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VECTOR–VALUED FUZZY MEASURES ON FUZZY QUANTUM POSETS

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ABSTRACT. The notion of a Hilbert space-valued fuzzy measure on fuzzy quantum posets is studied. Some results about the relation among fuzzy measures, Hilbert space-valued fuzzy measures and fuzzy morphism are mentioned, too.

I. Introduction

Vector-valued measures on orthocomplemented lattices or on quantum logics have been studied by several authors, e.g. [2], [3], [4], [5], [9]. In [11] the authors have proved that a state m on L can be expressed in the form $\|\xi(a)\|^2 = m(a)$, where ξ is a vector-valued measure on a quantum logic L, if and only if there exists a kernel function $K\colon L \times L \to \mathbb{R}$ satisfying some properties. In [5] a representation of a vector-valued measure on L by a morphism $\Phi\colon L \to L(H)$, ($L(H)$ being the lattice of closed subspaces of H), via $\xi(a) = \Phi(a) \cdot x$ (where $x \in H$) is pointed out. In the present paper similar results are given on fuzzy quantum posets. Moreover, the existence of a kernel function K in several cases is mentioned, too.

Let a, b be two fuzzy elements from $[0,1]^\Omega$, where Ω is a given non-void set.

(i) a and b are said to be *orthogonal* and we write $a \perp b$, if and only if

$$a + b \le 1 .$$

(ii) a and b are said to be *fuzzy orthogonal* and we write $a \perp_F b$ if and only if

$$a \cap b := \inf(a,b) \le 1/2 .$$

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It is evident that if $a \perp b$ then $a \perp_F b$. Moreover, if $a \cup a^\perp = b \cup b^\perp$ then $a \perp b$ if and only if $a \perp_F b$.

Let Ω be a non-void set and $M \subseteq [0;1]^\Omega$ such that:

(i) If $\mathbf{1}(\omega) = 1$ for any $\omega \in \Omega$, then $\mathbf{1} \in M$;

(ii) if $a \in M$, then $a^\perp := 1 - a \in M$;

(iii) if $\mathbf{1/2}(\omega) = 1/2$ for any $\omega \in \Omega$, then $\mathbf{1/2} \notin M$.

A couple (Ω, M) is said to be a *type I, type II fuzzy quantum poset* if M is closed with respect to a union of any sequence of fuzzy sets mutually fuzzy orthogonal, orthogonal, respectively.

If M is closed with respect to any sequence of fuzzy sets of M, then (Ω, M) is said to be a *fuzzy quantum space.*

It is obvious that a fuzzy quantum space is a fuzzy quantum poset type I, II and a fuzzy quantum poset type I is type II but the converse is not true, in general, (See [9]).

Let (Ω, M) be a type I (type II) fuzzy quantum poset. By a *fuzzy measure of type I (type II) on M* we understand a mapping $m\colon M \to [0;\infty)$ satisfying:

(i) $m(a) + m(a^\perp) = m(1)$ for any $a \in M$,

(ii) $m\Big(\bigcup\limits_{n=1}^{\infty} a_n\Big) = \sum\limits_{n=1}^{\infty} m(a_n)$,

for every sequence $\{a_n\}_{n=1}^{\infty} \subseteq M$, $a_n \perp_F a_k$, $(a_n \perp a_k)$ for $n \neq k$, resp.

If $m(1) = 1$, then m is called to be a *fuzzy state of type I (type II) on M*.

Let m be a fuzzy measure of type I on a type I fuzzy quantum poset (Ω, M). It is known that a type I fuzzy quantum poset is a type II. Therefore, if we consider (Ω, M) as a type II, then we can prove that m is a fuzzy measure of type II, too. Based on this note from here we can understand a fuzzy measure of type i on a type i fuzzy quantum poset by a fuzzy measure on a type i fuzzy quantum poset, $i = 1, 2$.

