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ON φ-CONVERGENCE AND φ-DENSITY

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ABSTRACT. We study φ-convergence (a special type of summability method introduced in [SCHOENBERG, I. J.: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361 375]), φ-density of subsets of integers (which is equivalent to the φ-convergence of the set's indicator function) and $\mathfrak{I}_\varphi$-convergence (convergence according to the ideal of all sets with φ-density zero in the sense as defined in [KOSTYRKO, P. ŠALÁT, T. WILCIŃSKY, W.: *I-convergence*, Real Anal. Exchange **26** (2000-01), 669 686]). We analyze the relation of φ-density and other types od densities, in particular asymptotic, logarithmic, and uniform density. We prove the following properties:

- φ-density can attain only values 0 and 1 (whenever it exists).
- If φ-density exists for a set, then asymptotic and logarithmic densities also exist and attain the same value.
- There is a set with φ-density zero which does not have uniform density.
- There is a sequence which is φ-convergent, but the sequence of its absolute values is not.
- $\mathfrak{I}_\varphi$-convergence is strictly weaker than φ-convergence.

1. Introduction

In [12], Schoenberg, motivated by studying integrability of generalizations of the Dirichlet function introduced a special type of summability method, called φ-convergence. According to his definition, a sequence $(x_n)_{n=1}^{\infty}$ of real numbers φ-converges to $\xi \in \mathbb{R}$ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{d|n} \varphi(d) x_d = \xi ,$$

where φ denotes the Euler function. Schoenberg also studied its basic properties and showed that φ-convergence is weaker than classical convergence and

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stronger than statistical convergence. In addition, E r d ő s in [3] provided an example of a sequence which is not convergent, but is φ-convergent.

In this paper we use φ-convergence to define φ-density of sets of positive integers as a φ-limit of the set's indicator function. We study the relation of φ-density to other types of densities, in particular asymptotic density, logarithmic density and uniform density. We show that φ-density can attain only values 0 and 1, and its existence implies also the existence of asymptotic density and logarithmic density, but not the existence of uniform density. Later we analyze the ideal generated by all sets with φ-density zero and study the convergence according to this ideal in the sense introduced in [6]. We show that this type of convergence is strictly weaker than φ-convergence.

The rest of the paper is organized as follows. In Section 2 we recall some well-known facts and notations. In Section 3 we illustrate the general approach how to create densities and ideals using infinite matrices. In Section 4 we introduce S h o e n b e r g's φ-convergence and study its basic properties. In Section 5 we introduce φ-density. Sections 6 and 7 contain the main results of this paper. In Section 6 we analyze the relation of φ-density to other types of densities. In Section 7 we study the ideal of all sets with φ-density zero and the convergence according to this ideal.

2. Definitions, notation, and preliminaries

Recall some well-known facts and notations.

2.1. Asymptotic density.

Let $A \subseteq \mathbb{N}$. If $m, n \in \mathbb{R}$, then $A(m,n)$ denotes the number of elements of set $A \cap [m,n]$. Then we define

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{A(1,n)}{n}, \qquad \overline{d}(A) = \limsup_{n\to\infty} \frac{A(1,n)}{n}$$

and the numbers $\underline{d}(A)$, resp. $\overline{d}(A)$ we call the *lower*, resp. *upper asymptotic density* of the set A. If, in addition, $\underline{d}(A) = \overline{d}(A) = d(A)$, then we say that the number

$$d(A) = \lim_{n\to\infty} \frac{A(1,n)}{n} \tag{2.1}$$

is the *asymptotic density* of the set A.

2.2. Logarithmic density.

For $A \subseteq \mathbb{N}$ we define

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{1}{\ln n} \sum_{a\in A,\, a\le n} \frac{1}{a}, \qquad \overline{\delta}(A) = \limsup_{n\to\infty} \frac{1}{\ln n} \sum_{a\in A,\, a\le n} \frac{1}{a},$$

and the numbers $\underline{\delta}(A)$, resp. $\overline{\delta}(A)$ we call the *lower*, resp. *upper logarithmic density* of the set A. If, in addition $\underline{\delta}(A) = \overline{\delta}(A) = \delta(A)$, then we say that the number

$$\delta(A) = \lim_{n\to\infty} \frac{1}{\ln n} \sum_{a\in A,\, a\le n} \frac{1}{a},$$

is the *logarithmic density* of the set A.

2.3. Uniform density.

Another type of density we focus on, is the uniform density, introduced in [1]. For $j \in \mathbb{N}$ denote

$$\alpha_j = \min_{m\ge 0} A(m+1, m+j), \qquad \alpha^j = \max_{m\ge 0} A(m+1, m+j), \tag{2.2}$$

where maximum and minimum are taken for $m \in \mathbb{Z}$, $m \ge 0$. In [1] it is shown that the numbers

$$\underline{u}(A) = \lim_{j\to\infty} \frac{\alpha_j}{j}, \qquad \text{resp.} \qquad \overline{u}(A) = \lim_{j\to\infty} \frac{\alpha^j}{j}$$

exist. We call them the *lower*, resp. *upper uniform density* of the set A. If, in addition, $\underline{u}(A) = \overline{u}(A) = u(A)$, then we say that the number $u(A)$ is the *uniform density* of the set A.

2.4. Relations between densities.

It is well known that for an arbitrary set $A \subseteq \mathbb{N}$ the inequalities

$$0 \le \underline{u}(A) \le \underline{d}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{d}(A) \le \overline{u}(A) \le 1 \tag{2.3}$$

hold. Therefore, if $u(A)$ exists, so does also $d(A)$ and $d(A) = u(A)$; if $d(A)$ exists, so does $\delta(A)$ and $\delta(A) = d(A)$. In particular, if $u(A) = 0$, then also $d(A) = 0$; if $d(A) = 0$, then also $\delta(A) = 0$. It is also well known that the converses do not hold. As an example of a set which has asymptotic density but does not have uniform density we can consider the set

$$B = \bigcup_{k=1}^{\infty} \{10^k+1, 10^k+2, \dots, 10^k+k\} \tag{2.4}$$

for which $d(B) = 0$, but $\underline{u}(B) = 0$, $\overline{u}(B) = 1$.

2.5. Properties of the set of all prime numbers.

Let
$$\mathbb{P} = \{p_1 < p_2 < \cdots < p_k < \dots\}$$
be the set of all prime numbers. We will use this notation further. It is well known that $\sum_{p\in\mathbb{P}} p^{-1} = +\infty$ and the series $\sum_{p\in\mathbb{P}} p^{-2}$ converges. This implies
$$\prod_{p\in\mathbb{P}}\Big(1-\frac{1}{p}\Big) = 0\,, \qquad 0 < \prod_{p\in\mathbb{P}}\Big(1-\frac{1}{p^2}\Big) < 1\,. \tag{2.5}$$
It is also well known that the latter product is equal to $6/\pi^2$. In addition,
$$\delta(\mathbb{P}) = d(\mathbb{P}) = u(\mathbb{P}) = 0\,. \tag{2.6}$$
A proof of the equality $d(\mathbb{P}) = 0$ directly follows from the prime number theorem (see [9; p. 217]). This directly implies $\delta(\mathbb{P}) = 0$. A proof of $u(\mathbb{P}) = 0$ can be found in [1].

2.6. Euler function φ.

If n is a positive integer, then $\varphi(n)$ denotes the number of elements from $\{1, 2, \dots, n\}$ coprime to n. It is well known that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$ is the prime number decomposition of $n > 1$, then
$$\varphi(n) = n\Big(1-\frac{1}{p_1}\Big)\Big(1-\frac{1}{p_2}\Big)\dots\Big(1-\frac{1}{p_m}\Big) \tag{2.7}$$
and $\varphi(1) = 1$. It is also well known that the function φ is multiplicative, i.e., if $n_1, n_2 \in \mathbb{N}$ are coprime, then $\varphi(n_1 n_2) = \varphi(n_1)\varphi(n_2)$. Another important property of the Euler function is the equality
$$n = \sum_{d|n} \varphi(d)\,, \tag{2.8}$$
also known as the *Gauss Theorem*. See, for example, [9; pp. 48 50] for further details.

