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ASYMPTOTIC EQUIVALENCE OF DIFFERENCE EQUATIONS

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ABSTRACT. The purpose of this paper is the study of a generalized asymptotic equivalence between the solutions of the difference equations

$$y(n+1) = A(n)y(n) \tag{I}$$

and

$$x(n+1) = A(n)x(n) + F\big(n, x(n), Tx(n)\big)\,. \tag{II}$$

By means of the contraction mapping principle, we prove the existence of a homeomorphism H between the sets of bounded solutions of (I) and (II).

Introduction

The purpose of this paper is the study of a generalized asymptotic equivalence between the solutions of the difference equations

$$y(n+1) = A(n)y(n) \tag{I}$$

and

$$x(n+1) = A(n)x(n) + F\big(n, x(n), Tx(n)\big)\,. \tag{II}$$

By means of the contraction mapping principle, we prove the existence of a homeomorphism H between the sets of bounded solutions of (I) and (II). Moreover, we are going to investigate the (g,p) asymptotic equivalence between equations (I) and (II) such that to each bounded solution

$$x(n) = Hy(n) \qquad \text{of (II)}$$

we have

$$\big|g^{-1}(n)\big[y(n) - Hy(n)\big]\big| \in l_p\,.$$

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The relationship between the asymptotic behavior of a homogeneous differential equation and a nonhomogeneous perturbation of that differential equation has been widely investigated. The objective of this paper is to develop part of these problems for some classes of difference equations.

Our results extend some theorems obtained by Talpalaru [6], which proved an asymptotic relationship between the solutions of (I) and (II) using Schauder's fixed point theorem.

Notations and definitions

Denote by $\mathbb{N}^+_{n_0} = \{n_0, n_0+1, \dots\}$, where n_0 is a natural number or zero, $\mathbb{R}^k$ — the k-dimensional real euclidean space with the norm $|x| = \sum_{i=1}^{k} |x_i|$, $x = (x_1, \dots, x_k)$, M^k — the space of $k \times k$ matrices $A = (a_{ij})$ with the norm $|A| = \max_j \sum_{i=1}^{k} |a_{ij}|$, I — the identity matrix. We denote by $Q = Q(\mathbb{N}^+_{n_0}, \mathbb{R}^k)$ the space of all functions from $\mathbb{N}^+_{n_0}$ into $\mathbb{R}^k$, $B = B(\mathbb{N}^+_{n_0}, \mathbb{R}^k)$ — the Banach space in Q for all bounded functions from $\mathbb{N}^+_{n_0}$ to $\mathbb{R}^k$ with the norm

$$|x|_B = |x(n)|_B = \sup\{|x(n)| : \ n \in \mathbb{N}^+_{n_0}\}.$$

We will be interested in establishing an asymptotic relationship between the solutions of systems (I) and (II), where x, y are k-dimensional vectors, $A\colon \mathbb{N}^+_{n_0} \to M^k$ an invertible matrix function for $n \in \mathbb{N}^+_{n_0}$, $F\colon \mathbb{N}^+_{n_0} \times D \times D \to \mathbb{R}^k$ (D — a region in $\mathbb{R}^k$) is for any $n \in \mathbb{N}^+_{n_0}$ continuous with respect to the last two arguments, and T is a continuous operator from $Q(\mathbb{N}^+_{n_0}, D)$ into $Q(\mathbb{N}^+_{n_0}, D)$.

Let $Y(n)$ be a fundamental matrix of (I). The matrix $Y(n) = A(n-1)A(n-2) \dots A(n_0)$ is the fundamental matrix of (I) such that $Y(n_0) = I$.

We can impose various meanings on the operator T.

Let $g(n)$ be a nonsingular $k \times k$ matrix that $g^{-1}(n)$ exists for all $n \in \mathbb{N}^+_{n_0}$.

Definition 1. We will say that a function z is *g-bounded* on $\mathbb{N}^+_{n_0}$ if $\sup\{|g^{-1}(n)z(n)| < \infty, \ n \in \mathbb{N}^+_{n_0}\}$.