II. Vector-valued measure on fuzzy quantum poset

DEFINITION 2.1. *Let (Ω, M) be a type I (type II) fuzzy quantum poset, H be a Hilbert space. An H-valued fuzzy measure on M is a mapping $\xi\colon M \to H$ such that:*

(i) $\xi(a \cup a^\perp) = \xi(1)$ *for any* $a \in M$,

(ii) *if* $a \perp_F b$ $(a \perp b)$, *then* $\xi(a) \perp \xi(b)$,

(iii) *if* $\{a_i\}_{i=1}^{\infty} \subset M$, $a_i \perp_F a_j$ $(a_i \perp a_j)$, *then*

$$\xi\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} \xi(a_i),$$

where the series on the right-hand converges in the norm in H *.* ξ *is called an* H*-valued fuzzy state if* $\|\xi(1)\| = 1$ *.*

It is evident that if ξ is an H-valued fuzzy measure, then the mapping m defined via:

$$m(a) = \|\xi(a)\|^2 \qquad \text{for any} \quad a \in M \tag{2.1}$$

is a fuzzy measure on M .

On a fuzzy quantum space, every fuzzy measure can be expressed in the form (2.1). Indeed, suppose (Ω, M) to be a fuzzy quantum space, m to be a fuzzy measure on M .

Let $K(M)$ be a family of all $A \subseteq \Omega$ for which there exists $a \in M$ such that

$$\{a > 1/2\} \subseteq A \subseteq \{a \geq 1/2\}\,, \tag{2.2}$$

where $\{a > 1/2\} := \{\omega \in \Omega;\ a(\omega) > 1/2\}$, analogically for $\{a \geq 1/2\}$ (See also P i a s e c k i [10]). Due to Theorem 2.1 by D v u r e č e n s k i j [1] and Remark of Theorem 2.7 [8], $K(M)$ is a σ-algebra and $P_m\colon K(M) \to [0,\infty)$ defined via $P_m(A) = m(a)$, where A, a satisfy (2.2), is a usual probability on $K(M)$.

Consider $\xi\colon M \to L_2\big(\Omega, K(M), P_m\big)$, $a \mapsto I_{\langle a>1/2\rangle}$.

It is easy to see that ξ is an $L_2\big(\Omega, K(M), P_m\big)$-valued fuzzy measure with (2.1).

THEOREM 2.2. *Let* (Ω, M) *be a type I (II) of fuzzy quantum poset and* m *be a fuzzy measure on* M *. Then there is a real Hilbert space* H *and an* H*-valued fuzzy measure* ξ *on* M *with* (2.1) *if and only if there is a mapping* $G\colon M \times M \to \mathbb{R}$ *such that:*

$$\begin{aligned} &\text{(i)} && G(a,b) = 0 \ \text{for every } a \perp_F b \ (a \perp b),\\ &\text{(ii)} && G(a,b) = G(b,a) \ \text{for any } a,b \in M,\\ &\text{(iii)} && G(a,b) = m(a) \ \text{if } a \leq b,\\ &\text{(iv)} && \sum_{i,j} \alpha_i \alpha_j G(a_i,a_j) \geq 0 \ \text{for any } \alpha_i \in \mathbb{R},\ a_i \in M,\ i \leq n,\ n \geq 1. \end{aligned} \tag{2.3}$$

P r o o f . If ξ exists, we put $G(a,b) = \big(\xi(a), \xi(b)\big)$. Then it is evident that (i), (ii), (iv) hold. (iii) follows from the observation that if $a \leq b$, then $\xi(a) + \xi(a^\perp \cap b) = \xi(a) + \xi(1) - \xi(a \cup b^\perp) = \xi(a) + \xi(1) - \xi(a) + \xi(b^\perp) = \xi(b)$.

Conversely, let G with properties (i) – (iv) be given. Then there is a measure space (X, S, P) and a centered Gaussian process $\{\xi(a);\ a \in M\}$ with the covariance function equal to G (See L o e v e [7, p. 489]). We claim to show that $a \to \xi(a)$ is an $L_2(X,S,P)$-valued fuzzy measure in question.

(i) implies $(\xi(a),\xi(b))=0$ for any $a\perp_F b$ $(a\perp b)$.

For any $a\in W_1(M)=\{a\in M;\ \ a=a\cup a^\perp\}$ we have that:

$$\begin{aligned}\|\xi(a)-\xi(1)\|^2&=\|\xi(a)\|^2+\|\xi(1)\|^2-2(\xi(a),\xi(1))\\&=m(a)+m(1)-2m(a)=0\,.\end{aligned}$$

Hence, $\xi(a)=\xi(1)$, which entails $\xi(a\cup a^\perp)=\xi(1)$ for any a $\in M$.

