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Mathematica Slovaca, Vol. 39 (1989), No. 1, 81--89

Persistent URL: http://dml.cz/dmlcz/130769

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1. Introduction

In [10], Roy A. Johnson studied atomic and nonatomic measures. In the present paper some generalizations of these results are presented, both in the case of nonnegative measures and for a more general type of set functions with different ranges. The main results of the paper are: An expression of the set of all atoms of a set function which is the sum of countably many set functions (Theorem 1), further its semigroup valued version (Theorem 5). The problem raised by Johnson [10, p. 651] is solved. By Theorem 2 the sum of countably many atomic measures is an atomic measure.

The results were obtained by means of the abstract definition of an atom (see [3], [4], [13]).

2. Definitions and notations

Throughout the paper $(X, \mathscr{S})$ will denote a measurable space with a σ-ring $\mathscr{S}$ of subsets of X.

Let $\mathscr{E}$ be a family of subsets of X. In what follows the symbol "$\mathscr{E}C$" is used in the sense of [6] and means that every family of pairwise disjoint elements from $\mathscr{E}$ is at most countable (therefore $\emptyset \notin \mathscr{E}$). If $A \subset X$, then we use the symbol $A \mid \mathscr{E}$ in the Hahn sense [8], i.e. $A \mid \mathscr{E} = \{E \in \mathscr{E}\colon E \subset A\}$. The symbol $A^{\perp}$ stands for $X - A$, N denotes the set of positive integers.

In the following we shall work with subfamilies $\mathscr{M}$ of a σ-ring $\mathscr{S}$. Frequently we shall use some of the following conditions in connection with

(i) $\mathscr{M} \neq \emptyset$,

(ii) $E \in \mathscr{M},\ F \in \mathscr{S} \Rightarrow E \cap F \in \mathscr{M}$,

(iii) $E,\ F \in \mathscr{M} \Rightarrow E \cup F \in \mathscr{M}$,

(iv) $E_k \in \mathscr{M},\ k \in N \Rightarrow \bigcup_{k=1}^{\infty} E_k \in \mathscr{M}$,

(v) $E, F \in \mathscr{M}$, $E \cap F = \emptyset \Rightarrow E \cup F \in \mathscr{M}$.

(vi) $\emptyset \in \mathscr{M}$.

Definition 1. *A subfamily $\mathscr{M}$ of a σ-ring $\mathscr{S}$ is called:*

(j) *hereditary in $\mathscr{S}$ if it satisfies* (ii),

(jj) *an ideal if it satisfies* (i), (ii), (iii),

(jjj) *a σ-ideal if it satisfies* (i), (ii), (iv),

(jw) *a generalized ideal (briefly a g-ideal) if it satisfies* (v), (vi).

Definition 2. *Let $\mathscr{N}$ be a subfamily of a σ-ring $\mathscr{S}$ and $E \in \mathscr{S}$. Then the family $\mathscr{N}_E = \{A \in \mathscr{S} : E \cap A \in \mathscr{N}\}$ is called "the contraction of the family $\mathscr{N}$ by E".*

Definition 3. *For $\mathscr{N} \subset \mathscr{S}$ we denote $\mathscr{A}(\mathscr{N}) = \bigcap_{E \in \mathscr{S}} (\mathscr{N}_E \cup \mathscr{N}_{E'}) - \mathscr{N}$. Then any element of $\mathscr{A}(\mathscr{N})$ is called an atom.*

If for every $B \in (\mathscr{S} - \mathscr{N})$ there exists $A \in B \,|\, \mathscr{A}(\mathscr{N})$, then $\mathscr{N}$ is called atomic and if $\mathscr{A}(\mathscr{N}) = \emptyset$, then $\mathscr{N}$ is called nonatomic.

Definition 4. *Let G be a commutative semigroup with a neutral element 0 and let $\mu : \mathscr{S} \to G$ be a set function. Then the family $\mathscr{N} = \{E \in \mathscr{S} : \mu(E) = 0\}$ will be called the null system of the set function μ.*

Remark 1. The notion $\mathscr{N}_E$ was motivated by the notion of contraction ν_E of a measure ν by E e.g., [2, p. 12]. For if ν is a semigroup valued set function defined on $\mathscr{S}$ with the null system $\mathscr{N}$, then the set function ν_E has the null system equal to $\mathscr{N}_E$, so there is valid: $\mathscr{N}_E = \{G \in \mathscr{S} : \nu_E(G) = 0\}$.