2.7. Mőbius function μ.

For any positive integer n define
$$\mu(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ 0 & \text{if } p^2 \mid n \text{ for some prime number } p\,, \\ (-1)^r & \text{if } n = q_1 q_2 \cdots q_r\,, \text{ where } q_1, q_2, \dots, q_r \\ & \text{are pairwise different prime numbers.} \end{cases}$$
Obviously, the function μ is multiplicative and
$$\sum_{d|n} d\mu(d) = (1-q_1)\cdots(1-q_r)\,, \tag{2.9}$$
where $n = q_1^{\alpha_1} \cdots q_r^{\alpha_r}$ is the prime number decomposition of n. See, for example, [9; pp. 111–113] for further details.

2.8. Statistical and uniform statistical convergence.

Using asymptotic and uniform density, we can define two types of convergence: statistical and uniform statistical convergence. The concept of statistical convergence was introduced in [12] and [4], and developed in [2] and [11].

Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\xi \in \mathbb{R}$. We say that this sequence *statistically* (resp. *uniformly statistically*) *converges* to the the number ξ if $d(A_\varepsilon) = 0$ (resp. $u(A_\varepsilon) = 0$) for every $\varepsilon > 0$, where $A_\varepsilon = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$. Then we also say that the sequence $(x_n)_{n=1}^{\infty}$ $\mathfrak{I}_d$*-converges* (resp. $\mathfrak{I}_u$*-converges*)[1] to the number ξ, and we write $\mathfrak{I}_d\text{-}\lim x_n = \xi$ (resp. $\mathfrak{I}_u\text{-}\lim x_n = \xi$).[2] The number ξ we call the *statistical* (resp. the *uniform statistical*) *limit* of the sequence $(x_n)_{n=1}^{\infty}$.

From (2.3) it is obvious that if the sequence $(x_n)_{n=1}^{\infty}$ uniformly statistically converges to the number ξ, then it converges to ξ also statistically. The converse is not true; consider the sequence $(x_n)_{n=1}^{\infty}$ such that $x_n = \chi_B(n)$ for all $n = 1, 2, \dots$, where B is the set from (2.4) and χ_B denotes the characteristic function (indicator) of the set B.

2.9. Ideals.

A more general type of convergence covering also statistical and uniform statistical convergence can be obtained using ideals on the set of positive integers. We say that a non-empty family of sets $\mathfrak{I} \subseteq 2^{\mathbb{N}}$ is an *ideal* if it has the following two properties:

(1) *Heredity*: If $B \in \mathfrak{I}$ and $A \subseteq B$, then $A \in \mathfrak{I}$.
(2) *Additivity*: If $A, B \in \mathfrak{I}$, then $A \cup B \in \mathfrak{I}$.

We call the ideal $\mathfrak{I}$ *non-trivial* if $\mathbb{N} \notin \mathfrak{I}$. Moreover, we say that the ideal $\mathfrak{I}$ is *admissible* if it is non-trivial and it contains all finite subsets of $\mathbb{N}$.

In the following we provide some examples of ideals:

(1) If $A \subseteq \mathbb{N}$, then 2^A is an ideal. For $A \neq \mathbb{N}$ it is non-trivial, but it is not admissible. In particular, for $A = \emptyset$ we have an ideal $2^A = 2^\emptyset = \{\emptyset\}$.
(2) The family of all finite subsets of the set $\mathbb{N}$ is an admissible ideal. We denote it $\mathfrak{I}_f$. It is obvious that $\mathfrak{I}_f \subseteq \mathfrak{I}$ for every admissible ideal $\mathfrak{I}$.
(3) The family of all sets with asymptotic density (resp. uniform density) zero is an admissible ideal; we denote it $\mathfrak{I}_d$ (resp. $\mathfrak{I}_u$). Using the above results, it is obvious that $\mathfrak{I}_u \subseteq \mathfrak{I}_d$. In addition, the set B from (2.4) is an element of $\mathfrak{I}_d \setminus \mathfrak{I}_u$, so $\mathfrak{I}_u$ is proper subset of $\mathfrak{I}_d$.

[1] Similarly we can define $\mathfrak{I}_\delta$-convergence using logarithmic density.
[2] In literature, the notation $\lim \operatorname{stat} x_n = \xi$ is more common than $\mathfrak{I}_d\text{-}\lim x_n = \xi$.

In the above examples we mentioned also some relations between ideals. In summary, we get

$$\mathfrak{I}_f \subseteq \mathfrak{I}_u \subseteq \mathfrak{I}_d \subseteq 2^{\mathbb{N}}. \tag{2.10}$$

Based on the above, we know that all inclusions are strict.

2.10. $\mathfrak{I}$-convergence.

Let $\mathfrak{I}$ be an ideal, $(x_n)_{n=1}^{\infty}$ a sequence of real numbers and $\xi \in \mathbb{R}$. We say that this sequence $\mathfrak{I}$-*converges* to the number ξ if

$$A_\varepsilon = \{n \in \mathbb{N} : \ |x_n - \xi| \geq \varepsilon\} \in \mathfrak{I}$$

for every $\varepsilon > 0$. The number ξ is then called the $\mathfrak{I}$-*limit* of the sequence $(x_n)_{n=1}^{\infty}$ and we write $\mathfrak{I}\text{-}\lim x_n = \xi$.

The notion of $\mathfrak{I}$-convergence was introduced in [6], where its basic properties are also proved. For our purposes, it is enough to know that if $\mathfrak{I}$ is admissible ideal, then $\mathfrak{I}$-limit is unique. Moreover, if $\mathfrak{I}_1$, $\mathfrak{I}_2$ are two admissible ideals such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$, then $\mathfrak{I}_1\text{-}\lim x_n = \xi$ implies $\mathfrak{I}_2\text{-}\lim x_n = \xi$. Since every admissible ideal $\mathfrak{I}$ contains the ideal $\mathfrak{I}_f$, then $\lim\limits_{n\to\infty} x_n = \xi$ implies[3] $\mathfrak{I}\text{-}\lim x_n = \xi$. See [6] for more details.

If we consider ideals $\mathfrak{I}_d$, resp. $\mathfrak{I}_u$, then $\mathfrak{I}_d$-convergence, resp. $\mathfrak{I}_u$-convergence is equivalent to statistical, resp. uniform statistical convergence. This is reflected also by the notation.

Now we state and prove a proposition about $\mathfrak{I}$-convergence of sequences of zeros and ones, which we will use later.

PROPOSITION 2.1. *Let $\mathfrak{I}$ be an admissible ideal and $\xi \in \mathbb{R}$. If $(x_n)_{n=1}^{\infty}$ is a sequence of zeros and ones such that $\mathfrak{I}\text{-}\lim x_n = \xi$, then $\xi \in \{0,1\}$.*

Proof. Using the assumption, we have $A_\varepsilon = \{n \in \mathbb{N} : \ |x_n - \xi| \geq \varepsilon\} \in \mathfrak{I}$ for each $\varepsilon > 0$. If $\xi \notin \{0,1\}$, choose $\varepsilon = \min\{\frac{1}{2}|\xi|, \frac{1}{2}|1-\xi|\} > 0$. If $x_n = 0$, then $|x_n - \xi| = |\xi| > \frac{1}{2}|\xi| \geq \varepsilon$, so $n \in A_\varepsilon$. If $x_n = 1$, then $|x_n - \xi| = |1-\xi| > \frac{1}{2}|1-\xi| \geq \varepsilon$, and so $n \in A_\varepsilon$. We obtained $A_\varepsilon = \mathbb{N}$, which means that $\mathbb{N} \in \mathfrak{I}$. However, this is a contradiction with the non-triviality of the ideal $\mathfrak{I}$. □

2.11. Infinite matrices and summability methods.

Recall the notion of matrix summability methods as described in [10] and [5]. Let

$$\mathbf{T} = (t_{nk})_{n,k=1}^{\infty} = \begin{pmatrix} t_{11} & t_{12} & \dots & t_{1k} & \dots \\ t_{21} & t_{22} & \dots & t_{2k} & \dots \\ \vdots & \vdots & \ddots & \vdots & \\ t_{n1} & t_{n2} & \dots & t_{nk} & \dots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix}, \tag{2.11}$$

[3] This is based on the fact that $\mathfrak{I}_f$-convergence is equivalent to classical convergence.

be an infinite matrix whose elements are real numbers. Further, we will simply denote a sequence $(x_n)_{n=1}^{\infty}$ as $\mathbf{x}$, i.e., $\mathbf{x} = (x_n)_{n=1}^{\infty}$. For $n = 1, 2, \dots$ construct the following series

$$y_n = \sum_{k=1}^{\infty} t_{nk} x_k \,. \tag{2.12}$$

Whenever all these series converge, we obtain a new sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ and we write $\mathbf{y} = \mathbf{T}\mathbf{x}$. If, in addition, $\lim_{n\to\infty} y_n = \xi$, then we say that the sequence $\mathbf{x}$ is $\mathbf{T}$-*summable* (*summable by matrix* $\mathbf{T}$) to the number ξ. Then, the number ξ is called $\mathbf{T}$-*limit* of the sequence $\mathbf{x}$; write $\mathbf{T}\text{-}\lim x_n = \xi$. Moreover, we say that the matrix $\mathbf{T}$ is *regular* if every convergent sequence $\mathbf{x}$ is $\mathbf{T}$-summable and $\mathbf{T}\text{-}\lim x_n = \lim_{n\to\infty} x_n$.