Now, if $a\perp_F b$ $(a\perp b)$, then

$$\begin{aligned}&\|\xi(a\cup b)-\xi(a)-\xi(b)\|^2\\&=\|\xi(a\cup b)\|^2+\|\xi(a)\|^2+\|\xi(b)\|^2-2(\xi(a\cup b),\xi(a))\\&\qquad\qquad-2(\xi(a\cup b),\xi(b))+2(\xi(a),\xi(b))\\&=G(a\cup b,a\cup b)+G(a,a)+G(b,b)-2G(a\cup b,a)-2G(a\cup b,b)+2G(a,b)\\&=m(a\cup b)+m(a)+m(b)-2m(a)-2m(b)=0\,.\end{aligned}$$

Thus,

$$\xi(a\cup b)=\xi(a)+\xi(b)\,.$$

By induction we have $\xi(a_1\cup\cdots\cup a_n)=\sum\limits_{i=1}^{n}\xi(a_i)$, whenever $a_i\perp_F a_j$ $(a_i\perp a_j)$, $i,j=1,2,\dots,n$, $i\neq j$.

Now, let $a=\bigcup\limits_{i=1}^{\infty}a_i$, $a_i\perp_F a_j$ $(a_i\perp a_j)$, $i,j=1,2,\dots$. Similarly,

$$\begin{aligned}\Big\|\xi(a)-\sum_{i=1}^{n}\xi(a_i)\Big\|&=\|\xi(a)\|^2+\sum_{i=1}^{n}\|\xi(a_i)\|^2-2\sum_{i=1}^{n}(\xi(a),\xi(a_i))\\&=m(a)-\sum_{n=1}^{n}m(a_i)\to 0\qquad\text{when}\quad n\to\infty\,.\end{aligned}$$

Hence, $\xi(a)=\sum\limits_{i=1}^{\infty}\xi(a_i)$. □

THEOREM 2.3. *Let (Ω,M) be a type I (II) of fuzzy quantum poset. Let m be a fuzzy state on M such that:*

(i) *if $a\perp_F b$ $(a\perp b)$ and $\max(m(a),m(b))\le 1/2$, then*

$$m(a)\cdot m(b)=0\,; \tag{2.4}$$

(ii) *if $m(a)<1/2$ and there is $b\in M$ such that $a<b$, $1/2\le m(b)<1$, then $m(a)=0$.*

Then, there is a function $G\colon M\times M\to\mathbb{R}$ *with* (2.3). *Therefore there exists a real Hilbert space-valued fuzzy state* ξ *with* (2.1).

P r o o f. Put

$$M_1=\{a\in M;\ \ m(a)\ge 1/2\},$$
$$M_0=\{a\in M;\ \ m(a)<1/2\}.$$

Consider $G\colon M\times M\to\mathbb{R}$ defined via

$$G(a,b)=G(b,a)=\begin{cases}\min\big(m(a),m(b)\big) & \text{if } a,b\in M_0 \text{ or } a,b\in M_1,\\ 0 & \text{if } a\in M_0,\ \ b\in M_1,\ \ m(b)<1,\\ m(a) & \text{if } a\in M_0,\ \ m(b)=1.\end{cases}$$

We claim to show that G fulfils (2.3). The properties (i), (ii), (iii) are evident.

Calculate $\sum\alpha_i\alpha_j G(a_i,a_j)$, with given $a_1,\dots,a_n$, $n\in\mathbb{N}$. They can be numbered such that $0=m(a_1)\le m(a_2)\le\dots\le m(a_n)=1$. Then the matrix of the above quadric can be written in the following form:

$$\left[\begin{array}{cccccccccccccc}
0 & & 0 & & & & 0 & & & & & 0 & & 0\\
 & \mathbf{0} & & & & \mathbf{0} & & & & \mathbf{0} & & & \mathbf{0} & \\
0 & & 0 & & & & 0 & & & & & 0 & & 0\\
 & & & m_{k+1} & \cdots & \cdots & m_{k+1} & & & & & m_{k+1} & \cdots & m_{k+1}\\
 & & & \vdots & m_{k+2} & \cdots & m_{k+2} & & & & & m_{k+2} & \cdots & m_{k+2}\\
 & & & \vdots & \vdots & \cdots & \cdots & & & \mathbf{0} & & \cdots & \cdots & \cdots\\
 & & & \vdots & \vdots & \vdots & & & & & & & & \\
0 & & 0 & m_{k+1} & m_{k+2} & \vdots & m_h & & & & & m_h & \cdots & m_h\\
 & & & & & & & m_{h+1} & \cdots & \cdots & \cdots & m_{h+1} & \cdots & m_{h+1}\\
 & & & & & & & & m_{h+2} & \cdots & \cdots & m_{h+2} & \cdots & m_{h+2}\\
 & \mathbf{0} & & & & \mathbf{0} & & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
 & & & & & & & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots\\
0 & & 0 & m_{k+1} & m_{k+2} & & m_h & m_{h+1} & m_{h+2} & \vdots & \vdots\ 1 & \cdots & \cdots & 1\\
 & & & & & & & \vdots & \vdots & \vdots & \vdots\ \vdots & 1 & \cdots & 1\\
 & & & & & & & \vdots & \vdots & \vdots & \vdots\ \vdots & \vdots & & \\
0 & & 0 & m_{k+1} & m_{k+2} & & m_h & m_{h+1} & m_{h+2} & \vdots & \vdots\ 1 & 1 & & 1
\end{array}\right],$$

where $m(a_i)=m_i$, $m(a_h)=m_h<1/2$, $m_{h+1}\ge 1/2$.

Without loss of generality we may assume that:

$$m_{k+1} < m_{k+2} < \cdots < m_h < m_{h+1} < \dots .$$

Indeed, if $m_j = m_{j+1}$, we replace m_{j+1} by $m'_{j+1} = m_j + \varepsilon$ with ε small. By calculating the determinant of the first corner matrixes we can prove that:

$$\begin{bmatrix}
\varepsilon_1 & & & & & & & & & & & \\
 & & \mathbf{0} & & \mathbf{0} & & & & \mathbf{0} & & \mathbf{0} & \\
\mathbf{0} & & & & & & & & & & & \\
 & \varepsilon_k & & & & & & & & & & \\
 & & m_{k+1} & \cdots & \cdots & m_{k+1} & & & & m_{k+1} & \cdot\cdot & m_{k+1} \\
 & & \vdots & m_{k+2} & \cdots & m_{k+2} & & & & m_{k+1} & \cdot\cdot & m_{k+2} \\
\mathbf{0} & & \vdots & \vdots & \cdots & \cdots & & & \mathbf{0} & \cdots & \cdots & \cdots \\
 & & \vdots & \vdots & \vdots & & & & & & & \\
 & & m_{k+1} & m_{k+2} & & m_h & & & & m_h & \cdot\cdot & m_h \\
 & & & & & & m_{h+1} & \cdots & \cdots & m_{h+1} & \cdot\cdot & m_{h+1} \\
 & & & & & & \vdots & m_{h+2} & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & & & \mathbf{0} & & & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots \\
 & & & & & & \vdots & \vdots & \vdots & & & \\
 & & m_{k+1} & m_{k+2} & & m_h & m_{h+1} & \vdots & \vdots & 1-\varepsilon_{k+1} & \cdot\cdot & 1-\varepsilon_{k+1} \\
\mathbf{0} & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots \\
 & & m_{k+1} & m_{k+2} & & m_h & m_{h+1} & \vdots & \vdots & 1-\varepsilon_{k+1} & \vdots & 1
\end{bmatrix}$$

is a matrix of a positively definite quadric. Therefore, limiting $\varepsilon_1, \varepsilon_2, \dots \to 0$, we see that $\sum \alpha_i \alpha_j G(a_i, a_j) \geq 0$ for any $\{\alpha_i\} \subseteq \mathbb{R}$. This means that G fulfils (iv) of (2.3). □

COROLLARY 2.4. *Let (Ω, M) be a type II fuzzy quantum poset and m be a $\{0,1\}$-valued fuzzy state on M. Then m can be expressed in the form* (2.1).

P r o o f. It is clear that m fulfils the condition (2.4). □

COROLLARY 2.5. *Let (Ω, M) be a type II fuzzy quantum poset such that $a, b \in M$, $a \le b$, $a \ne b \ne 1$ and $b \ge 1/2$ imply $a \le 1/2$. Then, every fuzzy state on M can be expressed in the form (2.1). In particular, suppose C to be a q-σ-algebra of subsets of given set Ω such that $A, B \in C$, $A \subset B \ne \Omega$, $A \ne B$ implies $A = \emptyset$. Then (Ω, M), where $M = \{I_A;\ \ A \in C\}$, fulfils the above condition.*

Proof. It is evident that every fuzzy state on M always fulfils (2.4). □

Example 2.6. Put $\Omega = \{1,2,3,4\}$. Let C be system of all even subsets of Ω, then (Ω, M) fulfils the condition of Corollary 2.5.