Definition 2. We shall say that two systems (I) and (II) are (g,p) $(p \geq 1)$ *asymptotically equivalent* on $\mathbb{N}^+_{n_0}$ if for each solution y of (I) there exists a solution x of (II) such that

$$\big|g^{-1}(n)\big[x(n) - y(n)\big]\big| \in l_p, \tag{III}$$

and conversely.

Let B_g be the space of all functions $x\colon \mathbb{N}^+_{n_0} \to \mathbb{R}^k$ such that

$$|x|_g = \sup_{n \in \mathbb{N}^+_{n_0}} |g^{-1}(n)x(n)| < +\infty .$$

The following theorems will be used in our subsequent discussion:

THEOREM 1. ([1], [2]) *Let C be the Banach space of bounded functions $x\colon J \to Y$ (where Y is a finite dimensional linear space) with the norm $\|x\| = \sup\{|x(t)| : t \in J = \langle t_0, \infty)\}$. Let $G\colon C \to C$ be a contraction, and V_1, V_2 non-empty subsets of C such that $(I - G)V_2 \in V_1$, where I is the identity operator. If $H\colon V_1 \to V_2$ satisfies relation $Hy(t) = y(t) + GHy(t)$, $t \in J$, $y \in V_1$, then H is a homeomorphism of V_1 into V_2.*

THEOREM 2. ([5]) *Suppose that Z is a mapping from a complete metric space $\langle X, d\rangle$ into itself and*

$$d\big(Z(x), Z(y)\big) \le q_0(a,b)d(x,y)$$

for each $(x,y) \in X$ such that $a \le d(x,y) \le b$, where $q_0(a,b) < 1$ for $b \ge a > 0$. Then there exists a unique $u \in X$ such that $u = Z(u)$.

A preliminary result

The following lemma will be used in the sequel.

LEMMA 1. *Let the following conditions be satisfied:*

1° $g(n)$ *is a $k \times k$ matrix such that $g^{-1}(n)$ exists for all $n \in \mathbb{N}^+_{n_0}$,*
2° $\varphi(n)$ *is a positive function for $n \in \mathbb{N}^+_{n_0}$,*
3° $Y(n)$ *is a non-singular matrix for all $n \in \mathbb{N}^+_{n_0}$,*
4° P *is a projection ($P^2 = P$),*
5° $\left(\sum\limits_{s=n_0}^{n} |g^{-1}(n)Y(n)PY^{-1}(s)\varphi(s)|^q\right)^{\frac{1}{q}} \le K < \infty$, $n \in \mathbb{N}^+_{n_0}$, $q \ge 1$, $K = \text{const}$.
6° $\sum\limits_{n=n_0}^{\infty} \exp\Big(-K^{-q} \sum\limits_{s=n_0}^{n} |\varphi^{-1}(s)g(s)|^{-q}\Big) < \infty$, $\varphi^{-1}(s) = \dfrac{1}{\varphi(s)}$, $p+q = pq$.

Then

$$\lim_{n\to\infty} |g^{-1}(n)Y(n)P| = 0, \tag{1}$$

$$|g^{-1}(n)Y(n)P| \in l_p, \qquad p \ge 2. \tag{2}$$

Proof. We follow first the proof due to T. G. Hallam [3] for a differential equation:

Let

$$h(n) = \big(\varphi(n)\big)^q |Y(n)P|^{-q}.$$

Then from the identity

$$\begin{aligned} Y(n)P \sum_{s=n_0}^{n} h(s) &= \sum_{s=n_0}^{n} Y(n)Ph(s) \\ &= \sum_{s=n_0}^{n} |\varphi^{-1}(s)Y(s)P|^{-q} Y(n)PY^{-1}(s)\varphi(s)\varphi^{-1}(s)Y(s)P, \end{aligned}$$

it follows by using Hölder's inequality that

$$\begin{aligned} &|g^{-1}(n)Y(n)P| \sum_{s=n_0}^{n} h(s) \\ &\le \sum_{s=n_0}^{n} |\varphi^{-1}(s)Y(s)P|^{-q} |g^{-1}(n)Y(n)PY^{-1}(s)\varphi(s)| |\varphi^{-1}(s)Y(s)P| \\ &\le \left(\sum_{s=n_0}^{n} |g^{-1}(n)Y(n)PY^{-1}(s)\varphi(s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n_0}^{n} |\varphi^{-1}(s)Y(s)P|^{(1-q)p}\right)^{\frac{1}{p}} \\ &= \left(\sum_{s=n_0}^{n} |g^{-1}(n)Y(n)PY^{-1}(s)\varphi(s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n_0}^{n} |\varphi^{-1}(s)Y(s)P|^{-q}\right)^{\frac{1}{p}}. \end{aligned}$$

