Peter Vojtáš

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A NOTE ON THE EFFECTIVENESS OF TESTS FOR THE ABSOLUTE CONVERGENCE AND DIVERGENCE OF INFINITE SERIES

PETER VOJTÁŠ

ABSTRACT. We prove that for every $b \in \ell^1$ the set of all series which fit the comparison test for absolute convergence with b as parameter is dense and of first category in ℓ^1. We consider the analogous question for tests of absolute divergence in both ℓ^∞ and ℓ^2.

Most of the known tests for the absolute convergence of infinite series are special cases of the comparison test (see e. g. [2]). We show that for every $b \in \ell^1$ the set of all series which fit the comparison test with b as parameter is dense and of first category in ℓ^1 (with the usual norm). This generalizes the result of J. Belasová, J. Ewert and T. Šalát ([1]), who proved the corresponding result for the d'Alembert test (i. e. the ratio test), the Cauchy test (i. e. the root test) and the Raabe ratio test. We consider the analogous question for tests concerning absolute divergence for spaces ℓ^∞ and ℓ^2. We mention other possibilities of measuring the effectiveness of tests.

1. Notation

We denote the infinite series of real numbers $a = \{a_n\}$; $\limsup$ and $\liminf$ are understood "when n tends to infinity". We consider spaces ℓ^p for $1 \leqq p < \infty$ with the usual norm $\|a\|_p = (\sum |a_n|^p)^{\frac{1}{p}}$ and the space ℓ^∞ with the norm $\|a\|_\infty = \sup |a_n|$; $K_p(a,\varepsilon)$ denotes the open ball in ℓ^p with the centre a and the radius ε. In what follows TC (TD respectively) stands for the test for (absolute) convergence (divergence respectively). We denote the sets of series

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decided by appropriate tests as follows:

$$RatioTC = \left\{a\colon \limsup\left|\frac{a_{n+1}}{a_n}\right| < 1\right\}$$
$$RootTC = \left\{a\colon \limsup \sqrt[n]{|a_n|} < 1\right\}$$
$$RaabeTC = \left\{a\colon \liminf n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) > 1\right\}$$

and for $b \in \ell^1$

$$CompTC(b) = \left\{a\colon (\exists n_0)(\forall n \geqq n_0)(|a_n| < |b_n|)\right\}.$$

Similarly

$$RatioTD = \left\{a\colon \liminf\left|\frac{a_{n+1}}{a_n}\right| > 1\right\}$$
$$RootTD = \left\{a\colon \liminf \sqrt[n]{|a_n|} > 1\right\}$$
$$RaabeTD = \left\{a\colon \limsup n\left(\left|\frac{a_n}{a_{n+1}}\right| - 1\right) < 1\right\}$$

and for $b \in \ell^\infty - \ell^1$

$$CompTD(b) = \left\{a\colon (\exists n_0)(\forall n \geqq n_0)(|b_n| < |a_n|)\right\}.$$

2. Tests for absolute convergence

To motivate our main theorem, first we discuss the relationship of some tests to the comparison test. Let us mention that for every $a \in RatioTC$ there is a q, $0 < q < 1$, such that $|a_n| = o(q^n)$ (i.e. a is asymptotically smaller than the very geometric series); $a \in RootTC$ if and only if there is q, $0 < q < 1$, such that $a_n = o(q^n)$. Moreover, $a \in RaabeTC$ implies that there is $\alpha > 0$ such that $a_n = o\left(\frac{1}{n^{1+\alpha}}\right)$ (i. e. a is asymptotically smaller than the very harmonic series). So for $a \in RootTC \cup RaabeTC$ we have always $a_n = o\left(\frac{1}{n\log^2 n}\right)$. As $a_n = o\left(\frac{1}{n\log^2 n}\right)$ implies $a \in CompTC\left(\frac{1}{n\log^2 n}\right)$ and the series $\left\{\frac{1}{n\log^2 n}\right\}_{n=1}^{\infty}$ converges, the following theorem covers the result in [1].

