P$. This case can be dealt with as the previous one.

Now suppose that for every path uvw of C, $u, w \in H$ implies $v \in H$. For every $x \in C$ let F_x be the set consisting of the two vertices of C which are farthest from x, and let $A = \{x \in C\colon F_x \subseteq V(C) \setminus V(H)\}$. Since for every vertex $u \in V(C) \setminus V(H)$ at least one of its neighbours is also in $V(C) \setminus V(H)$, it follows that

$$V(C) \setminus V(H) = \bigcup_{x \in A} F_x .$$

Hence the graph G obtained by adding $|A|$ new vertices x_a, $a \in A$, to C and joining each x_a to a has clearly H as the center (see Figure 2). ∎

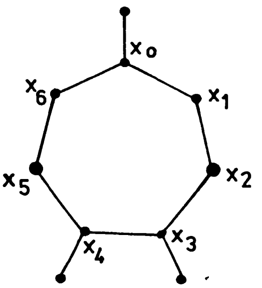

Figure 2. $V(H) = \{x_2, x_5\}$, $A = \{x_0, x_3, x_4\}$; the center is shown in dark vertices.

Now we pass on to centroids. Similarly as in the case of centers we define $\mathscr{M}(C)$ to be the collection of all induced subgraphs of C which are centroids of unicyclic graphs from $\mathscr{U}(C)$.

Theorem 8. *(a) If $|C|$ is odd then $\mathscr{M}(C)$ consists of all induced subgraphs of C.*

(b) If $|C|$ is even and $H \subseteq C$ then $H \in \mathscr{M}(C)$ if and only if either $H = C$ or H contains no pair of antipodic vertices of C, i.e. points at distance $|C|/2$ from each other.

Proof. We begin with some simple observations. Suppose that $C = x_0 x_1 \ldots x_{m-1} x_0$ and $G \in \mathscr{U}(C)$. For $0 \leq i \leq m-1$ let T_i be the set of vertices of the maximal tree in G which contains x_i and no other vertices of C.

(i) Obviously,

$$\mathrm{m}_G(x_i) = \mathrm{m}_G(x_i\,;\, T_i) + \mathrm{m}_G(x_i\,;\, T_{i+1}) + \ldots + \mathrm{m}_G(x_i\,;\, T_{i+m-1})$$

(throughout the proof additions of indices are modulo m) and

$$m_G(x_i\,;\,T_{i+p})=\begin{cases} p|T_{i+p}|+m_G(x_{i+p}\,;\,T_{i+p}) & p\leq m/2\\ (m-p)|T_{i+p}|+m_G(x_{i+p}\,;\,T_{i+p}) & p>m/2.\end{cases}$$

These two facts imply that

$$m_G(x_i)=\begin{cases} c+|T_{i+1}|+2|T_{i+2}|+\ldots+k|T_{i+k}| \\ \quad +(k-1)|T_{i+k+1}|+\ldots+|T_{i+2k-1}| & m=2k\\ c+|T_{i+1}|+2|T_{i+2}|+\ldots+k|T_{i+k}| \\ \quad +k|T_{i+k+1}|+\ldots+|T_{i+2k}| & m=2k+1,\end{cases}$$

where $c=m_G(x_0;\,T_0)+\ldots+m_G(x_{m-1}\,;\,T_{m-1})$.

Now, let $F\in\mathcal{U}(C)$ be a unicyclic graph which differs from G only in the number of pendant vertices at each x_i, $0\leq i\leq m-1$. Denote by b_i the difference between the numbers of pendant vertices adjacent to x_i in F and G, respectively. The following three observations are simple consequences of (i).

(ii) If $m=2k+1$, then for $i=0, 1, \ldots, 2k$

$$m_F(x_i)-m_G(x_i)=b_i+2b_{i+1}+\ldots+(k+1)b_{i+k}+(k+1)b_{i+k+1}+\ldots+2b_{i+2k}.$$

In particular, if $k\geq 2$, $b_j=2p$, $b_{j+1}=-2p$, $b_{j+k}=p$, $b_{j+k+2}=-p$ and all other b_i's are equal to 0 then $m_F(x_j)-m_G(x_j)=-p$, $m_F(x_{j+1})-m_G(x_{j+1})=p$ and $m_F(x_i)-m_G(x_i)=0$, for $i\neq j$, $j+1$.

(iii) If $m=2k$, then for $i=0, 1, \ldots, 2k-1$

$$m_F(x_i)-m_G(x_i)=b_i+2b_{i+1}+\ldots+kb_{i+k-1}+(k+1)b_{i+k}+kb_{i+k+1}+\ldots+2b_{i+2k-1}.$$

In particular, if $k\geq 3$, $b_j=b_{j+k-1}=p$, $b_{j+1}=b_{j+k}=-p$ and all other b_i's are equal to 0, then $m_F(x_j)-m_G(x_j)=-2p$, $m_F(x_{j+k})-m_G(x_{j+k})=2p$ and $m_F(x_i)-m_G(x_i)=0$ for $i\neq j$, $j+k$.

(iv) If $b_0=\ldots=b_{m-1}$, then $m_G(x_i)-m_G(x_j)=m_F(x_i)-m_F(x_j)$ for every $0\leq i$, $j\leq m-1$. In particular, $M(G)=M(F)$.

We are ready to prove the theorem. In the proof we shall use the symbol $m(G)$ to denote the minimum distance of a vertex in G, i.e. $m(G)=\min\{m_g(u)\colon u\in G\}$.

(a) Let $m=2k+1$. If $k=1$, then suitable unicyclic graphs are shown in Figure 3.