Hence

$$|g^{-1}(n)Y(n)P| \le \left(\sum_{s=n_0}^{n} h(s)\right)^{-\frac{1}{q}} \left(\sum_{s=n_0}^{n} |g^{-1}(s)Y(n)PY^{-1}(s)\varphi(s)|^q\right)^{\frac{1}{q}},$$

and, by 5°, we have

$$|g^{-1}(n)Y(n)P| \le K\left(\sum_{s=n_0}^{n} h(s)\right)^{-\frac{1}{q}}. \tag{3}$$

Use the notation

$$\mu(n) = \sum_{s=n_0}^{n} h(s),$$

then

$$|g^{-1}(n)Y(n)P| \le K\big(\mu(n)\big)^{-\frac{1}{q}}. \tag{4}$$

Since

$$|\varphi^{-1}(s)Y(s)Ph(s)| \leq |\varphi^{-1}(s)Y(s)P||\varphi^{-1}(s)Y(s)P|^{-q} = |\varphi^{-1}(s)Y(s)P|^{1-q},$$

we have

$$|\varphi^{-1}(s)Y(s)Ph(s)|^p \leq |\varphi^{-1}(s)Y(s)P|^{-q} = h(s).$$

From the above and 5°, it follows that

$$\begin{aligned} |\varphi^{-1}(n)Y(n)P|\left(\sum_{s=n_0}^{n} h(s)\right)^{\frac{1}{q}} &= \left(\sum_{s=n_0}^{n} |\varphi^{-1}(n)Y(n)P|^{-q}h(s)\right)^{\frac{1}{q}} \\ \leq |\varphi^{-1}(n)g(n)|\left(\sum_{s=n_0}^{n} |g^{-1}(n)Y(n)PY^{-1}(s)\varphi(s)|^q\right)^{\frac{1}{q}} \\ \leq K|\varphi^{-1}(n)g(n)|. \end{aligned} \tag{5}$$

Hence

$$K^{-q}|\varphi^{-1}(n)g(n)|^{-q} \leq h(n)\left(\sum_{s=n_0}^{n} h(s)\right)^{-1}.$$

Since $h(n) = \mu(n) - \mu(n-1)$, it follows that

$$\mu(n) - \mu(n-1) = \left(\varphi^{-1}(n)|Y(n)P|\right)^{-q} \geq K^{-q}\left(\sum_{s=n_0}^{n} h(s)\right)|\varphi^{-1}(n)g(n)|^{-q},$$

and so

$$\mu(n)\left[1 - K^{-q}|\varphi^{-n}(n)g(n)|^{-q}\right] \geq \mu(n-1) \qquad \text{for} \quad n \in \mathbb{N}^+_{n_0}. \tag{6}$$

Using the well-known inequality $1 - u \leq \exp(-u)$, we obtain from (6)

$$\mu(n) \geq \mu(n_0)\exp\left[K^{-q}\sum_{s=n_0}^{n} |\varphi^{-1}(s)g(s)|^{-q}\right]. \tag{7}$$

Note that 6° implies that

$$\sum_{s=n_0}^{\infty} |\varphi^{-1}(s)g(s)|^{-q} = \infty.$$

Thus $\lim \mu(n) = \infty$ as $n \to \infty$, and then (3) yields (1) and

$$\sum_{n=n_0}^{N} |g^{-1}(n)Y(n)P|^p \leq K^p \sum_{n=n_0}^{N} \left(\mu(n)\right)^{-\frac{p}{q}}.$$

By (7), we have

$$\sum_{n=n_0}^{N} |g^{-1}(n)Y(n)P|^p$$
$$\leq K^p\big(\mu(n_0)\big)^{1-p} \sum_{n=n_0}^{N} \exp\left[k^{-q}(1-p) \sum_{s=n_0+1}^{n} |\varphi^{-1}(s)g(s)|^{-q}\right],$$

which, by (6), gives (2). □

Lemma 2. *Let $h(n) \geq 0$ for $n \in \mathbb{N}^+_{n_0}$, and let $\sum\limits_{k=n_0}^{\infty} kh(k) < \infty$. Then $\sum\limits_{k=n}^{\infty} h(k) \in l_p$, $n \in \mathbb{N}^+_{n_0}$, $p > 1$.*

Proof.

