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NOTE ON THE OSCILLATION OF DIFFERENTIAL EQUATION WITH ADVANCED ARGUMENT

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We want to consider the oscillatory behaviour of solutions of the nonlinear differential equation with advanced argument

$$(1) \qquad y^{(n)}(t)+p(t)f(y(g(t)))=0, \qquad n\geq 2,$$

where:

a) $p(t)$ is continuous and nonnegative on $[t_0, \infty)$;

b) $g(t)$ is a nondecreasing continuous function on $[t_0, \infty)$ and such that $t<g(t)$;

c) $f(u)$ is a continuous function on $(-\infty, \infty)$ such that $uf(u)>0$ for $u\neq 0$.

A solution $y(t)$ of the equation (1) is called oscillatory if it has arbitrarily large zeros, and it is called nonoscillatory otherwise.

We introduce the notation:

$$M_f=\max\left\{\limsup_{y\to\infty}\frac{y}{f(y)}, \limsup_{y\to-\infty}\frac{y}{f(y)}\right\}\geq 0.$$

We restrict our consideration to those solutions $y(t)$ of (1) which exist on some interval $[T_y, \infty)$ and satisfy

$$\sup\{|y(t)|: t_0\leq t<\infty\}>0 \text{ for any } t_0\in[T_y, \infty).$$

Lemma 1 (Kiguradze) [1]. *Let $y(t)$ be a solution of the equation* (1) *satisfying the condition*

$$y(t)> \text{ for } t\in[t_0, \infty)$$

and let $y^{(n)}(t)\leq 0$ for $t\in[t_0, \infty)$.

Then there exist a $t_1\in[t_0, \infty)$ and an integer $l\in\{0, 1, \dots, n\}$ such that $l+n$ is odd and

$$(2_l) \qquad y^{(i)}(t)>0 \text{ for } t\in[t_1, \infty) \qquad (i=0, \dots, l-1),$$

$$(-1)^{i+l}y^{(i)}(t)>0 \text{ for } t\in[t_1, \infty) \qquad (i=l, \dots, n-1).$$

An analogous statement can be made if $y(t)<0$ and $y^{(n)}(t)\geq 0$ for $t\in[t_0, \infty)$.

The next lemma characterizes the oscillatory behaviour of bounded solutions.

Lemma 2. *Suppose that the conditions* a)—c) *are satisfied and, in addition,*

$$\int^{\infty} t^{n-1} p(t)\,\mathrm{d}t = \infty . \tag{3}$$

Then every bounded solution of equation (1) *is oscillatory if n is even, and every bounded solution of equation* (1) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \dots, n-1$, *if n is odd.*

Proof. Let $y(t)$ be a bounded and positive solution of equation (1) on $[t_0, \infty)$. From the equality

$$y^{(j)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(i)}(s) + $$
$$+ \frac{(-1)^{n-j}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} y^{(n)}(u)\,\mathrm{d}u,$$

$s \geq t \geq t_0$, with regard to equation (1) we get

$$y^{(j)}(t) = \sum_{i=j}^{n-1} (-1)^{i-j} \frac{(s-t)^{i-j}}{(i-j)!} y^{(i)}(s) + $$
$$+ \frac{(-1)^{n-j+1}}{(n-j-1)!} \int_t^s (u-t)^{n-j-1} p(u) f(y(g(u)))\,\mathrm{d}u. \tag{4}$$

Let n be even. Since $y(t)$ is a positive and bounded solution of equation (1), in view of Lemma 1 we have $l = 1$ and for $j = 1$, from (4) we get

$$y'(t) \geq \frac{1}{(n-2)!} \int_t^{\infty} (u-t)^{n-2} p(u) f(y(g(u)))\,\mathrm{d}u.$$

Integrating the last inequality from T to t, $t > T \geq t_0$, we obtain

$$y(t) \geq \frac{1}{(n-1)!} \int_T^t (u-T)^{n-1} p(u) f(y(g(u)))\,\mathrm{d}u.$$

Let $y(t) \to c > 0$ as $t \to \infty$. Since $y(t)$ is nondecreasing, $\frac{c}{2} \leq y(t) < c$ for $t \geq t_1 \geq T$. Then there exist positive constants c_1, c_2 such that $c_1 \leq f(y(g(t))) \leq c_2$, $t \geq t_1$. As $t \to \infty$, we have

$$c > \frac{c_1}{(n-1)!} \int_{t_1}^{\infty} (u-T)^{n-1} p(u)\,\mathrm{d}u,$$

which is a contradiction to (3).