Remark 2.7. Theorem 2.3, Corollary 2.4, 2.5 and Example 2.6 are still in validity if a fuzzy state m is replaced by any fuzzy measure, in which 1 and $1/2$ are replaced by $m(1)$ and $m(1)/2$.

III. A representation of a vector-valued fuzzy measure

Let H be a Hilbert space, $L(H)$ be the set of all orthogonal projections in H. Then, $L(H)$ is a logic and it coincides with the logic of closed subspaces of H (See [12, p. 190-192]).

Now, let (Ω, M) be a type I (II) fuzzy quantum poset.

A mapping $\Phi\colon M \to L(H)$ is called a *fuzzy morphism* if:

(i) $\Phi(a \cup a^\perp) = \Phi(1)$ for any $a \in M$,
(ii) $a, b \in M$, $a \perp_F b$ $(a \perp b)$ implies $\Phi(a) \perp \Phi(b)$.

A fuzzy morphism $\Phi\colon M \to L(H)$ is a *fuzzy σ-morphism* if $\Phi\Big(\bigcup_{i=1}^{\infty} a_i\Big) = \bigvee_{i=1}^{\infty} \Phi(a_i)$ for any sequence $\{a_i\}_{i=1}^{\infty}$ of mutually fuzzy orthogonal (orthogonal) elements of M.

According to Kruszynski [6], two H-valued fuzzy measures ξ, η on M are said to be biorthogonal if for every $a, b \in M$, $a \perp_F b$ $(a \perp b)$ we have $\big(\xi(a), \eta(b)\big) = 0$.

It is evident that ξ, η are biorthogonal if and only if $\alpha\xi + \beta\eta$ is also an H-valued fuzzy measure for any nonnegative real numbers α, β.

A family N of H-valued fuzzy measures on M is said to be *biorthogonal* if every two measures $\xi, \eta \in N$ are biorthogonal. A biorthogonal family N is a *maximal biorthogonal* family if every H-valued fuzzy measure on M, which is biorthogonal to every member of N, necessarily belongs to N. It is clear that every maximal biorthogonal family is a linear space. Obviously, a maximal biorthogonal family is maximal with respect to the ordering by the set inclusion in the class of biorthogonal families of H-valued fuzzy measures. Hence, every biorthogonal family of H-valued fuzzy measures is contained in some maximal family.

THEOREM 3.1. *Let (Ω, M) be a type I (II) of fuzzy quantum poset and let H be a real Hilbert space, $\Phi\colon M \to L(H)$ be a fuzzy σ-morphism. Then:*

(i) *If $v \in H$, then the mapping ξ_v defined via*

$$\xi_v(a) = \Phi(a)v \qquad \text{for any} \quad a \in M \tag{3.1}$$

is an H-valued fuzzy measure on M.

(ii) *$N = \{\xi_v ;\ \ v \in \Phi(1)H\}$ is a biorthogonal family of H-valued fuzzy measures on M.*

(iii) *N is a maximal biorthogonal family of $\Phi(1)H$-valued fuzzy measures on M.*

Proof. (i), (ii) follow immediately from the definitions.

(iii): Let η be a $\Phi(1)H$-valued fuzzy measure orthogonal to ξ_v, for any $v \in \Phi(1)H$. This means that $\eta(a) \perp \Phi(a^\perp)v$ for any $v \in \Phi(1)H$ and $a \in M$. So $\eta(a) \perp \Phi(a^\perp)H$. On the other hand, $\Phi(1)H = \Phi(a)H + \Phi(a^\perp)H$, and $\Phi(a)H \perp \Phi(a^\perp)H$.

Hence, $\eta(a) \in \Phi(a)H$ for any a $\in M$. So, $\eta(a) = \Phi(a)\eta(a) = \Phi(a)\eta(a) + \Phi(a)\eta(a^\perp) = \Phi(a)\eta(1)$, which entails $\eta \in N$. □

The following result for a fuzzy quantum poset is similar to Proposition 3.6 by Kruszynski [6] and Theorem 2.7 by Pulmannová and Dvurečenskij [11].