From this we can easily obtain that the set of all atoms of a set function ν with the null system $\mathscr{N}$ is exactly equal to the set $\mathscr{A}(\mathscr{N})$ while the notion of a ν-atom is understood in the following sense:

A is an atom of the set function ν if $\nu(A) \neq 0$ and if for all $E \in \mathscr{S}$ there holds: $\nu(A \cap E) = 0$ or $\nu(A - E) = 0$.

Thus the results obtained in the paper evidently are valid for atoms of a set function ν having $\mathscr{N}$ as a null system.

For applications of the results obtained for subfamilies of $\mathscr{S}$, see Section 4 of this paper and Chapter II of [4, p. 61].

3. Results

Throughout the paper we shall need the following properties of subfamilies of $\mathscr{S}$, those of a contraction of the family by the set and those of the set of all atoms of a subfamily.

The proofs of Lemma 1 to Lemma 7 are rather straight-forward.

Lemma 1. *Let $\mathscr{M}$, $\mathscr{N}$, $\mathscr{M}_n$ be subfamilies of $\mathscr{S}$ and let $E \in \mathscr{S}$. Then we have:*

(a) $\left(\bigcap_{n=1}^{\infty} \mathscr{M}_n\right)_E = \bigcup_{n=1}^{\infty} (\mathscr{M}_n)_E$,

(b) $\mathcal{M} \subset \mathcal{N} \Rightarrow \mathcal{M}_E \subset \mathcal{N}_E$,

(c) $A \in \mathcal{A}(\mathcal{N})$, $B \in A \,|\, (\mathcal{S} - \mathcal{N}) \Rightarrow A - B \in \mathcal{N}$,

(d) $\mathcal{A}(\mathcal{N}) \cap \mathcal{N} = \emptyset$.

If $\mathcal{N}$ is hereditary, then we have:

(e) $\mathcal{N} \subset \mathcal{N}_E$,

(f) $\mathcal{N}_E$ *is hereditary.*

If $\mathcal{N}$ is a g-ideal, then we have:

(g) $E \in (\mathcal{S} - \mathcal{N})$, $F \in E \,|\, \mathcal{N} \Rightarrow E - F \notin \mathcal{N}$,

(h) $E \in \mathcal{A}(\mathcal{N})$, $F \in E \,|\, \mathcal{N} \Rightarrow E - F \in \mathcal{A}(\mathcal{N})$.

Throughout the paper, (a) to (h) will be reserved for the above indicated conclusions of Lemma 1.

Lemma 2. *Let $\mathcal{M}$, $\mathcal{N}$, $\mathcal{M}_n$ be subfamilies of $\mathcal{S}$ for $k \in N$. Then there holds:*

(1) $\mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right) \subset \bigcup_{k=1}^{\infty} (\mathcal{A}(\mathcal{M}_k) \cup \mathcal{M}_k)$,

(2) $\mathcal{A}(\mathcal{M} \cap \mathcal{N}) \subset \mathcal{A}(\mathcal{M}) \cup \mathcal{A}(\mathcal{N})$.

Lemma 3. *Let $\mathcal{M}$, $\mathcal{N}$ be subfamilies of $\mathcal{S}$, $\mathcal{N}$ be hereditary, then*

(1) $\mathcal{A}(\mathcal{M}) \cap \mathcal{N} \subset \mathcal{A}(\mathcal{N} \cap \mathcal{N})$,

(2) $\mathcal{A}(\mathcal{M} \cap \mathcal{N}) \cap \mathcal{N} = \mathcal{A}(\mathcal{M}) \cap \mathcal{N}$.