The following lemma (known as the *Toeplitz Theorem*) contains necessary and sufficient condition for regularity of a matrix (see [5; p. 43, Theorem 2]).

LEMMA 2.2. *Matrix $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ is regular if and only if the following three conditions hold:*

(1) *There exists $M > 0$ such that for every $n = 1, 2, \dots$ the following inequality holds:*

$$\sum_{k=1}^{\infty} |t_{nk}| \le M \,;$$

(2) $\lim_{n\to\infty} t_{nk} = 0$ *for every* $k = 1, 2, \dots$ *;*

(3) $\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{nk} = 1$.

Remark 2.3. An interesting question is whether there is a matrix by which every sequence of real numbers is summable. The answer to that question is negative. Steinhaus in [13] proved a stronger result that for any regular matrix $\mathbf{T}$ there exists a sequence of zeros and ones which is not summable by matrix $\mathbf{T}$.

3. Densities created by matrices

In this section we study the relation between infinite matrices and ideals. Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be non-negative[4] regular matrix. Let $A \subseteq \mathbb{N}$ be an arbitrary set and χ_A its characteristic function. Then, the series

$$h_{\mathbf{T}}^{(n)}(A) = \sum_{k=1}^{\infty} t_{nk} \chi_A(k) \,, \qquad n = 1, 2, \dots \,, \tag{3.1}$$

[4]This means that all its elements are non-negative real numbers.

converges. This we obtained using the condition (1) from Lemma 2.2. Moreover, from the condition (3), we get $\lim_{n\to\infty} h_{\mathbf{T}}^{(n)}(\mathbb{N}) = 1$ for every $n \in \mathbb{N}$.

DEFINITION 3.1. Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix and $A \subseteq \mathbb{N}$. Denote $h_{\mathbf{T}}^{(n)}(A)$ as in (3.1). If the limit

$$h_{\mathbf{T}}(A) = \lim_{n\to\infty} h_{\mathbf{T}}^{(n)}(A)\,, \tag{3.2}$$

exists, then the number $h_{\mathbf{T}}(A)$ is called $\mathbf{T}$-*density* of the set A.

Remark 3.2. Obviously, $h_{\mathbf{T}}(A)$ is only another notation for $\mathbf{T}\text{-}\lim \chi_A(n)$.

EXAMPLE 3.3. If we consider matrix $\mathbf{T}$ to be

$$\mathbf{T}_d = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

we obtain the asymptotic density.[5] The matrix $\mathbf{T}_d$ is in the literature usually called the *Cesàro matrix.*

Remark 3.4. According to Remark 2.3 for every non-negative regular matrix $\mathbf{T}$ there exists a set which does not have $\mathbf{T}$-density. If we consider a sequence $(x_n)_{n=1}^{\infty}$ of zeros and ones which is not summable by matrix $\mathbf{T}$, then the set $\{n \in \mathbb{N} :\ x_n = 1\}$ does not have $\mathbf{T}$-density.

The following proposition describes basic properties of densities defined above. Its proof is routine and we leave it out.

PROPOSITION 3.5. *Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix and $A, B \subseteq \mathbb{N}$. Then the following statements hold:*

(1) *If $h_{\mathbf{T}}(A)$ exists, then $0 \le h_{\mathbf{T}}(A) \le 1$.*
(2) *If $A \subseteq B$ and $h_{\mathbf{T}}(A)$, $h_{\mathbf{T}}(B)$ exist, then $h_{\mathbf{T}}(A) \le h_{\mathbf{T}}(B)$.*
(3) *If $A \subseteq B$ and $h_{\mathbf{T}}(B) = 0$, then also $h_{\mathbf{T}}(A) = 0$.*
(4) *If $h_{\mathbf{T}}(A) = h_{\mathbf{T}}(B) = 0$, then also $h_{\mathbf{T}}(A \cup B) = 0$.*
(5) *If $h_{\mathbf{T}}(A)$ exists, then also $h_{\mathbf{T}}(\mathbb{N}\setminus A)$ exists, and $h_{\mathbf{T}}(A) + h_{\mathbf{T}}(\mathbb{N}\setminus A) = 1$.*

Remark 3.6. The last proposition directly implies that the family of sets $\mathfrak{I}_{h_{\mathbf{T}}} = \{A \subseteq \mathbb{N} :\ h_{\mathbf{T}}(A) = 0\}$ is an ideal. In addition, the condition (2) from Lemma 2.2 implies that $\mathfrak{I}_{h_{\mathbf{T}}}$ is admissible.

[5] Similarly we can obtain the logarithmic density.

DEFINITION 3.7. Let $\mathbf{T}$ be a non-negative regular matrix and $h_{\mathbf{T}}$ be $\mathbf{T}$-density. Then we say that the ideal $\mathfrak{I}_{h_{\mathbf{T}}} = \{A \subseteq \mathbb{N} : h_{\mathbf{T}}(A) = 0\}$ is *created by the matrix* $\mathbf{T}$.

Using the ideal $\mathfrak{I}_{h_{\mathbf{T}}}$, we can also define $\mathfrak{I}_{h_{\mathbf{T}}}$-convergence. Here, we can ask about the relation of $\mathbf{T}$-summability and $\mathfrak{I}_{h_{\mathbf{T}}}$-convergence.

THEOREM 3.8. *Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix, $\mathbf{x} = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers, and $\xi \in \mathbb{R}$. Then, the following statements hold:*

(1) *If $\mathbf{T}\text{-}\lim |x_n - \xi|^p = 0$ for some $p > 0$, then $\mathfrak{I}_{h_{\mathbf{T}}}\text{-}\lim x_n = \xi$.*
(2) *If the sequence $\mathbf{x}$ is bounded and such that $\mathfrak{I}_{h_{\mathbf{T}}}\text{-}\lim x_n = \xi$, then $\mathbf{T}\text{-}\lim |x_n - \xi|^p = 0$ for each $p > 0$.*

Proof. We will prove each part separately.

(1) For $\varepsilon > 0$ denote $A_\varepsilon = \{n \in \mathbb{N} : |x_n - \xi| \geq \varepsilon\}$ and let

$$z_n = \sum_{k=1}^{\infty} t_{nk}|x_k - \xi|^p \qquad \text{for} \quad n = 1, 2, \dots .$$

Using the assumption, we get $\lim\limits_{n\to\infty} z_n = 0$. Obviously, $0 \leq h_{\mathbf{T}}^{(n)}(A_\varepsilon)$ for every $n = 1, 2, \dots$. Using the following sequence of inequalities we get an upper bound:

$$\begin{aligned} h_{\mathbf{T}}^{(n)}(A_\varepsilon) &= \sum_{k \in A_\varepsilon} t_{nk} = \sum_{k:\,|x_k - \xi| \geq \varepsilon} t_{nk} \\ &= \frac{1}{\varepsilon^p} \sum_{k:\,|x_k - \xi| > \varepsilon} t_{nk}\varepsilon^p \\ &\leq \frac{1}{\varepsilon^p} \sum_{k:\,|x_k - \xi| > \varepsilon} t_{nk}|x_k - \xi|^p \\ &\leq \frac{1}{\varepsilon^p} \sum_{k=1}^{\infty} t_{nk}|x_k - \xi|^p = \frac{1}{\varepsilon^p} z_n . \end{aligned}$$

Altogether, we have $0 \leq h_{\mathbf{T}}^{(n)}(A_\varepsilon) \leq z_n/\varepsilon^p$, and so $h_{\mathbf{T}}(A_\varepsilon) = \lim\limits_{n\to\infty} h_{\mathbf{T}}^{(n)}(A_\varepsilon) = 0$. This means that $A_\varepsilon \in \mathfrak{I}_{h_{\mathbf{T}}}$ for each $\varepsilon > 0$, and therefore $\mathfrak{I}_{h_{\mathbf{T}}}\text{-}\lim x_n = \xi$.