Let n be odd. In view of the fact that $y(t)$ is bounded, $l = 0$ and from the equality (4) for $j = 0$ we get

$$y(T)-y(t)\geqslant\frac{1}{(n-1)!}\int_T^t(u-T)^{n-1}p(u)f(y(g(u)))\,du,\quad t\geqslant T\geqslant t_0.$$

Let $y(t)\to L>0$ as $t\to\infty$. Since $y(t)$ is a nonincreasing solution of the equation (1), then $L<y(t)\leqslant 2L$ for $t\geqslant t_1\geqslant T$. Then there exist positive constants L_1, L_2 such that $L_1\leqslant f(y(g(t)))\leqslant L_2$, $t\geqslant t_1$. As $t\to\infty$, we get

$$y(T)>y(T)-L\geqslant\frac{L_1}{(n-1)!}\int_{t_1}^{\infty}(u-T)^{n-1}p(u)\,du,$$

which is a contradiction to (4), so $\lim_{t\to\infty}y(t)=0$. The proof of Lemma 2 is complete.

In this paper the theorems have specific character for differential equations with advanced argument. The assertions of these theorems are not true for the corresponding ordinary differential equations.

Theorem 1. *Suppose that the conditions* a)—c) *are satisfied,* $M_f<\infty$ *and in addition*

$$\limsup_{t\to\infty}\int_t^{g(t)}(s-t)^{n-1}p(s)\,ds>M_f(n-1)!. \tag{5}$$

Then every solution of equation (1) *is oscillatory if n is even, and every solution of equation* (1) *is oscillatory or* $\lim_{t\to\infty}y^{(i)}(t)=0$, $i=0, 1, \ldots, n-1$, *if n is odd.*

Proof. Let $y(t)$ be a nonoscillatory solution of the equation (1). Without loss of generality we may suppose that $y(t)$ is eventually positive on $[t_0, \infty)$.

Suppose that n is even and $l=1$. From (4) with regard to Lemma 1 for $j=1$ we obtain

$$y'(t)\leqslant\frac{1}{(n-2)!}\int_t^{\infty}(u-t)^{n-2}p(u)f(y(g(u)))\,du,\quad t\geqslant t_0.$$

Integration of the last inequality from t to $g(t)$, $t>t_0$, yields

$$y(g(t))\geqslant\frac{1}{(n-1)!}\int_t^{g(t)}(u-t)^{n-1}p(u)f(y(g(u)))\,du. \tag{6}$$

We remind that the condition (5) implies (3). If now $y(t)$ increases to a finite limit as $t\to\infty$, then similarly as in the proof of Lemma 2 we get a contradiction to (3).

Let $y(t)$ increase to infinity as $t\to\infty$. From (6) we get

$$y(g(t))\geqslant\frac{y(g(t))}{(n-1)!}\int_t^{g(t)}(u-t)^{n-1}p(u)\frac{f(y(g(u)))}{y(g(u))}\,du,$$

$$(n-1)!\geqslant\inf_{g(t)\geqslant u\geqslant t}\frac{f(y(g(u)))}{y(g(u))}\int_t^{g(t)}(u-t)^{n-1}p(u)\,du,$$

$$(n-1)! \sup_{y(g(g(t)))\geq z > y(g(t))} \frac{z}{f(z)} \geq \int_t^{g(t)} (u-t)^{n-1} p(u)\, du,$$

$$(n-1)! \limsup_{z\to\infty} \frac{z}{f(z)} \geq \limsup_{t\to\infty} \int_t^{g(t)} (u-t)^{n-1} p(u)\, du,$$

which is a contradiction to the condition (5).

Let n be odd and $l=0$. In view of Lemma 1, from (4) for $j=0$, $t>t_0$, we have

$$y(t_0)-y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^{t} (u-t_0)^{n-1} p(u) f(y(g(u)))\, du.$$

Since $y'(t)\leq 0$ for $t>t_0$, $y(t)$ decreases to limit $L\geq 0$ as $t\to\infty$. Let $L>0$. Then similarly as in the proof of Lemma 2 we get a contradiction to (3), so $\lim_{t\to\infty} y(t)=0$.