THEOREM 3.2. *Let (Ω, M) be a type I (II) of a fuzzy quantum poset and let H be a real Hilbert space. Let N be a maximal biorthogonal family of H-valued fuzzy measures on M. For any $a \in M$ put $N(a) = \{\xi(a) ;\ \ a \in M\}$. Then, the following statements hold:*

(i) *For every* $a \in M$, $N(a)$ *is a closed linear subspace of* H;

(ii) *for every* $a, b \in M$, $a \perp_F b$ $(a \perp b)$, *we have* $N(a) \perp N(b)$ *and* $N(a \cup b) = N(a) \vee N(b)$, *i.e.* $\Phi(a \cup b) = \Phi(a) + \Phi(b)$, *where* $\Phi(a)$ *denotes the projection on* $N(a)$. *In addition, for every sequence* $\{a_i\}_{i=1}^{\infty}$ *of mutually orthogonal elements of* M *we have:*

$$\Phi\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} \Phi(a_i);$$

(iii) *for every* $\xi \in N$ *we have* $\xi(a) = \Phi(a)\xi(1)$, *for* $a \in M$;

(iv) $\Phi(a \cup a^{\perp}) = \Phi(1)$, *for any* $a \in M$.

In other words, Φ *is a fuzzy* σ*-morphism on* M *and* ξ *is represented in the form* (3.1).

P r o o f. (i), (ii) can be proved in the same way as the Proposition 3.5 [6]. (iv) is evident from the definition of $N(a)$, $a \in M$.

(iii): For every $\xi \in N$, a $\in M$, we have $\xi(1) = \xi(a \cup a^{\perp}) = \xi(a) + \xi(a^{\perp})$, where $\xi(a) \in N(a)$ and $\xi(a^{\perp}) \perp N(a)$, since $\xi(a^{\perp}) \perp \eta(a)$ for any $\eta \in N$. Hence, $\xi(a) = \Phi(a)\xi(1)$. □

R e m a r k. In view of Theorems 3.1 and 3.2, it can be pointed out that there is a one-to-one correspondence between the set of all maximal biorthogonal families of H-valued fuzzy measures on M and the set of morphisms Φ from M into $L(H)$ such that $\Phi(1)H = H$.

REFERENCES

[1] DVUREČENSKIJ, A.: *Models of fuzzy quantum spaces.* (Slovak) In: Proceedings PROBASTAT'89, MÚ SAV, Bratislava, 1989, pp. 96–96.

[2] DVUREČENSKIJ, A.: *On existence of probability measures on fuzzy measurable spaces,* Fuzzy Sets and Systems **43** (1991), 173–181.

[3] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *Random measures on a logic,* Demonstratio Math. **14** (1989), 305–320.

[4] DVUREČENSKIJ, A.—PULMANNOVÁ, S.: *State on splitting subspaces and completeness of inner product spaces,* Internat. J. Theoret. Phys. **27** (1988), 1059–1067.

[5] HAMHALTER, J.—PTÁK, P.: *Hilbert Space Valued States on Quantum Logics.* Preprint, ČVUT, Praha, 1989.

[6] KRUSZYNSKI, P: *Vector measures on orthocomplemented lattices,* Math. Proc. A **91** (1988), 427–442.

[7] LOEVE, M.: *Probability Theory.* (Russian translation: Teorija rešetok), Izd. Inostr. Lit., Moskva, 1962.

[8] LONG, L. B.: *Fuzzy quantum posets and their states,* Acta Math. Univ. Comenian. **58–59** (1991), 231-238.

[9] LONG, L. B.: *A new approach to representation of observables on fuzzy quantum posets*, Appl. Math. **37** (1992), 357–368.

[10] PIASECKI, K.: *On some relation between fuzzy probability measure and fuzzy P-measure*, BUSEFAL **23** (1985), 73–77.

[11] PULMANNOVÁ, S.—DVUREČENSKIJ, A.: *Quantum logics, vector valued measures and representation*, Ann. Inst. H. Poincaré Probab. Statist. **53** (1990), 83–95.

[12] PTÁK, P.—PULMANNOVÁ, S.: *Quantum Logics.* (Slovak), Veda, Bratislava, 1989.

[13] PYKACZ, J.: *Quantum logics and soft fuzzy probability spaces*, BUSEFAL **32** (1987), 150–157.

[14] RIEČAN, B.: *A new approach to some notions of statistical quantum mechanics*, BUSEFAL **35** (1988), 4–6.

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