(*) If $\sum\limits_{k=n_0}^{\infty} kh(k) < \infty$, then $\sum\limits_{k=n_0}^{\infty} h(k) < \infty$.

Moreover,

$$\sum_{n=n_0}^{\infty}\left(\sum_{k=n}^{\infty} h(k)\right) = \sum_{k=n_0}^{\infty}\left(\sum_{n=n_0}^{\infty} h(k)\right) = \sum_{k=n_0}^{\infty}(k+1-n_0)h(k) < \infty. \tag{8}$$

From (*), we have that $\lim\limits_{n\to\infty} \dfrac{\left(\sum\limits_{k=n}^{\infty} h(k)\right)^p}{\sum\limits_{k=n}^{\infty} h(k)} = 0$, and hence, by (8), the proof of the lemma follows from the comparison principle. □

Asymptotic equivalence

We now prove our main results.

Theorem 3. *If:*

1° *$r\colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing function with respect to each variable separately and such that*

$$\sup\left\{\tfrac{r(u,v)}{\max(u,v)}\,,\ a \leq u\,,\ v \leq b\,,\ 0 < a \leq b\right\} < 1\,,$$

2° *there exist supplementary projections* P_i $(i = 1, 2)$ *and a constant* $K > 0$ *such that*

$$\left(\sum_{s=n_0}^{n-1} |g^{-1}(n)Y(n)P_1Y^{-1}(s+1)|^q \right)^{\frac{1}{q}}$$

$$+ \left(\sum_{s=n}^{\infty} |g^{-1}(n)Y(n)P_2Y^{-1}(s+1)|^q \right)^{\frac{1}{q}} \leq K < \infty \qquad \text{for} \quad n \in \mathbb{N}_{n_0}^+ ,$$

3° *there exist a nonnegative function* h *defined on* $\mathbb{N}_{n_0}^+$ *and positive constants* α, K_1 *such that*

$$\begin{aligned} &\left|F\big(n,u(n),Tu(n)\big) - F\big(n,v(n),Tv(n)\big)\right| \\ &\qquad \leq h(n) r\Big(\big|g^{-1}(n)\big[u(n)-v(n)\big]\big|, \big|Tu(n)-Tv(n)\big|\Big), \end{aligned}$$

$$|T(u) - T(v)| \leq \alpha \big|g^{-1}(n)\big[u(n) - v(n)\big]\big| ,$$

$0 < \alpha \leq 1$, $|u|, |v| < \infty$, $h \in l_p$, $K\left(\sum\limits_{n=n_0}^{\infty} h^p(n) \right)^{\frac{1}{p}} \leq K_1 \leq 1$, $p > 1$, $p + q = pq$,

4° $F(n,0,0) \in l_p$, $p > 1$,

then there exists a homeomorphism H *from the set of* g*-bounded solutions of* (I) *into the* g*-bounded solutions of* (II).

Proof. Let $y = y(n)$ be a g-bounded solution of (I) on $\mathbb{N}_{n_0}^+$. Then there exists a constant $a > 0$ such that $y \in B_{g,a}$, where

$$B_{g,a} = \Big\{ z \in Q : \ \sup\big(|g^{-1}(n)z(n)|\big) \leq a , \ \ n \in \mathbb{N}_{n_0}^+ \Big\} .$$

Define the operator R for $x \in B_{g,2a}$ by

$$\begin{aligned} Rx(n) = y(n) &+ \sum_{s=n_0}^{n-1} Y(n)P_1Y^{-1}(s+1)F\big(s,x(s),Tx(s)\big) \\ &- \sum_{s=n}^{\infty} Y(n)P_2Y^{-1}(s+1)F\big(s,x(s),Tx(s)\big) \qquad \text{for} \quad n \in \mathbb{N}_{n_0}^+ . \end{aligned} \tag{9}$$