(2) Let $L > 0$ be such that $|x_n| \leq L$ for every $n = 1, 2, \dots$. For $p > 0$ denote $L_p = (L + |\xi|)^p$. Then, obviously $|x_n - \xi|^p \leq L_p$ for $n = 1, 2, \dots$. In addition, from the regularity of the matrix $\mathbf{T}$ and from the condition (1) from Lemma 2.2, we obtain that $\sum\limits_{k=1}^{\infty} |t_{nk}| \leq M$ for some $M > 0$.

Denote A_ε and z_n in the same way as in the previous part. According to the assumption, we get $\lim_{n\to\infty} h_{\mathbf{T}}^{(n)}(A_\varepsilon) = 0$. The non-negativeness of the matrix $\mathbf{T}$ implies $0 \le z_n$ for every $n = 1, 2, \ldots$. An upper bound for z_n is obtained using the following inequalities:

$$\begin{aligned}
z_n &= \sum_{k=1}^{\infty} t_{nk}|x_k - \xi|^p \\
&= \sum_{k\colon |x_k-\xi|\ge\varepsilon} t_{nk}|x_k - \xi|^p + \sum_{k\colon |x_k-\xi|<\varepsilon} t_{nk}|x_k - \xi|^p \\
&\le \sum_{k\colon |x_k-\xi|\ge\varepsilon} t_{nk}L_p + M\varepsilon^p \\
&= L_p h_{\mathbf{T}}^{(n)}(A_\varepsilon) + M\varepsilon^p\,.
\end{aligned}$$

Taking $n \to \infty$, we get

$$0 \le \liminf_{n\to\infty} z_n \le \limsup_{n\to\infty} z_n \le M\varepsilon^p\,,$$

which holds for every $\varepsilon > 0$, and hence $\lim_{n\to\infty} z_n = 0$.

The proof is complete. □

THEOREM 3.9. *Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix. Then for each sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ of real numbers and for each $\xi \in \mathbb{R}$, the following statement holds: If $\mathbf{T}\text{-}\lim |x_n - \xi| = 0$, then $\mathbf{T}\text{-}\lim x_n = \xi$.*

Proof. For any $K \in \mathbb{N}$ we can write

$$\sum_{k=1}^{K} t_{nk}x_k = \sum_{k=1}^{K} t_{nk}(x_k - \xi) + \xi \sum_{k=1}^{K} t_{nk}\,.$$

Since the series $\sum_{k=1}^{\infty} t_{nk}|x_k - \xi|$ converges, so does also the first term on the right-hand side (for $K \to \infty$). The second term converges because of the regularity of the matrix $\mathbf{T}$, using the condition (1) from Lemma 2.2. Therefore the limit of the right-hand side, and also of the left-hand side, exists and the following equality holds:

$$\sum_{k=1}^{\infty} t_{nk}x_k = \sum_{k=1}^{\infty} t_{nk}(x_k - \xi) + \xi \sum_{k=1}^{\infty} t_{nk}\,,$$

for $n = 1, 2, \ldots$. Hence

$$\begin{aligned}
\left|\sum_{k=1}^{\infty} t_{nk}x_k - \xi\right| &\le \left|\sum_{k=1}^{\infty} t_{nk}(x_k - \xi)\right| + |\xi| \cdot \left|\sum_{k=1}^{\infty} t_{nk} - 1\right| \\
&\le \sum_{k=1}^{\infty} t_{nk}|x_k - \xi| + |\xi| \cdot \left|\sum_{k=1}^{\infty} t_{nk} - 1\right|.
\end{aligned}$$

For $n \to \infty$, the right-hand side converges to 0. Therefore also the left-hand side converges to 0. We obtained $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{\infty} t_{nk}x_k = \xi$, which means $\mathbf{T}\text{-}\lim x_n = \xi$. $\square$

COROLLARY 3.10. *Let $\mathbf{T} = (t_{nk})_{n,k=1}^{\infty}$ be a non-negative regular matrix. Then for every bounded sequence $\mathbf{x} = (x_n)_{n=1}^{\infty}$ of real numbers and $\xi \in \mathbb{R}$, the following statement holds: If $\mathfrak{I}_{h_{\mathbf{T}}}\text{-}\lim x_n = \xi$, then $\mathbf{T}\text{-}\lim x_n = \xi$.*

Proof. The statement is a direct corollary of Theorem 3.8.(2) and the previous theorem. $\square$

Remark 3.11. The converse statement to the statement from the previous theorem does not hold, for example, for the Cesàro matrix $\mathbf{T}_d$ from Example 3.3 and the sequence $\left((-1)^n\right)_{n=1}^{\infty}$. We can easily check that $\mathbf{T}_d\text{-}\lim (-1)^n = 0$, but $\mathbf{T}_d\text{-}\lim |(-1)^n - 0| = \mathbf{T}_d\text{-}\lim 1 = 1$.

4. φ-convergence

In this section we study a special matrix summability method also called φ-convergence. It was introduced by Schoenberg in [12]. To define it, consider the infinite matrix $\Phi = \left(\phi_{nk}\right)_{n,k=1}^{\infty}$ such that

$$\phi_{nk} = \begin{cases} \frac{\varphi(k)}{n} & \text{if } k \mid n\,, \\ 0 & \text{if } k \nmid n\,. \end{cases} \tag{4.1}$$

In [12], Schoenberg proved that the matrix Φ is regular and he called the Φ-summability a φ-convergence.

DEFINITION 4.1. Let $\mathbf{x} = (x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\xi \in \mathbb{R}$. We say that this sequence *φ-converges* to the number ξ if $\Phi\text{-}\lim x_n = \xi$, i.e., if $\lim\limits_{n\to\infty} y_n = \xi$, where

$$y_n = \frac{1}{n} \sum_{d|n} \varphi(d) x_d\,. \tag{4.2}$$

The number ξ is then called the *φ-limit* of the sequence $\mathbf{x}$ and we write $\varphi\text{-}\lim x_n = \xi$.

From the regularity of the matrix Φ we obtain that $\lim\limits_{n\to\infty} x_n = \xi$ implies $\varphi\text{-}\lim x_n = \xi$. The following example shows that the converse is not true. We introduce a sequence which is φ-convergent, but is not convergent. The problem to find such sequence was formulated by Šalát and Strauch as problem 6090 in AMM 1976, p. 385. The following example contains the solution submitted by Erdős in [3].

EXAMPLE 4.2. Let $\mathbb{P} = \{p_1 < p_2 < \cdots < p_k < \dots\}$ be the set of all primes. Let

$$x_n = \begin{cases} 1 & \text{if } n = 2\cdot 3\cdot 5\cdots p_k\,, \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $\boldsymbol{x} = (x_n)_{n=1}^{\infty}$ is not convergent, but it is φ-convergent. See [3] for details.

The following lemma contains an interesting result about convergence of certain subsequences of a φ-convergent sequence. Its proof can be found in [12; Theorem 2].

LEMMA 4.3. *Let $x = (x_n)_{n=1}^{\infty}$ and $\xi \in \mathbb{R}$ be such that $\varphi\text{-}\lim x_n = \xi$, and let $(n_i)_{i=1}^{\infty}$ be an increasing sequence of positive integers. Then,*

$$\liminf_{i\to\infty} \frac{\varphi(n_i)}{n_i} > 0 \tag{4.3}$$

implies that

$$\lim_{i\to\infty} x_{n_i} = \xi\,.$$

The previous theorem can be used as a criterion whether a sequence is not φ-convergent. We illustrate it on the following example from [12; p. 367, Remark 1].