Let $l\in\{2, \ldots, n-1\}$. With regard to Lemma 1 from (4) for $j=l$, $t>t_0$, we have

$$y^{(l)}(t) \geq \frac{1}{(n-l-1)!} \int_t^{\infty} (u-t)^{n-l-1} p(u) f(y(g(u)))\, du.$$

By integrating the last inequality from t_0 to t, $t>t_0$, we obtain

$$y^{(l-1)}(t) \geq \frac{(t-t_0)^{n-l}}{(n-l)!} \int_t^{\infty} p(u) f(y(g(u)))\, du.$$

Repeating this procedure we get

$$y'(t) \geq \frac{(t-t_0)^{n-2}}{(n-2)!} \int_t^{\infty} p(u) f(y(g(u)))\, du.$$

We integrate last inequality from t to $g(t)$, $t>t_0$,

$$y(g(t)) \geq \frac{1}{(n-2)!} \int_t^{g(t)} p(u) f(y(g(u))) \int_t^{u} (s-t_0)^{n-2}\, ds\, du,$$

$$y(g(t)) \geq \frac{1}{(n-1)!} \int_t^{g(t)} (u-t)^{n-1} p(u) f(y(g(u)))\, du,$$

which is the inequality (6). The proof now proceeds as above, when $y(t)$ increases to infinity. This completes the proof.

Corollary 1. *We consider the differential equation*

$$y^{(n)}(t)+p(t)y(g(t))=0. \tag{7}$$

Suppose that the conditions a), b) *are satisfied and in addition*

$$\limsup_{t\to\infty} \int_t^{g(t)} (s-t)^{n-1} p(s)\, ds > (n-1)!. \tag{8}$$

Then every solution of the equation (7) *is oscillatory if n is even, and every solution of the equation* (7) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t)=0$, $i=0, 1, \ldots, n-1$, *if n is odd.*

It can occur that the ordinary differential equation has a nonoscillatory solution, but if the corresponding differential equation with advanced argument has a solution, then this solution is oscillatory.

Example 1. The ordinary differential equation

$$y''(t)+\frac{1}{4t^2}\,y(t)=0, \quad t>0,$$

has a nonoscillatory solution $y(t)=t^{\frac{1}{2}}$, but the corresponding differential equation with advanced argument

$$y''(t)+\frac{1}{4t^2}\,y(149t)=0, \quad t>0,$$

in view of the condition (8), has every solution oscillatory.

Theorem 2. *Suppose that the conditions* a)—c) *are satisfied,* $M_f<\infty$ *and in addition*

$$\text{(9)} \qquad \limsup_{t\to\infty} \int_t^{g(t)} \int_s^{\infty} (u-s)^{n-2} p(u)du \, \mathrm{d}s > M_f(n-2)!\,.$$

Then the equation (1) *has no solution satisfying* (2_l), *and* $\lim_{t\to\infty} y^{(i)}(t)=0$, $i=0, 1, \ldots, n-1$, *for every solution of the equation* (1) *which satisfies* (2_0).

Proof. Let $y(t)$ be a positive solution of the equation (1) on $[t_0, \infty)$. Let $l=1$. Then n is even and from (4) for $j=1$ we get

$$\text{(10)} \qquad y'(t) \geq \frac{1}{(n-2)!} \int_t^{\infty} (u-t)^{n-2} p(u) f(y(g(u)))\, \mathrm{d}u.$$

Integrating from t to $g(t)$, $t>t_0$, we obtain

$$\text{(11)} \qquad y(g(t)) \geq \frac{1}{(n-2)!} \int_t^{g(t)} \int_s^{\infty} (u-s)^{n-2}\, p(u) f(y(g(u)))du \, \mathrm{d}S.$$

We remind that the condition (9) implies (3). Otherwise if

$$\int^{\infty} t^{n-1} p(t)\, \mathrm{d}t < \infty,$$

then

$$0 < \limsup_{t\to\infty} \int_t^{g(t)} \int_s^{\infty} (u-s)^{n-2} p(u)du \, \mathrm{d}s \leq$$

$$\leq \limsup_{t\to\infty} \int_t^\infty \int_s^\infty (u-s)^{n-2} p(u)\,du\,ds = \limsup_{t\to\infty} \frac{1}{n-1} \int_t^\infty (u-t)^{n-1} p(u)\,du \leq$$

$$\leq \limsup_{t\to\infty} \frac{1}{n-1} \int_t^\infty (u-t_0)^{n-1} p(u)\,du = 0,$$

which is a contradiction.

Let $y(t)$ increase to a finite limit as $t\to\infty$. We integrate (10) from t_0 to t,

$$y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^t (u-t_0)^{n-1} p(u) f(y(g(u)))\,du.$$

Similarly as in the proof of Lemma 2 we get a contradiction to (3).