Write $d_i(n,s) = g^{-1}(n)Y(n)P_iY^{-1}(s+1)$, $i=1,2$, then

$$
\begin{aligned}
&|g^{-1}(n)Rx(n)| \\
&\le a + \sum_{s=n_0}^{n-1} \left|d_1(n,s)F\big(s,x(s),Tx(s)\big)\right| + \sum_{s=n}^{\infty}\left|d_2(n,s)F\big(s,x(s),Tx(s)\big)\right| \\
&\le a + \sum_{s=n_0}^{n-1} |d_1(n,s)|\Big(h(s)r\big(|g^{-1}(s)x(s)|,\, |g^{-1}(s)x(s)|\big)\Big) \\
&\quad + \sum_{s=n}^{\infty} |d_2(n,s)|\Big(h(s)r\big(|g^{-1}(s)x(s)|,\, |g^{-1}(s)x(s)|\big)\Big) \\
&\quad + \sum_{s=n_0}^{n-1} |d_1(n,s)||F(s,0,0)| + \sum_{s=n}^{\infty} |d_2(n,s)||F(s,0,0)| \\
&\le a + r(2a,2a)\left\{ \sum_{s=n_0}^{n-1} |d_1(n,s)|h(s) + \sum_{s=n}^{\infty} |d_2(n,s)|h(s) \right\} \\
&\quad + \sum_{s=n_0}^{n-1} |d_1(n,s)||F(s,0,0)| + \sum_{s=n}^{\infty} |d_2(n,s)||F(s,0,0)| \\
&\le a + r(2a,2a)\left\{ \left(\sum_{s=n_0}^{n-1} |d_1(n,s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n_0}^{n-1} h^p(s)\right)^{\frac{1}{p}} \right. \\
&\quad + \left(\sum_{s=n}^{\infty} |d_2(n,s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n}^{\infty} h^p(s)\right)^{\frac{1}{p}} \\
&\quad + \left(\sum_{s=n_0}^{n-1} |d_1(n,s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n_0}^{n-1} |F(s,0,0)|^p\right)^{\frac{1}{p}} \\
&\quad \left. + \left(\sum_{s=n}^{\infty} |d_2(n,s)|^q\right)^{\frac{1}{q}} \left(\sum_{s=n}^{\infty} |F(s,0,0)|^p\right)^{\frac{1}{p}} \right\}.
\end{aligned}
$$

If we choose n_0 such that

$$
r(2a,2a)K\left(\sum_{n=n_0}^{\infty} h^p(s)\right)^{\frac{1}{p}} \le \frac{a}{2}
$$

and

$$
K\left(\sum_{n=n_0}^{\infty} |F(s,0,0)|^p\right)^{\frac{1}{p}} \le \frac{a}{2},
$$

we have that R maps $B_{g,2a}$ into itself. Now, using Theorem 2, we are going to demonstrate that the operator R has a unique fixed point in $B_{g,2a}$.

For $x_1, x_2 \in B_{g,2a}$, we have

$$\begin{aligned}
&\left|g^{-1}(n)\left[Rx_1(n) - Rx_2(n)\right]\right| \\
&\le \sum_{s=n_0}^{n-1} |d_1(n,s)|\left|F\big(s, x_1(s), Tx_1(s)\big) - F\big(s, x_2(s), Tx_2(s)\big)\right| \\
&\quad + \sum_{s=n}^{\infty} |d_2(n,s)|\left|F\big(s, x_1(s), Tx_1(s)\big) - F\big(s, x_2(s), Tx_2(s)\big)\right| \\
&\le K\left(\sum_{n=n_0}^{\infty} h^p(n)\right)^{\frac{1}{p}} r\big(|x_1 - x_2|_g, |x_1 - x_2|_g\big).
\end{aligned}$$