Definition 5. *The set $A \in \mathcal{A}(\mathcal{M}) \cap \mathcal{A}(\mathcal{N})$ will be called $\mathcal{M} \cap \mathcal{N}$-decomposable if there exists $E \in \mathcal{S}$ such that $A \cap E \notin \mathcal{M}$ and $A \cap E^{\perp} \notin \mathcal{N}$. In the opposite case we shall say that $A \in \mathcal{A}(\mathcal{M}) \cap \mathcal{A}(\mathcal{N})$ is $\mathcal{M} \cap \mathcal{N}$-indecomposable.*

Definition 6. *The set $A \in \bigcap_{i \in I} \mathcal{A}(\mathcal{N}_i)$ will be called pairwise indecomposable for $i \in I$ if for every $i, j \in I$ A is $\mathcal{N}_i \cap \mathcal{N}_j$-indecomposable.*

Remark 2. If $A \in \mathcal{A}(\mathcal{M}) \cap \mathcal{A}(\mathcal{N})$ where $\mathcal{M} = \mathcal{N}$, then by (c), A is $\mathcal{M} \cap \mathcal{N}$-indecomposable.

Lemma 4. *Let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be a sequence of subfamilies of $\mathcal{S}$ such that $A \in \bigcap_{k=1}^{\infty} \mathcal{A}(\mathcal{N}_k)$. Then $A \in \mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{N}_k\right)$ iff A is pairwise indecomposable for $k \in N$.*

Theorem 1. *Let $\{\mathcal{M}_k\}_{k=1}^{\infty}$ be a sequence of hereditary subfamilies of $\mathcal{S}$, then*

$$\mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right) = \bigcup_{\emptyset \neq M \subset N} \left[\left\{A \in \bigcap_{k \in M} \mathcal{A}(\mathcal{M}_k) \colon A \text{ is pairwise indecomp. for } k \in M\right\} \cap \left(\bigcap_{k \in N - M} \mathcal{M}_k\right)\right].$$

Proof. We subsequently use Lemma 2, (1) (1. equality), the distributive law [11, § 19, (10)] and (d) (2. equality), Lemma 3, (2) for $\mathcal{M} = \bigcap_{k \in M} \mathcal{M}_k$ and $\mathcal{N} = \bigcap_{k \in N - M} \mathcal{M}_k$ (3. equality) and Lemma 4 (last equality) so that we get:

$$\mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right) = \mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right) \cap \left[\bigcap_{k=1}^{\infty} (\mathcal{A}(\mathcal{M}_k) \cup \mathcal{M}_k)\right] =$$

$$= \mathscr{A}\left(\bigcap_{k=1}^{\infty} \mathscr{M}_k\right) \cap \bigcup_{0 \neq M \subset N} \left[\bigcap_{k \in M} \mathscr{A}(\mathscr{M}_k) \cap \left(\bigcap_{k \in N - M} \mathscr{M}_k\right)\right] =$$

$$= \bigcup_{0 \neq M \subset N} \left\{\left[\bigcap_{k \in M} \mathscr{A}(\mathscr{M}_k) \cap \mathscr{A}\left(\bigcap_{k \in M} \mathscr{M}_k\right)\right] \cap \left(\bigcap_{k \in N - M} \mathscr{M}_k\right)\right\} =$$

$$= \bigcup_{0 \neq M \subset N} \left[\left\{A \in \bigcap_{k \in M} \mathscr{A}(\mathscr{M}_k) \colon A \text{ is pairwise indecomp. for } k \in M\right\} \cap \right.$$

$$\left. \cap \left(\bigcap_{k \in N - M} \mathscr{M}_k\right)\right].$$

If we consider only two subfamilies, we get as a special case the following consequence, which is a generalization of the theorem on the sum of two nonatomic measures ([10, Theorem 1.1.]).

Corollary 1. *Let $\mathscr{M}$, $\mathscr{N}$ be hereditary subfamilies of $\mathscr{S}$. Then*

$$\mathscr{A}(\mathscr{M} \cap \mathscr{N}) = \{A \in \mathscr{A}(\mathscr{M}) \cap \mathscr{A}(\mathscr{N}) \colon A \text{ is } \mathscr{M} \cap \mathscr{N}\text{-indecomp.}\} \cup$$
$$\cup (\mathscr{A}(\mathscr{M}) \cap \mathscr{N}) \cup (\mathscr{A}(\mathscr{N}) \cap \mathscr{M}).$$

Lemma 5. *If $\mathscr{N}$ is hereditary and $A \in \mathscr{A}(\mathscr{N})$, then $A \mid \mathscr{S} \subset \mathscr{A}(\mathscr{N}) \cup \mathscr{N}$.*

The two following lemmas characterize the notions of $\mathscr{M} \cap \mathscr{N}$-decomposability and $\mathscr{M} \cap \mathscr{N}$-indecomposability.