EXAMPLE 4.4. Consider the sequence $(x_n)_{n=1}^{\infty}$ such that $x_n = \chi_{\mathbb{P}}(n)$, $n = 1, 2, \dots$. Then

$$\lim_{i\to\infty} x_{p_i} = 1\,, \qquad \lim_{i\to\infty} x_{2p_i} = 0\,,$$

and since

$$\frac{\varphi(p_i)}{p_i} = 1 - \frac{1}{p_i}\,, \qquad \frac{\varphi(2p_i)}{2p_i} = \frac{1}{2}\left(1 - \frac{1}{p_i}\right)\,,$$

Lemma 4.3 implies that this sequence is not φ-convergent.

Now we focus on the relation between φ-convergence and $\mathfrak{I}$-convergence, where $\mathfrak{I}$ is an admissible ideal. The following theorem contains a sufficient condition for $\mathfrak{I}$ so that every φ-convergent sequence is also $\mathfrak{I}$-convergent. Its proof is a generalization for the proof for the ideal $\mathfrak{I}_d$ in [12].

Theorem 4.5. *Let $\mathfrak{I}$ be an admissible ideal containing every set $S = \{n_1 < n_2 < \dots < n_i < \dots\} \subseteq \mathbb{N}$ such that*

$$\lim_{i\to\infty} \frac{\varphi(n_i)}{n_i} = 0. \tag{4.4}$$

Then for every sequence $(x_n)_{n=1}^{\infty}$ of real numbers and for every $\xi \in \mathbb{R}$ the following statement holds: If $\varphi\text{-}\lim x_n = \xi$, then $\mathfrak{I}\text{-}\lim x_n = \xi$.

Proof. For $\varepsilon > 0$ consider the set $A_\varepsilon = \{n \in \mathbb{N} : \ |x_n - \xi| \geq \varepsilon\}$. It is sufficient to prove that $A_\varepsilon \in \mathfrak{I}$. If A_ε is empty or finite, then obviously $A \in \mathfrak{I}$, because $\mathfrak{I}$ is admissible ideal. Further assume that A_ε is infinite and let $A_\varepsilon = \{n_1 < n_2 < \dots < n_i < \dots\}$. Then the equality $\lim\limits_{i\to\infty} x_{n_i} = \xi$ does not hold. From Lemma 4.3 we obtain

$$\liminf_{i\to\infty} \frac{\varphi(n_i)}{n_i} = 0.$$

We will prove by contradiction that the condition (4.4) holds. Otherwise,

$$\limsup_{i\to\infty} \frac{\varphi(n_i)}{n_i} > 0,$$

and there is a subsequence $(n_i')_{i=1}^{\infty}$ of the sequence $(n_i)_{i=1}^{\infty}$ such that

$$\liminf_{i\to\infty} \frac{\varphi(n_i')}{n_i'} > 0.$$

Then, from Lemma 4.3, we get $\lim\limits_{i\to\infty} x_{n_i'} = \xi$. This is contradiction, because the definition of A_ε implies that $|x_{n_i'} - \xi| \geq \varepsilon$ for each $i \in \mathbb{N}$. So, we have proved that the condition (4.4) holds. Therefore, by assumption, $A_\varepsilon \in \mathfrak{I}$. $\square$

The following lemma says that the ideal $\mathfrak{I}_d$ fulfills the sufficient condition from the previous theorem. Its proof can be found in [12; Lemma 2, Theorem 3]. The lemma implies that every φ-convergent sequence is also statistically convergent.

Lemma 4.6. *If $S \subseteq \mathbb{N}$, $S = \{n_1 < n_2 < \dots < n_i < \dots\}$ is such that $\lim\limits_{i\to\infty} \varphi(n_i)/n_i = 0$, then $d(S) = 0$.*

Corollary 4.7. *Let $(x_n)_{n=1}^{\infty}$ be a sequence of real numbers and $\xi \in \mathbb{R}$. If $\varphi\text{-}\lim x_n = \xi$, then also $\mathfrak{I}_d\text{-}\lim x_n = \xi$.*

5. φ-density

In this section, we focus on a particular type of density based on φ-convergence. To define it, we use the approach from Section 3. Consider the matrix Φ as defined in (4.1). This matrix is non-negative, regular and has finite rows (i.e., each row contains only a finite number of non-zero elements). Let A be an arbitrary subset of $\mathbb{N}$ and let $\chi_A\colon \mathbb{R} \to \{0,1\}$ be its characteristic function. Denote

$$d_\varphi^{(n)}(A) = \frac{1}{n}\sum_{d|n}\varphi(d)\chi_A(d)\,, \qquad n = 1,2,\dots\,. \tag{5.1}$$

DEFINITION 5.1. Let $A \subseteq \mathbb{N}$. Denote $d_\varphi^{(n)}(A)$ as in (5.1). If the limit

$$d_\varphi(A) = \lim_{n\to\infty} d_\varphi^{(n)}(A) = \lim_{n\to\infty}\frac{1}{n}\sum_{d|n}\varphi(d)\chi_A(d) \tag{5.2}$$

exists, then the number $d_\varphi(A)$ is called the *φ-density* of the set A.

In other words, φ-density is the same as Φ-density, and obviously also the same as the φ-limit of the sequence $\bigl(\chi_A(n)\bigr)_{n=1}^\infty$. Basic properties of φ-density are given by Proposition 3.5.

EXAMPLE 5.2. Let $\mathbb{P}$ be the set of all prime numbers. Consider the set

$$E = \{2 < 2\cdot 3 < \dots < 2\cdot 3 \cdots p_k < \dots\}\,.$$

Example 4.2 shows that $d_\varphi(E) = 0$. Therefore E is an infinite set with φ-density zero.

Now, we examine the image of φ-density. Consider first the sets of φ-density zero. According to Proposition 3.5, the family $\mathfrak{I}_\varphi = \bigl\{A \subseteq \mathbb{N} :\ d_\varphi(A) = 0\bigr\}$ of all sets with φ-density zero is an ideal. Note that $\mathfrak{I}_\varphi$ is just another notation for the ideal $\mathfrak{I}_{h_\Phi}$ in the sense of Definition 3.7.

For asymptotic density it can be shown that, for each $t \in [0,1]$, there is a set $A \subseteq \mathbb{N}$ such that $d(A) = t$. However, a similar statement does not hold for φ-density, as the following theorem shows.

THEOREM 5.3. *If $A \subseteq \mathbb{N}$ is such that $d_\varphi(A)$ exists, then $d_\varphi(A) \in \{0,1\}$.*

Proof. Consider a sequence $(x_n)_{n=1}^\infty$ such that $x_n = \chi_A(n)$ for $n = 1,2,\dots$. Then $\varphi\text{-}\lim x_n = d_\varphi(A)$, and from Corollary 4.7 we get that $\mathfrak{I}_d\text{-}\lim x_n = d_\varphi(A)$. In addition, $(x_n)_{n=1}^\infty$ is a sequence of zeros and ones which, according to Proposition 2.1, means that $d_\varphi(A) \in \{0,1\}$. $\square$

COROLLARY 5.4. *Let $A \subseteq \mathbb{N}$ be such that $d_\varphi(A)$ exists. Then either $d_\varphi(A) = 0$ (i.e., $A \in \mathfrak{I}_\varphi$) or $d_\varphi(\mathbb{N}\setminus A) = 0$ (i.e., $\mathbb{N}\setminus A \in \mathfrak{I}_\varphi$).*

6. The relation of φ-density and other types of densities

In this section we explore the relation of φ-density and other types of densities introduced in Section 2. We show that if a set has φ-density, then it has also asymptotic density (therefore also logarithmic density) and they are equal. This statement was proved already by Schoenberg in [12]. Moreover, we show that there is a set which has asymptotic, logarithmic and uniform density, but it does not have φ-density. We also construct a set which has φ-density, but does not have uniform density. This construction was first time introduced in author's Master's Thesis [7].