Hence

$$|Rx_1 - Rx_2|_g \le r\big(|x_1 - x_2|_g, |x_1 - x_2|_g\big).$$

Thus we can apply Theorem 2, which yields the existence of a unique $x \in B_{g,2a}$ such that $x = Rx$. An easy computation shows that the fixed point $x(n) = Rx(n)$, $n \in \mathbb{N}^+_{n_0}$ is a solution of (II). Let $B_{g,I}$ and $B_{g,II}$ denote the species of g-bounded solutions of (I) and (II), respectively. We define the mapping $H\colon B_{g,I} \to B_{g,II}$ as follows: for every $y \in B_{g,I}$, Hy is the fixed point of the contraction R. This means $Hy(n) = RHy(n)$. We prove that H is homeomorphism. For this purpose, let $y_1, y_2 \in B_{g,I}$ be such that $Hy_1 = Hy_2$. Then we obtain $y_1 = y_2$. Moreover, H is continuous.

Next we define the inverse mapping of H, $H^{-1}\colon B_{g,II} \to B_{g,I}$, by

$$H^{-1}x(n) = x(n) - R_1x(n),$$

where

$$\begin{aligned}
R_1x(n) = &\sum_{s=n_0}^{n-1} Y(n)P_1Y^{-1}(s+1)F\big(s, x(s), Tx(s)\big) \\
&- \sum_{s=n}^{\infty} Y(n)P_2Y^{-1}(s+1)F\big(s, x(s), Tx(s)\big).
\end{aligned}$$

H^{-1} is one to one, continuous mapping. □

THEOREM 4. *If:*

1° *the assumptions of Theorem 3 hold,*

2° $\sum_{n=n_0}^{\infty} |P_1 Y^{-1}(n+1)| h(n) < +\infty$, $\sum_{n=n_0}^{\infty} |P_1 Y^{-1}(n+1)| |F(n,0,0)| < +\infty$,

3° $\sum_{n=n_0}^{\infty} n h(n) < \infty$, $\sum_{n=n_0}^{\infty} n |F(n,0,0)| < \infty$,

then $\left| g^{-1}(n) \left[Hy(n) - y(n) \right] \right| \in l_p$.

P r o o f. From (8) and the assumptions of the theorem, we have

$$\begin{aligned}
&\left| g^{-1}(n) \left[Hy(n) - y(n) \right] \right| \\
&\le \sum_{s=n_0}^{n-1} |g^{-1}(n) Y(n) P_1 Y^{-1}(s+1)| \left| F\big(s, Hy(s), THy(s)\big) \right| \\
&\quad + \sum_{s=n}^{\infty} |g^{-1}(n) Y(n) P_2 Y^{-1}(s+1)| \left| F\big(s, Hy(s), THy(s)\big) \right| \\
&\le |g^{-1}(n) Y(n) P_1| \left\{ r(2a,2a) \sum_{s=n_0}^{n-1} |P_1 Y^{-1}(s+1)| h(s) \right. \qquad (10) \\
&\quad \left. + \sum_{s=n_0}^{n-1} |P_1 Y^{-1}(s+1)| |F(s,0,0)| \right\} \\
&\quad + r(2a,2a) \sum_{s=n}^{\infty} |g^{-1}(n) Y(n) P_2 Y^{-1}(s+1)| h(s) \\
&\quad + \sum_{s=n}^{\infty} |g^{-1}(n) Y(n) P_2 Y^{-1}(s+1)| |F(s,0,0)| .
\end{aligned}$$

Hence

$$\begin{aligned}
&|g^{-1}(n) Y(n) P_1| \left\{ r(2a,2a) \sum_{s=n_0}^{n-1} |P_1 Y^{-1}(s+1)| h(s) \right. \\
&\qquad\qquad \left. + \sum_{s=n_0}^{n-1} |P_1 Y^{-1}(s+1)| |F(s,0,0)| \right\} \\
&\le |g^{-1}(n) Y(n) P_1| \left\{ r(2a,2a) \sum_{s=n_0}^{\infty} |P_1 Y^{-1}(s+1)| h(s) \right. \\
&\qquad\qquad \left. + \sum_{s=n_0}^{\infty} |P_1 Y^{-1}(s+1)| |F(s,0,0)| \right\} .
\end{aligned}$$