Lemma 6. *Let $\mathscr{M}$, $\mathscr{N}$ be hereditary and $A \in \mathscr{A}(\mathscr{M}) \cap \mathscr{A}(\mathscr{N})$. Then the following conditions are equivalent:*

(1) *A is $\mathscr{M} \cap \mathscr{N}$-decomposable;*

(2) *there exists $E \in \mathscr{S}$ such that $A \cap E \in \mathscr{A}(\mathscr{M})$ and $A \cap E^{\perp} \in \mathscr{A}(\mathscr{N})$;*

(3) *there exists $E \in \mathscr{S}$ such that $A \cap E \in \mathscr{A}(\mathscr{M} \cap \mathscr{N})$ and $A \cap E^{\perp} \in \mathscr{A}(\mathscr{M} \cap \mathscr{N})$;*

(4) *there exists $E \in \mathscr{S}$ such that $A \cap E \notin \mathscr{M} \cap \mathscr{N}$ and $A \cap E^{\perp} \notin \mathscr{M} \cap \mathscr{N}$.*

Lemma 7. *Let $\mathscr{M}$, $\mathscr{N}$ be hereditary and $A \in \mathscr{A}(\mathscr{M}) \cap \mathscr{A}(\mathscr{N})$. Then the following conditions are equivalent:*

(5) *A is $\mathscr{M} \cap \mathscr{N}$-indecomposable,*

(6) *$A \in \mathscr{A}(\mathscr{M} \cap \mathscr{N})$.*

Moreover, if $\mathscr{N}$ is an ideal, then the conditions (5), (6) *are equivalent with* (7):

(7) *$A \mid \mathscr{A}(\mathscr{N}) \subset \mathscr{A}(\mathscr{M})$.*

Lemma 8. *Let $\{\mathscr{N}_k\}_{k=1}^{\infty}$ be a sequence of σ-ideals such that $B \in \bigcap_{k=1}^{\infty} \mathscr{A}(\mathscr{N}_k)$. Then there exists $A \in B \mid \mathscr{A}\left(\bigcap_{k=1}^{\infty} \mathscr{N}_k\right)$.*

Proof. We introduce on the index set N the equivalence relation R as follows: $(i, j) \in R$ iff B is $\mathscr{M}_i \cap \mathscr{M}_j$-indecomposable. Evidently R is reflexive and symmetric. We will show that it is transitive too. Let $(i, j) \in R$ and $(j, k) \in R$. By Lemma 7 we have $A \mid \mathscr{A}(\mathscr{N}_i) \subset \mathscr{A}(\mathscr{N}_j)$ and $A \mid \mathscr{A}(\mathscr{N}_j) \subset \mathscr{A}(\mathscr{N}_k)$. From this we obtain that $A \mid \mathscr{A}(\mathscr{N}_i) \subset \mathscr{A}(\mathscr{N}_k)$, so $(i, k) \in R$; thus we have proved the transitivity of R.

Thus R is an equivalence on N and so it defines a partition $\{K_i\}_{i\in I}$ of the set $N\left(\text{i.e. } K_i \text{ are nonempty pairwise disjoint subsets of } N \text{ such that } \bigcup_{i\in I} K_i = N\right)$.

Put $\mathcal{N}^i = \bigcap_{k\in K_i} \mathcal{N}_k$ for every $i\in I$. For any fixed $i\in I$, due to Lemma 4 and Lemma 7, (7) there is valid $B|\mathcal{A}(\mathcal{N}_k) = B|\mathcal{A}(\mathcal{N}^i)$ for all $k\in K_i$, therefore the set of atoms of $B|\mathcal{A}(\mathcal{N}^i)$ is the same as that of $B|\mathcal{A}(\mathcal{N}_k)$, where $\mathcal{N}_k$ is the arbitrary σ-ideal from the class $\{\mathcal{N}_q: q\in K_i\}$.