THEOREM 6.1. *Let $A \in \mathbb{N}$ be such set that $d_\varphi(A)$ exists. Then $d(A)$ also exists and $d(A) = d_\varphi(A)$.*

Proof. Consider a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n = \chi_A(n)$ for $n = 1, 2, \dots$. Then $\varphi\text{-}\lim x_n = d_\varphi(A)$. Moreover, Corollary 4.7 yields $\mathfrak{I}_d\text{-}\lim x_n = d_\varphi(A)$. Then, using Example 3.3 and Theorem 3.8, we get $\mathbf{T}_d\text{-}\lim |x_n - d_\varphi(A)| = 0$. In addition, Theorem 3.9 yields

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x_i = \mathbf{T}_d\text{-}\lim x_n = d_\varphi(A)\,.$$

According to the definition of the sequence $(x_n)_{n=1}^{\infty}$, the term on the left-hand side is equal $d(A)$. Thus we obtain that $d(A)$ exists and $d(A) = d_\varphi(A)$. □

COROLLARY 6.2. *Let $A \in \mathbb{N}$ be such set that $d_\varphi(A)$ exists. Then $\delta(A)$ also exists and $\delta(A) = d_\varphi(A)$.*

COROLLARY 6.3. *The following statement holds: $\mathfrak{I}_\varphi \subseteq \mathfrak{I}_d \subseteq \mathfrak{I}_\delta$.*

EXAMPLE 6.4. We introduce a set which has uniform density (and therefore also asymptotic and logarithmic density), but does not have φ-density. Consider the set $\mathbb{P}$ of all prime numbers. According to (2.6), the equality $\delta(\mathbb{P}) = d(\mathbb{P}) = u(\mathbb{P}) = 0$ holds. Moreover, according to Example 4.4, the set $\mathbb{P}$ does not have φ-density, which means that none of the following inclusions $\mathfrak{I}_d \subseteq \mathfrak{I}_\varphi$, $\mathfrak{I}_\delta \subseteq \mathfrak{I}_\varphi$, $\mathfrak{I}_u \subseteq \mathfrak{I}_\varphi$ holds.

THEOREM 6.5. *There exists a set B such that $d_\varphi(B) = 0$, but $u(B)$ does not exist.*

Proof. First we construct a sequence $(z_k)_{k=1}^{\infty}$ of positive integers using the following steps. For $k = 1$ define $z_1 = 7$. If we have constructed the numbers $z_1, \dots, z_{k-1}$ $(k \geq 2)$, we define $m_1(k) = 1$ and using induction, we construct the numbers $m_1(k), \dots, m_{k+1}(k)$. Having constructed the numbers

$m_1(k), \dots, m_i(k)$ (where $i \le k$), there exists a positive integer $m_{i+1}(k)$ such that $m_{i+1}(k) > m_i(k)$ and

$$\prod_{j=m_i(k)}^{m_{i+1}(k)-1} \Bigl(1 - \frac{1}{p_j}\Bigr) < \frac{1}{k \cdot 2^k} .$$

Moreover, for $i = 1, 2, \dots, k$ denote

$$a_i(k) = \prod_{j=m_i(k)}^{m_{i+1}(k)-1} p_j = p_{m_i(k)} p_{m_i(k)+1} \cdots p_{m_{i+1}(k)-1} .$$

Then

$$\frac{\varphi(a_i(k))}{a_i(k)} = \prod_{j=m_i(k)}^{m_{i+1}(k)-1} \Bigl(1 - \frac{1}{p_j}\Bigr) < \frac{1}{k \cdot 2^k} . \tag{6.1}$$

Then, for all positive integers i, l such that $1 \le i < l \le k$, the equality $\bigl(a_i(k), a_l(k)\bigr) = 1$ holds, where (b, c) means the greater common divisor of integers b, c. According to the *Chinese reminder theorem*, there exists a positive integer z_k such that

$$z_k \equiv i \pmod{a_i(k)} \qquad \text{for} \quad i = 1, 2, \dots, k , \tag{6.2}$$

and

$$z_k > z_{k-1}^2 . \tag{6.3}$$

Then, for every $i = 1, 2, \dots, k$ we have $a_i(k) \mid z_k - i$, which implies

$$\frac{\varphi(z_k - i)}{z_k - i} \le \frac{\varphi\bigl(a_i(k)\bigr)}{a_i(k)} < \frac{1}{k \cdot 2^k} . \tag{6.4}$$

Having constructed the sequence $(z_k)_{k=1}^{\infty}$, consider for every $k \in \mathbb{N}$ the set

$$B_k = \{z_k - i : \ i = 1, 2, \dots, k\} . \tag{6.5}$$

Using (6.4), we obtain

$$\frac{\varphi(n)}{n} < \frac{1}{k \cdot 2^k} \qquad \text{for} \quad n \in B_k . \tag{6.6}$$

Now, define $B = \bigcup\limits_{k=1}^{\infty} B_k$. We can easily see that for α_j, α^j from (2.2), the equalities $\alpha_j = 0$, $\alpha^j = j$ hold for every $j \in \mathbb{N}$. Then $\underline{u}(B) = 0$, $\overline{u}(B) = 1$, which means than the set B does not have uniform density.

Consider a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n = \chi_B(n)$ for $n = 1, 2, \dots$. We will show that $\varphi\text{-}\lim x_n = 0$, i.e.,

$$y_n = \frac{1}{n} \sum_{d \mid n} \varphi(d) x_d \to 0 . \tag{6.7}$$

For given $n \in \mathbb{N}$ $(n \geq z_1)$, there is unique $k \in \mathbb{N}$ such that

$$z_k \leq n < z_{k+1}\,. \tag{6.8}$$

Then we can rewrite (6.7) as

$$y_n = \frac{1}{n}\sum_{i=1}^{k-1} \sum_{d\in B_i\,,\, d|n} \varphi(d)x_d + \frac{1}{n}\sum_{d\in B_k\,,\, d|n} \varphi(d)x_d + \frac{1}{n}\sum_{d\in B_{k+1}\,,\, d|n} \varphi(d)x_d\,, \tag{6.9}$$

where $x_d = 1$ for $d \in B$. For each of the above sums we will find an upper bound. Using (6.5) and (6.6) for $i = 1, 2, \dots, k$ we get

$$\begin{aligned} \frac{1}{n}\sum_{d\in B_i\,,\, d|n} \varphi(d)\, x_d &\leq \frac{1}{n}\sum_{d\in B_i} \varphi(d) \leq \frac{1}{n}\sum_{d=z_i-i}^{z_i-1} \frac{d}{i\cdot 2^i} \\ &= \frac{1}{n}\cdot i \cdot \frac{d}{i\cdot 2^i} \leq \frac{z_i}{n}\cdot\frac{1}{2^i}\,. \end{aligned} \tag{6.10}$$

Moreover, using (6.3) we obtain

$$\frac{z_i}{n} \leq \frac{z_{k-1}}{z_k} \leq \frac{1}{z_{k-1}} \qquad \text{for} \quad i = 1, 2, \dots, k-1\,, \tag{6.11}$$

which yields

$$\frac{1}{n}\sum_{d\in B_i\,,\, d|n} \varphi(d)x_d \leq \frac{z_i}{n}\cdot\frac{1}{2^i} \leq \frac{1}{z_{k-1}}\cdot\frac{1}{2^i}\,.$$

Then we get an upper bound for the first sum from (6.9):

$$\frac{1}{n}\sum_{i=1}^{k-1} \sum_{d\in B_i\,,\, d|n} \varphi(d)x_d \leq \sum_{i=1}^{k-1} \frac{1}{z_{k-1}}\cdot\frac{1}{2^i} \leq \frac{1}{z_{k-1}}\,. \tag{6.12}$$

Moreover, using (6.10) and (6.8) we obtain an upper bound for the second sum from (6.9):

$$\frac{1}{n}\sum_{d\in B_k\,,\, d|n} \varphi(d)x_d \leq \frac{z_k}{n}\cdot\frac{1}{2^k} \leq \frac{1}{2^k}\,. \tag{6.13}$$

Now consider the third sum from (6.9). Obviously, if $d \mid n$ for some $d \in B_{k+1}$, then necessary $d = n$. In this case

$$\frac{1}{n}\sum_{d\in B_{k+1}\,,\, d|n} \varphi(d)x_d = \frac{\varphi(n)}{n} \leq \frac{1}{(k+1)\cdot 2^{k+1}}\,, \tag{6.14}$$

because $n \in B_{k+1}$. Otherwise, i.e., when $d \nmid n$ for all $n \in B_{k+1}$, the above sum is equal to zero.