Let $y(t)$ increase to infinity as $t\to\infty$. From (11) we get

$$(n-2)! \sup_{z\geq y(g(t))} \frac{z}{f(z)} \geq \int_t^{g(t)} \int_s^\infty (u-s)^{n-2} p(u)\,du\,ds,$$

$$(n-2)! \limsup_{z\to\infty} \frac{z}{f(z)} \geq \limsup_{t\to\infty} \int_t^{g(t)} \int_s^\infty (u-s)^{n-2} p(u)\,du\,ds,$$

which is a contradiction to condition (9).

Let $l=0$. Then n is odd and from (4) for $j=0$ we have

$$y(t_0) - y(t) \geq \frac{1}{(n-1)!} \int_{t_0}^t (u-t_0)^{n-1} p(u) f(y(g(u)))\,du.$$

Let $\lim_{t\to\infty} y(t) = L > 0$. In view of the fact that the condition (9) implies (3), similarly as in the proof of Lemma 2 we get a contradiction to (3). So $\lim_{t\to\infty} y(t) = 0$.

Corollary 2. *We consider the differential equation*

$$y''(t) + p(t) f(y(g(t))) = 0.$$

Suppose that the conditions a)—c) *are satisfied,* $M_f < \infty$ *and in addition*

$$\limsup_{t\to\infty} \int_t^{g(t)} \int_s^\infty p(u)\,du\,ds > M_f. \tag{12}$$

Then every solution of this equation is oscillatory.

Example 2. We cannot decide about the oscillatory character of solutions of the differential equation with advanced argument

$$y''(t) + \frac{1}{4t^2} y(81t) = 0, \quad t > 0,$$

with regard to the condition (8). But in view of condition (12) every solution of this equation is oscillatory.

Theorem 3. *Suppose that the conditions* a)—c) *are satisfied,* $M_f < \infty$ *and in addition*

$$(13)\qquad \limsup_{t\to\infty} \int_t^{g(t)} (s-t)s^{n-2}p(s)\,ds > M_f(n-1)!.$$

Then the equation (1) *has no solution satisfying* (2_l), $l \in \{2, \ldots, n-1\}$.

Proof. Let $y(t)$ be a positive solution of the equation (1) on $[t_0, \infty)$ which satisfies (2_l), $l \in \{2, \ldots, n-1\}$. Similarly as in the proof of Theorem 1 we get

$$y'(t) \geq \frac{(t-t_0)^{n-2}}{(n-2)!} \int_t^{\infty} p(u)f(y(g(u)))\,du.$$

We integrate the last inequality from t to $g(t)$, $t > t_0$,

$$y(g(t)) \geq \frac{1}{(n-2)!} \int_t^{g(t)} p(u)f(y(g(u))) \int_t^u (s-t_0)^{n-2}\,ds\,du,$$

$$y(g(t)) \geq \frac{1}{(n-1)!} \int_t^{g(t)} (u-t)(u-t_0)^{n-2}p(u)f(y(g(u)))\,du.$$

From the last inequality we have

$$(n-1)!\limsup_{z\to\infty} \frac{z}{f(z)} \geq \limsup_{t\to\infty} \int_t^{g(t)} (u-t)(u-t_0)^{n-2}p(u)\,du,$$

which is a contradiction to (13). The proof is complete.

Theorem 4. *Suppose that the conditions* a)—c), (9), (13), $M_f < \infty$ *are satisfied. Then every solution of the equation* (1) *is oscillatory if n is even, and every solution of the equation* (1) *is oscillatory or* $\lim_{t\to\infty} y^{(i)}(t) = 0$, $i = 0, 1, \ldots, n-1$, *if n is odd.*

The proof follows from the Theorems 2, 3.

The above results are new. The sufficient condition [2, Th. 8.4] which guarantees that every solution of the equation from the example 2 is oscillatory

$$\int_0^{\infty} \beta_0^{1-\varepsilon}(t)p(t)\,dt = \infty, \quad \varepsilon > 0, \quad \beta_0(t) = \min\{t, g(t)\},$$

is not satisfied. But the condition (12) is satisfied.

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ЗАМЕТКА О КОЛЕБЛЕМОСТИ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ С ОПЕРЕЖАЮЩИМ АРГУМЕНТОМ

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Резюме

В работе приведены достаточные условия для того, чтобы каждое решение уравнения (1) при четном n являлось колеблющимся, а при нечетном n, либо колеблющимся, либо удовлетворяло условию

$$\lim_{t\to\infty} y^{(i)}(t) = 0, \quad i = 0, \ldots, n-1.$$