So we have an at most countable family $\{\mathcal{N}^i\}_{i\in I}$ of σ-ideals such that $B\in\mathcal{A}(\mathcal{N}^i)$ and for all $i\neq j$ B is $\mathcal{N}^i\cap\mathcal{N}^j$-decomposable.

According to Lemma 6, (2) and (c) we get that for all $i, j\in I$, $i\neq j$ there exists sets B_{ij}, B_{ji} such that $B_{ij}\cap B_{ji} = \emptyset$, $B_{ij}\cup B_{ji} = B$, $B_{ij}\in\mathcal{A}(\mathcal{N}^i)\cap\mathcal{N}^j$, $B_{ji}\in \in\mathcal{A}(\mathcal{N}^j)\cap\mathcal{N}^i$. Put $A^i = \bigcap_{i\neq j\in I} B_{ij}$ for $i\in I$. Then

$$B - A^i = B - \bigcap_{i\neq j\in I} B_{ij} = \bigcup_{i\neq j\in I} (B - B_{ij}) = \bigcup_{i\neq j\in I} B_{ji}\in\mathcal{N}^i$$

for all $i\in I$.

Therefore by (h), $A^i\in\mathcal{A}(\mathcal{N}^i)$ and thus $\{A^i\}_{i\in I}$ is a family of pairwise disjoint sets such that $A^i\in \bigcap_{i\neq j\in I}\mathcal{N}^j$. By Lemma 3, (1) $A^i\in\mathcal{A}\left(\bigcap_{i\in I}\mathcal{N}^i\right)$. Because $\bigcap_{i\in I}\mathcal{N}^i = \bigcap_{k=1}^{\infty}\mathcal{N}_k$, we obtain that for every $i\in I$, $A^i\in\mathcal{A}\left(\bigcap_{k=1}^{\infty}\mathcal{N}_k\right)$. So we can put $A = A^i$ for arbitrary $i\in I$ and we obtain A from the conclusion of the lemma.

Theorem 2. *Let $\{\mathcal{N}_k\}_{k=1}^{\infty}$ be a sequence of atomic σ-ideals, then $\bigcap_{k=1}^{\infty}\mathcal{N}_k$ is an atomic σ-ideal, too.*

Proof. Let $C\notin\bigcap_{k=1}^{\infty}\mathcal{N}_k$. Denote $M = \{k\in N: C\notin\mathcal{N}_k\}$. Obviously $M\neq\emptyset$. To proof the theorem it suffices to find $A\in C|\mathcal{A}\left(\bigcap_{k\in M}\mathcal{N}_k\right)$, because in this case with respect to the fact $A\in\bigcap_{k\in N-M}\mathcal{N}_k$, by Lemma 3, (1) we get $A\in\mathcal{A}\left(\bigcap_{k=1}^{\infty}\mathcal{N}_k\right)$.

Thus we may suppose $\emptyset\neq M\subset N$ and $C\notin\mathcal{N}_k$ for all $k\in M$. Since $\mathcal{N}_k$ are atomic for all $k\in M$, there exists $C_k\in C|\mathcal{A}(\mathcal{N}_k)$. From the family of atoms $\{C_k\}_{k\in M}$ we form the family of atoms $\{B_k\}_{k\in M}$, $B_k\in C|\mathcal{A}(\mathcal{N}_k)$ by putting

(1) $B_k = C_k - (\cup\{C_i: C_i\cap C_k\in\mathcal{N}_k\}\cup\{(C_k - C_i): (C_k - C_i)\in\mathcal{N}_k\})$.

We affirm that

(2) $\{B_k\}_{k\in M}$ is a family of atoms such that for all $p, q\in M$ there holds either $B_p = B_q$ or $B_p\cap B_q = \emptyset$.

Indeed in the case when $C_p\cap C_q\in\mathcal{N}_q$ we have $B_q\subset C_q - C_p$, and so $B_q\cap B_p = \emptyset$.