Now, to find an upper bound for y_n, we just consider upper bounds for the sums from (6.9). Using the inequalities (6.12), (6.13), and (6.14), we obtain

$$0 \le y_n \le \frac{1}{z_{k-1}} + \frac{1}{2^k} + \frac{1}{(k+1)\cdot 2^{k+1}}\,.$$

At the same time, (6.8) implies that if $n \to \infty$, then $k \to \infty$ and also $z_k \to \infty$. Therefore the right-hand side of the last inequality goes to zero when $n \to \infty$. This means that $\lim\limits_{n\to\infty} y_n = 0$, and therefore φ- $\lim x_n = 0$, which concludes the proof. □

7. The ideal of all sets with φ-density zero

In this section we analyze in detail the ideal $\mathfrak{I}_\varphi = \{A \subseteq \mathbb{N} : d_\varphi(A) = 0\}$. We start by considering those subsets of $\mathbb{N}$ for which (4.4) holds. Their close connection to φ-convergence was indicated in Section 4. Later, we analyze the relation of $\mathfrak{I}_\varphi$-convergence and φ-convergence.

THEOREM 7.1. *Let $S \subseteq \mathbb{N}$, $S = \{n_1 < n_2 < \dots < n_i < \dots\}$ be such set that $\liminf\limits_{i\to\infty} \varphi(n_i)/n_i > 0$. Then, for every set $A \in \mathfrak{I}_\varphi$, the set $A \cap S$ is finite.*

P r o o f. Denote $x_n = \chi_A(n)$ for $n = 1, 2, \dots$. If $d_\varphi(A) = 0$, then, using Lemma 4.3, we get $\lim\limits_{n\to\infty} x_{n_i} = 0$. Since $(x_{n_i})_{i=1}^\infty$ is a sequence of zeros and ones, there exists $i_0 \in \mathbb{N}$ such that $x_{n_i} = 0$ for any $i \ge i_0$. Therefore, $n_i \notin A$, which means that the set $A \cap S$ is finite. □

For an arbitrary $\varepsilon > 0$ denote $F_\varepsilon = \{n \in \mathbb{N} : \varphi(n)/n \ge \varepsilon\}$. Let $\mathfrak{I}_s$ be the family of all sets $S \subseteq \mathbb{N}$ for which the following statement holds: For any $\varepsilon > 0$, the set $S \cap F_\varepsilon$ is finite. Obviously, $S \subseteq \mathbb{N}$ is an element of $\mathfrak{I}_s$ if and only if S is finite or S is infinite, $S = \{n_1 < n_2 < \dots < n_i < \dots\}$, and

$$\lim_{n\to\infty,\, n\in S} \frac{\varphi(n)}{n} = \lim_{i\to\infty} \frac{\varphi(n_i)}{n_i} = 0\,.$$

Now we examine the relation of the ideal $\mathfrak{I}_\varphi$ and the family of sets $\mathfrak{I}_s$.

PROPOSITION 7.2. *The family of sets $\mathfrak{I}_s$ is an admissible ideal.*

P r o o f. The family $\mathfrak{I}_s$ contains all finite subsets of the set of all positive integers, so it is non-empty and if it is an ideal, it is also admissible.

Let $S \in \mathfrak{I}_s$, $S' \subseteq S$. Then for each $\varepsilon > 0$ we obtain $S' \cap F_\varepsilon \subseteq S \cap F_\varepsilon$, which is a finite set. Therefore also $S' \in \mathfrak{I}_s$, which proves the heredity.

Now, consider sets $S, S' \in \mathfrak{I}_s$ and an arbitrary $\varepsilon > 0$. By assumption the sets $S \cap F_\varepsilon$, $S' \cap F_\varepsilon$ are finite. Therefore, also the set $(S \cup S') \cap F_\varepsilon = (S \cap F_\varepsilon) \cup (S' \cap F_\varepsilon)$ is finite, which yields $S \cup S' \in \mathfrak{I}_s$ and proves the additivity. □

THEOREM 7.3. *The ideal $\mathfrak{I}_\varphi$ is a subset of $\mathfrak{I}_s$, but $\mathfrak{I}_s$ is not a subset of $\mathfrak{I}_\varphi$.*

Proof. Let $A \in \mathfrak{I}_\varphi$. If A is finite, then $A \in \mathfrak{I}_s$. Now consider A to be infinite, and let $A = \{n_1 < n_2 < \dots < n_i < \dots\}$. Obviously,

$$0 \le \frac{\varphi(n_i)}{n_i} \le \frac{1}{n_i} \sum_{d|n_i} \varphi(d)\chi_A(d) = d_\varphi^{(n_i)}(A).$$

For $i \to \infty$, the right-hand side converges to 0, so $\lim_{i\to\infty} \varphi(n_i)/n_i = 0$ and $A \in \mathfrak{I}_s$.

Now we provide an example of a set which is an element of $\mathfrak{I}_s \setminus \mathfrak{I}_\varphi$. Let $\mathbb{P}$ be the set of all prime numbers. For any $k \in \mathbb{N}$ denote

$$B_k = \left\{2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k} : \ \alpha_1, \alpha_2, \dots, \alpha_k \in \{1,2\}\right\} \tag{7.1}$$

and let $B = \bigcup_{k=1}^{\infty} B_k$. Then for every $k \in \mathbb{N}$ and for every $n \in B_k$:

$$\frac{\varphi(n)}{n} = \left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\cdots\left(1 - \tfrac{1}{p_k}\right).$$

Thus

$$\lim_{n\to\infty,\, n\in B} \frac{\varphi(n)}{n} = 0.$$

Now we will show that $d_\varphi(B)$ does not exist. Consider $n = 2\cdot 3\cdots p_k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned} d_\varphi^{(n)}(B) &= \frac{1}{2\cdot 3\cdots p_k} \sum_{d|2\cdot 3\cdots p_k} \varphi(d)\chi_B(d) \\ &= \frac{1}{2\cdot 3\cdots p_k}\left[(2-1) + (2-1)(3-1) + \dots + (2-1)\dots(p_k - 1)\right], \end{aligned}$$

which implies that $d_\varphi^{(n)}(B) \to 0$ for $n \to \infty$, $n = 2\cdot 3\cdots p_k$; see also Example 5.2.

Now consider $n = 2^2\cdot 3^2\cdots p_k^2$. Then

$$\begin{aligned} d_\varphi^{(n)}(B) &= \frac{1}{2^2\cdot 3^2\cdots p_k^2} \sum_{d|2^2\cdot 3^2\cdots p_k^2} \varphi(d)\chi_B(n) \\ &\ge \frac{1}{2^2\cdot 3^2\cdots p_k^2} \sum_{d\in B_k} \varphi(d) \\ &= \frac{1}{2^2\cdot 3^2\cdots p_k^2}\cdot(2\cdot 3\cdots p_k)\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\cdots\left(1 - \tfrac{1}{p_k}\right) \sum_{d'|2\cdot 3\cdots p_k} d' \\ &= \frac{1}{2\cdot 3\cdots p_k}\left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{3}\right)\cdots\left(1 - \tfrac{1}{p_k}\right)(1+2)(1+3)\cdots(1+p_k) \\ &= \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\dots\left(1 - \frac{1}{p_k^2}\right). \end{aligned}$$

Above, for $d \in B_k$, we expressed $\varphi(d) = d\cdot(1-1/2)\cdots(1-1/p_k) = d'\cdot 2\cdots p_k\cdot(1-1/2)\cdots(1-1/p_k)$, where $d' = d/(2\cdots p_k)$ is an arbitrary divisor of $2\cdots p_k$. We also used that $\sum\limits_{d'\mid 2\cdot 3\cdots p_k} d' = (1+2)(1+3)\cdots(1+p_k)$. Thus $d_\varphi^{(n)}(B)$ cannot converge to 0 when $n\to\infty$, $n = 2^2\cdot 3^2\cdots p_k^2$, because of (2.5).

The above results imply that $\lim\limits_{n\to\infty} d_\varphi^{(n)}(B)$ does not exist, and so the set B does not have φ-density. □

Now we analyze the relation of $\mathfrak{I}_\varphi$-convergence and φ-convergence. Reformulating Theorems 3.8 and 3.9 we obtain the following theorem.

THEOREM 7.4. *Let $(x_n)_{n=1}^\infty$ be an arbitrary sequence of real numbers and $\xi\in\mathbb{R}$. Then:*

(1) *If φ-$\lim|x_n-\xi| = 0$, then $\mathfrak{I}_\varphi$-$\lim x_n = \xi$.*
(2) *If $(x_n)_{n=1}^\infty$ is a bounded sequence such that $\mathfrak{I}_\varphi$-$\lim x_n = \xi$, then φ-$\lim|x_n-\xi| = 0$.*
(3) *If φ-$\lim|x_n-\xi| = 0$, then φ-$\lim x_n = \xi$.*

As we can see, for each bounded sequence, $\mathfrak{I}_\varphi$-convergence implies φ-convergence. Now we will prove that the converse does not hold.