Since (from Lemma 1) $|g^{-1}(n)Y(n)P_1| \in l_p$, it is evident that this first term in the inequality (10) belongs to l_p. Taking in to account the second term of the above inequality, we obtain

$$\begin{aligned}
&r(2a,2a)\sum_{s=n}^{\infty}|g^{-1}(n)Y(n)P_2Y^{-1}(s+1)|h(s)\\
&\qquad+\sum_{s=n}^{\infty}|g^{-1}(n)Y(n)P_2Y^{-1}(s+1)||F(s,0,0)|\\
&\le r(2a,2a)\left(\sum_{s=n}^{\infty}|g^{-1}(n)Y(n)P_2Y^{-1}(s+1)|^q\right)^{\frac{1}{q}}\left(\sum_{s=n}^{\infty}h^p(s)\right)^{\frac{1}{p}}\\
&\qquad+\left(\sum_{s=n}^{\infty}|g^{-1}(n)Y(n)P_2Y^{-1}(s+1)|^q\right)^{\frac{1}{q}}\left(\sum_{s=n}^{\infty}|F(s,0,0)|^p\right)^{\frac{1}{p}}\\
&\le Kr(2a,2a)\left(\sum_{s=n}^{\infty}h^p(s)\right)^{\frac{1}{p}}+K\left(\sum_{s=n}^{\infty}|F(s,0,0)|^p\right)^{\frac{1}{p}}.
\end{aligned}$$

Also from 3°, this second term belongs to l_p.

The proof of the theorem is complete. □

THEOREM 5. *If:*

1° *the assumptions of Theorem* 3 *hold,*

2° $\sum_{n=n_0}^{\infty}\exp\left(-K^{-q}\sum_{s=n_0}^{n}|g(s)|^{-q}\right)<\infty$,

then

$$\lim_{n\to\infty}\left|g^{-1}(n)\left[Hy(n)-y(n)\right]\right|=0. \tag{11}$$

P r o o f. To verify that (11) holds, observe that

$$\left|g^{-1}(n)\left[Hy(n)-y(n)\right]\right|\le A+B,$$

where

$$A=\sum_{n=n_0}^{n-1}\left|Y(n)P_1Y^{-1}(s+1)F\big(s,Hy(s),THy(s)\big)\right|,$$

$$B=\sum_{s=n}^{\infty}\left|Y(n)P_2Y^{-1}(s+1)F\big(s,Hy(s),THy(s)\big)\right|.$$

Using the assumptions of Theorem 3 and Hölder's inequality we get

$$B\le Kr(2a,2a)\left(\sum_{s=n}^{\infty}h^p(s)\right)^{\frac{1}{p}}+K\left(\sum_{s=n}^{\infty}|F(s,0,0)^p|\right)^{\frac{1}{p}}<\frac{\varepsilon}{2} \tag{12}$$

for $n \in \mathbb{N}_{n_1}^{+}$, where $n_1 \in \mathbb{N}_{n_0}^{+}$ is sufficiently large.

Moreover, for $n_2 \in \mathbb{N}_{n_1}^{+}$, from Lemma 1 and 1°, we have

$$\begin{aligned} A = & \sum_{s=n_0}^{n_2-1} \left| g^{-1}(n)Y(n)P_2Y^{-1}(s+1)F\big(s, Hy(s), THy(s)\big) \right| \\ & + \sum_{s=n_2}^{n-1} \left| g^{-1}(n)Y(n)P_1Y^{-1}(s+1)F\big(s, Hy(s), THy(s)\big) \right| \\ \le & \left| g^{-1}(n)Y(n)P_1 \right| \sum_{s=n_0}^{n_2-1} \left| P_1Y^{-1}(s+1)F\big(s, Hy(s), THy(s)\big) \right| \\ & + r(2a, 2a)K \left(\sum_{s=n_2}^{n-1} h^p(s) \right)^{\frac{1}{p}} + K \left(\sum_{s=n_2}^{n-1} |F(s,0,0)|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{2} \end{aligned} \tag{13}$$

for $n \in \mathbb{N}_{n_2}^{+}$ and n_2 sufficiently large. From (12) and (13) we obtain (11). □

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