In the opposite case $(C_p\cap C_q)\notin(\mathcal{N}_p\cap\mathcal{N}_q)$. Then by (c) it is easy to see that $B_p\cup B_q\subset C_p\cap C_q$ and thus for B_p, B_q there holds:

(3) $B_p = (C_p \cap C_q) - (\cup\{C_i\colon C_i \cap C_p \in \mathcal{N}_p\} \cup \{C_p \cap C_q - C_i\colon (C_p - C_i) \in \mathcal{N}_p\})$,
(4) $B_q = (C_p \cap C_q) - (\cup\{C_i\colon C_i \cap C_q \in \mathcal{N}_q\} \cup \{C_p \cap C_q - C_i\colon (C_q - C_i) \in \mathcal{N}_q\})$.
If for all $r \in M$ the following condition is satisfied
(5) $C_r \cap C_p \in \mathcal{N}_p$ iff $C_r \cap C_q \in \mathcal{N}_q$,
then, in this case, from (3) and (4) we get $B_p = B_q$.

If (5) is not satisfied, then there exists $r \in M$ such that $C_r \cap C_p \in \mathcal{N}_p$ but $(C_q - C_r) \in \mathcal{N}_q$. In this case $B_p \subset C_p - C_r$ and $B_q \subset C_q - (C_q - C_r) = C_q \cap C_r$ and thus $B_p \cap B_q = \emptyset$. So we have proved (2).

Thus we have the family of atoms $\{B_k\}_{k \in M}$ $\emptyset \neq M \subset N$ satisfying the property (2). Denote $I_q = \{i \in M\colon B_i = B_q\}$. By (2) $\{I_q\colon q \in M\}$ form a partition of the set M.

Let $q \in M$ be arbitrarily choosen. Then by Lemma 8 there exists $A \in B_q | \mathcal{A}\left(\bigcap_{i \in I_q} \mathcal{N}_i\right)$. Of course since $A \in \left(\bigcap_{i \in N - I_q} \mathcal{N}_i\right)$ by Lemma 3, (1) we have $A \in B_q | \mathcal{A}\left(\bigcap_{k \in N} \mathcal{N}_k\right)$ and so $A \in C | \mathcal{A}\left(\bigcap_{k \in N} \mathcal{N}_k\right)$.

Remark 3. Let ν be a measure and $\mathcal{N}$ be its null system. Then we shall say that a measure ν satisfies the countable chain condition (shortly CCC) if there $(\mathcal{S} - \mathcal{N})\,C$ holds. A finite measure satisfies CCC (see, e.g., [2, Section 44] or [5] Lemma 1 and Theorem 2). Thus the supposition $(\mathcal{M} - \mathcal{N})\,C$ in Lemma 9 is weaker than that of finiteness of the measure ν.

If ν is a σ-finite measure, then for all $E \in \mathcal{S}$ there exists a sequence E_n of pairwise disjoint sets such that $E = \bigcup_{n=1}^{\infty} E_n$. Then we have $\nu_E = \sum_{n=1}^{\infty} \nu_{E_n}$. Because ν_{E_n} are finite for their null system there $(\mathcal{M} - \mathcal{N}_{E_n})\,C$ holds. Then since $(\mathcal{S} - \mathcal{N}_E) = \bigcup_{n=1}^{\infty} (\mathcal{S} - \mathcal{N}_{E_n})$ we obtain $(\mathcal{S} - \mathcal{N}_E)\,C$. Thus the supposition of Theorem 4 that for all $E \in \mathcal{S}$ there holds that $(\mathcal{M} - \mathcal{N}_E)\,C$ is more general than the supposition of σ-finiteness of a measure ν.

For a proof and applications of Lemma 9 see Lemma 2, Corollary 1 and Corollary 2 from [4].

The proof of Lemma 10 is straightforward.