THEOREM 7.5. *There exists a bounded sequence $(x_n)_{n-1}^\infty$ such that φ-$\lim x_n = 0$, but φ-$\lim|x_n|$ does not exist.*

P r o o f . For $k = 1,2,\dots$ denote

$$B_k = \{2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k} :\ \alpha_1,\alpha_2,\dots,\alpha_k\in\{1,2\}\},$$

analogically as in (7.1). Further let

$$B_k^+ = \{2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k}\in B_k :\ 2\mid \alpha_1+\alpha_2+\cdots+\alpha_k\},$$
$$B_k^- = \{2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k}\in B_k :\ 2\nmid \alpha_1+\alpha_2+\cdots+\alpha_k\},$$

and denote $B^+ = \bigcup\limits_{k=1}^\infty B_k^+$ and $B^- = \bigcup\limits_{k=1}^\infty B_k^-$. Consider the sequence $(x_n)_{n\ 1}^\infty$ such that

$$x_n = \begin{cases} 1 & \text{if } n\in B^+,\\ -1 & \text{if } n\in B^-,\\ 0 & \text{otherwise.}\end{cases}$$

In the proof of Theorem 7.3 we have also proved that φ-$\lim|x_n|$ does not exist. Now we will show that φ-$\lim x_n = 0$.

For a positive integer n let $k(n)\in\mathbb{N}$ be the greatest integer such that n is divisible by first $k(n)$ primes. In case $2\nmid n$ let $k(n) = 0$ and $2^{\alpha_1}3^{\alpha_2}\cdots p_k^{\alpha_k} = 1$. Then there is a unique $d_n\in\mathbb{N}$ such that $n = 2^{\alpha_1}3^{\alpha_2}\cdots p_{k(n)}^{\alpha_{k(n)}} d_n$, where

$\alpha_1, \alpha_2, \dots, \alpha_k \in \{1,2\}$ and if $\alpha_i = 1$, then $p_i^2 \nmid n$. In other words, $\alpha_i = \min\{2, \max\{a \in \mathbb{Z} : \ a \geq 0,\ p_i^a \mid n\}\}$ for any $i = 1, 2, \dots, k(n)$. In this case also denote $\pi_n = 2^{\alpha_1} 3^{\alpha_2} \cdots p_{k(n)}^{\alpha_{k(n)}}$.

Let $\mathbf{y} = \Phi \mathbf{x}$, i.e., (4.2) holds. Then

$$\begin{aligned} y_n &= \frac{1}{n} \sum_{d|n} \varphi(d) x_d = \frac{1}{d_n} \cdot \frac{1}{\pi_n} \sum_{d|n} \varphi(d) x_d \\ &= \frac{1}{d_n} \cdot \frac{1}{\pi_n} \sum_{i=1}^{k} \sum_{d|\pi_n,\, d \in B_i} \varphi(d) x_d. \end{aligned} \tag{7.2}$$

Obviously, $d \mid 2^{\alpha_1} 3^{\alpha_2} \cdots p_i^{\alpha_i}$ for any $d \mid n$ such that $d \in B_i$. Consider an arbitrary $n' \in B_i$ and let $n' = r_1^2 \cdots r_l^2 r_{l+1} \cdots r_i$, where $(r_1, \dots, r_i)$ is some permutation of primes $2, 3, \dots, p_i$. For simplicity assume that $r_1 < \cdots < r_l$ and $r_{l+1} < \cdots < r_i$. Now, if $d \mid n'$, $d \in B_i$, then $d = r_1 \cdots r_i \cdot d'$, where d' is some divisor of $r_1 \cdots r_l$. Obviously, this defines a bijection between the sets $\{d \in \mathbb{N} : \ d \mid n',\ d \in B_i\}$ and $\{d' \in \mathbb{N} : \ d' \mid r_1 \cdots r_l\}$. Moreover, $x_d = (-1)^i \mu(d')$ and $\varphi(d) = (r_1 - 1) \cdots (r_i - 1) d'$. Further we obtain

$$\begin{aligned} \sum_{d|n',\, d \in B_i} \varphi(d) x_d &= r_1 \cdots r_i \Big(1 - \frac{1}{r_1}\Big) \cdots \Big(1 - \frac{1}{r_i}\Big) \sum_{d' | r_1 \cdots r_l} d' \mu(d') (-1)^i \\ &= (r_1 - 1) \cdots (r_i - 1) \cdot (r_1 - 1) \cdots (r_l - 1) \cdot (-1)^{i-l} \\ &= \big((r_1 - 1) \cdots (r_l - 1)\big)^2 \cdot (r_{l+1} - 1) \cdots (r_i - 1) \cdot (-1)^{i-l}. \end{aligned}$$

In the second equality we used (2.9). For simplicity denote $k = k(n)$. Then

$$\begin{aligned} \bigg| \frac{1}{\pi_n} \sum_{d|n} \varphi(d) x_d \bigg| &\leq \frac{1}{2^{\alpha_1} 3^{\alpha_2} \cdots p_k^{\alpha_k}} \sum_{i=1}^{k} \bigg| \sum_{d|\pi_n,\, d \in B_i} \varphi(d) x_d \bigg| \\ &= \frac{1}{2^{\alpha_1} 3^{\alpha_2} \cdots p_k^{\alpha_k}} \sum_{i=1}^{k} (2-1)^{\alpha_1} (3-1)^{\alpha_2} \cdots (p_i - 1)^{\alpha_i} \\ &= \sum_{i=1}^{k} \Big(1 - \frac{1}{2}\Big)^{\alpha_1} \Big(1 - \frac{1}{3}\Big)^{\alpha_2} \cdots \Big(1 - \frac{1}{p_i}\Big)^{\alpha_i} \frac{1}{p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k}} \\ &\leq \sum_{i=1}^{k-1} \frac{1}{p_{i+1} \cdots p_k} + \Big(1 - \frac{1}{2}\Big)\Big(1 - \frac{1}{3}\Big) \cdots \Big(1 - \frac{1}{p_k}\Big) \\ &= \Big(\frac{1}{p_k} + \frac{1}{p_{k-1} p_k} + \cdots + \frac{1}{3 \cdots p_{k-1} p_k}\Big) + \prod_{i=1}^{k} \Big(1 - \frac{1}{p_i}\Big) \end{aligned}$$

$$\leq \frac{1}{p_k}\Big(1+\frac{1}{3}+\frac{1}{3^2}+\dots\Big)+\prod_{i=1}^{k}\Big(1-\frac{1}{p_i}\Big)$$
$$=\frac{1}{p_k}\cdot\frac{3}{2}+\prod_{i=1}^{k}\Big(1-\frac{1}{p_i}\Big).$$

For $k = 1,2,\dots$ denote

$$q_k=\frac{3}{2}\cdot\frac{1}{p_k}+\prod_{i=1}^{k}\Big(1-\frac{1}{p_i}\Big)$$

and we obtain that $q_k \to 0$ for $k \to \infty$. Thus

$$|y_n| \leq \frac{1}{d_n} q_{k(n)},$$

and if $n \to \infty$, then either $d_n \to \infty$ or $k(n) \to \infty$. This means that $\lim_{n\to\infty} y_n = 0$, and therefore φ-$\lim x_n = 0$. □

COROLLARY 7.6. *There is a bounded sequence of real numbers $(x_n)_{n=1}^{\infty}$ such that φ-$\lim x_n = 0$, but $\mathfrak{I}_\varphi$-$\lim x_n$ does not exist.*

Proof. Consider the sequence $(x_n)_{n=1}^{\infty}$ from the previous theorem. If $\mathfrak{I}_\varphi$-$\lim x_n = \xi$, then using Theorem 7.4 we obtain φ-$\lim |x_n - \xi| = 0$, and also φ-$\lim x_n = \xi$. This means that $\xi = 0$, and therefore φ-$\lim |x_n| = 0$, which is a contradiction with the choice of the sequence $(x_n)_{n=1}^{\infty}$. □

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