THEOREM 1. *For every $b \in \ell^1$, the set $CompTC(b)$ is dense and of first category in ℓ^1.*

Proof. Denote $X(b,m) = \{a\colon (\forall i > m)(|a_i| < |b_i|)\}$. As $CompTC(b)$ is equal to $\bigcup_{m=0}^{\infty} X(b,m)$, it is enough to prove that $X(b,m)$ is meagre for every m. Hence, it is enough to show that

$$(\forall c \in \ell^1)(\forall \varepsilon > 0)\big(\exists\, d \in K(c,\varepsilon)\big)(\exists\, \delta > 0)\big(K(d,\delta) \cap X(b,m) = \emptyset\big).$$

Take c and ε arbitrary and put

$$k = \min\Big\{j\colon j > m \;\&\; (\forall i \geqq j)\Big(|b_i| < \frac{\varepsilon}{3}\Big)\Big\}.$$

Define $d_i = c_i$ for $i \neq k$

$$d_k = c_k + 2\,\mathrm{sgn}(c_k)\frac{\varepsilon}{3} \qquad \text{if} \quad c_k \neq 0$$

$$d_k = 2\frac{\varepsilon}{3} \qquad \text{if} \quad c_k = 0.$$

Then $\|d - c\| = 2\frac{\varepsilon}{3}$. Take $\delta = \frac{\varepsilon}{3}$ and suppose, on the contrary, that there is an $a \in K(d,\delta) \cap X(b,m)$. As $|a_k - d_k| \leqq \|a - d\| < \frac{\varepsilon}{3}$, we have $|a_k| > \frac{\varepsilon}{3}$. But $k > m$ and $|b_k| < \frac{\varepsilon}{3}$, a contradiction.

To prove the density, just notice that the very set contains the set of all sequences which are eventually equal to zero and this set is known to be dense in each ℓ^p for $0 < p < \infty$.

Consequently the set of all series determined by any reasonable test of absolute convergence in an explicit form (which is a special case of the comparison test) is a dense set of first category in ℓ^1 (e. g. tests of d'Alembert, Cauchy, Raabe, Kummer (with parameter), Bertrand, Gauss, in fact all tests in the hierarchies from [4] and [3]), see e. g. [2].

3. Tests for absolute divergence

We turn our attention to the tests for the absolute divergence of infinite series. In this case it is not immediately clear according to which space we should consider the effectiveness of tests. It turns out that natural and typical candidates are ℓ^∞ and, say ℓ^2.

As

$$Ratio\,TD \cap \ell^\infty = Root\,TD \cap \ell^\infty = \emptyset$$

and for all $a \in Raabe\,TD$ we have, e.g., $\dfrac{1}{n \log n} = o(a_n)$, so it suffices to discuss the strength of the comparison test in ℓ^∞ and ℓ^2.

THEOREM 2. *For every $b \in c_0 \setminus \ell^1$ the set $CompTD(b)$ is residual in ℓ^∞.*

Proof. Notice that the set of all $a \in \ell^\infty$ for which 0 is not an accumulation point of a is open and dense in ℓ^∞ and is contained in $CompTD(b)$.

The set $\ell^2 \setminus \ell^1$ is residual in ℓ^2 and, similarly as in the Section 2, the following theorem covers all reasonable tests for divergence which are a special case of the comparison test (see e. g. [2]).

THEOREM 3. *For every $b \in \ell^2 \setminus \ell^1$, the set $CompTD(b)$ is dense and of first category in ℓ^2.*

Proof. Denote $Y(b,m) = \{a \in \ell^2 \colon (\forall i > m)(|b_i| < |a_i|)\}$. We show that $Y(b,m)$ is always meagre. Similarly as in the proof of Theorem 1., take $c \in \ell^2$ and $\varepsilon > 0$ arbitrary and put

$$k = \min\left\{i \colon i > m,\ b_i \neq 0,\ |c_i| < \frac{\varepsilon}{3}\right\}.$$

Define $d_i = c_i$ for $i \neq k$ and $d_k = 0$. Then $\|d - c\|_2 = |c_k| < \frac{\varepsilon}{3}$ and take $\delta = \frac{|b_k|}{3} > 0$. Suppose that there is an a such that $a \in K_2(d,\delta) \cap Y(b,m)$. But

$$|a_k| = |a_k - d_k| \leqq \|a - d\|_2 < \delta < |b_k|,$$

a contradiction.

4. Discussion

Previous results show that the question of effectiveness of tests for absolute convergence and divergence in terms of category in the usual topology of ℓ^p is probably not adequate. We just mention several further possibilities: consider the effectiveness of tests according to measure, set-theoretic cardinal characteristics and/or the complexity in the hierarchy of projective sets.

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Mathematical Institute
Slovak Academy of Sciences
Jesenná 5
041 54 Košice
Czechoslovakia
e-mail: vojtas@csearn.bitnet