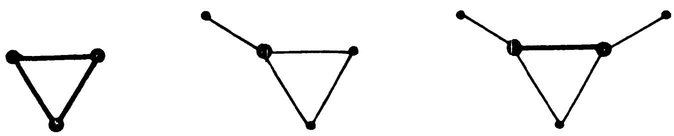

Figure 3. The centroids are indicated by dark vertices and hold lines.

So, assume that $k\geq 2$. Instead of proving (a) we shall prove the following stronger statement: *for every induced subgraph H of C there is $G\in\mathcal{U}\mathcal{I}(C)$ such* *that* $M(G) = H$. We apply induction on $|H|$. If $|H| = 1$, say $V(H) = \{x_i\}$, then the graph G obtained from C by adding a new vertex y_i and joining it to x_i has H as its centroid and, obviously, belongs to $\mathcal{U}\mathcal{I}(C)$. Suppose now, that $|H| = n > 1$ and the assertion holds for subgraphs of order less than n. If $n = 2k + 1$, i.e. if $H = C$, then C is a graph in $\mathcal{U}\mathcal{I}(C)$ having C as its centroid. So, let $n < 2k + 1$ and x_j be a vertex of H such that $x_{j+1} \notin H$. Define $H' = H - x_j$. Then, by the induction hypothesis, there is a graph $G' \in \mathcal{U}\mathcal{I}(C)$ such that $M(G') = H'$. Since $x_j \notin H'$, $\mathrm{m}_{G'}(x_j) > \mathrm{m}(G')$. Put $r = \mathrm{m}_{G'}(x_j) - \mathrm{m}(G')$. Let G'' be a graph obtained from G' by joining to each vertex of C $2r$ new pendant vertices. By (iv) $M(G') = M(G'') = H'$ and $\mathrm{m}_{G''}(x_j) - \mathrm{m}(G\ ) = r$. Let X (resp. Y) be a set consisting of any $2r$ (resp. r) pendant vertices adjacent to x_{j+1} (resp. to x_{j+k+2}) in G . Denote by G the graph obtained from G'' by deleting all edges $x_{j+1}x$, $x \in X$, and $x_{j+k+2}y$, $y \in Y$, and joining the vertices of X to x_j and the vertices of Y to x_{j+k}. It follows from Corollary 5 and (ii) that $M(G) = H$.

(b) Let $m = 2k$. Suppose first that $H \in \mathcal{M}(C)$ and $H \neq C$. Furthermore, suppose that H contains at least one pair of antipodic vertices of C. Since $H \neq C$, it follows that there is i, $0 \leqslant i \leqslant 2k - 1$, such that $x_i \in H$, $x_{i+k} \in H$ and $x_{i+k+1} \notin H$. Let us consider a unicyclic graph $G \in \mathcal{U}(C)$ such that $M(G) = H$. Clearly, $\mathrm{m}_G(x_i) \leqslant \mathrm{m}_G(x_{i+1})$ and $\mathrm{m}_G(x_{i+k}) < \mathrm{m}_g(x_{i+k+1})$. Hence, by (i),

$$|T_{i+1}| + \ldots + |T_{i+k}| \leqslant |T_{i+k+1}| + \ldots + |T_{i+2k}|$$

and

$$|T_{i+k+1}| + \ldots + |T_{i+2k}| < |T_{i+1}| + \ldots + |T_{i+k}|,$$

which yields a contradiction (T_i's are defined as at the beginning of the proof).

Suppose now, that $H \subseteq C$ contains no pair of antipodic vertices of C (the case $H = C$ is trivial). If $k = 2$, then suitable unicyclic graphs are shown in Figure 4.

Figure 4. The centroids are indicated by dark vertices and hold lines.

So, let us suppose that $k \geqslant 3$. As in (a), we shall prove the following stronger statement: *for every* $H \subseteq C$ *which contains no pair of antipodic vertices of* C *there is* $G \in \mathcal{U}\mathcal{I}(C)$ *such that* $M(G) = H$ *and* $\mathrm{m}_G(x_i) - \mathrm{m}_G(x_j)$ *is even for every* $0 \leqslant i$, $j \leqslant m - 1$. The assertion holds if $|H| = 1$, say $V(H) = \{x_i\}$, since the graph G obtained from C by joining to x_i two new vertices y_i and y_i' satisfies it, and the induction step follows from Corollary 6, (iii) and (iv) similarly as in (a). ■

REFERENCES

[1] HARARY, F.—NORMAN, R. Z.: The dissimilarity characteristic of Husimi trees. Ann Math. 58, 1953, 134—141.
[2] HARARY, F.: Graph Theory. Addison-Wesley, Reading, MA, 1969.
[3] MITCHEL HEDETNIEMI, S. L —HEDETNIEMI, S. T.—SLATER, P. J.: Centers and medians of $C_{(n)}$-trees. Utilitas Math. 21C, 1982, 225—235.
[4] JORDAN, C.: Sur les assemblages des lignes. J. Reine Anew. Math. 70, 1869, 185—190.
[5] PROSKUROWSKI, A.: Centers of maximal outerplanar graphs. J. Graph Theory. 4, 1980, 75—79.
[6] PROSKUROWSKI, A.: Centers of 2-trees. Ann. of Discrete Math. 9, 1980, 1—5.

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ЦЕНТРЫ И ЦЕНТРОИДЫ УНИЦИКЛИЧЕСКИХ ГРАФОВ

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Резюме

В настоящей работе характеризуются центры и центроиды унициклических графов.