Lemma 9. *Let $\mathcal{M}$ be a σ-ideal, $\emptyset \in \mathcal{N} \subset \mathcal{M}$ and $(\mathcal{M} - \mathcal{N})\,C$. Then there exists $F \in \mathcal{M}$ such that $\mathcal{N}_{F\perp} = \mathcal{M}$.*

Lemma 10. *Let $\mathcal{N} \subset \mathcal{S}$, G, $F \in \mathcal{S}$. Then $G \in \mathcal{A}(\mathcal{N}_F)$ iff $G \cap F \in \mathcal{A}(\mathcal{N})$.*

Theorem 3. *Let $\mathcal{M}$ be a σ-ideal on $\mathcal{S}$, $\emptyset \in \mathcal{M} \subset \mathcal{N}$ and let $(\mathcal{M} - \mathcal{N})\,C$ hold. Then there exists $F \in \mathcal{M}$ such that $A \in \mathcal{A}(\mathcal{M})$ iff $A - F \in \mathcal{A}(\mathcal{N})$.*

In particular $\{A - F\colon A \in \mathcal{A}(\mathcal{M})\} \subset \mathcal{A}(\mathcal{N})$,
$\{A\colon A - F \in \mathcal{A}(\mathcal{N})\} \subset \mathcal{A}(\mathcal{M})$.

Proof. By Lemma 9, there exists $F \in \mathcal{M}$ such that $\mathcal{M} = \mathcal{N}_{F\perp}$. By Lemma 10, $A \in \mathcal{A}(\mathcal{M})$ iff $A - F \in \mathcal{A}(\mathcal{N})$.

The following theorem is a generalization of Theorem 2.4 from [10]. Our proof is more straighttforward and it does not use singularity.

Theorem 4. *Let $\mathcal{M}$ be a σ-ideal, $\emptyset \in \mathcal{N} \subset \mathcal{M}$ and let for all $E \in \mathcal{S}$, $(\mathcal{M} - \mathcal{N}_E)\, C$ hold. Then we have:*

(1) *If $\mathcal{N}$ is nonatomic, then $\mathcal{M}$ is nonatomic.*

(2) *If $\mathcal{N}$ is atomic, then $\mathcal{M}$ is atomic.*

Proof. (1) Indirectly. Suppose $A \in \mathcal{A}(\mathcal{M})$. Then since $(\mathcal{M} - \mathcal{N}_A)\, C$, by Theorem 3 there would exist $F \in \mathcal{M}$ such that $(A - F) \in \mathcal{A}(\mathcal{N}_A)$. According to Lemma 10, $A - F \in \mathcal{A}(\mathcal{N})$, which is a contradiction with the nonatomicity of $\mathcal{N}$.

(2) Let $A \notin \mathcal{M}$. Take $F \in \mathcal{M}$ (from Theorem 3) such that $(\mathcal{N}_A)_{F\perp} = \mathcal{M}$. Then by (g) we have $A - F \notin \mathcal{M}$ and thus $A - F \notin \mathcal{N}$. Since $\mathcal{N}$ is atomic, there exists $B \in (A - F) \mid \mathcal{A}(\mathcal{N})$. As $B = B \cap (A - F)$ by Lemma 10, $B \in \mathcal{A}((\mathcal{N}_A)_{F\perp})$ and thus $B \in \mathcal{A}(\mathcal{M})$.

The following theorem is a semigroup valued version of Theorem 1, Theorem 2 and Theorem 1.2 from [10]. Indeed, if μ is a set function with values in a topological semigroup such that $\mu = \sum_{n=1}^{\infty} \mu_n$ and $\mathcal{M}$ resp. $\mathcal{M}_n$ are null system of μ and μ_n, respectively, then for the null system $\mathcal{M}$ of μ there holds $\mathcal{M} \subset \bigcap_{n=1}^{\infty} \mathcal{M}_n$.

Theorem 5. *Let $\{\mathcal{M}_k\}_{k=1}^{\infty}$ be a sequence of hereditary subfamilies of $\mathcal{S}$ and $\mathcal{M}$ be a σ-ideal such that $\bigcap_{k=1}^{\infty} \mathcal{M}_k \subset \mathcal{M}$ and let $(\mathcal{M} - \mathcal{M}_k)\, C$ hold for all $k \in N$. Then there exists $F \in \mathcal{M}$ such that*

(1) $\{A - F\colon A \in \mathcal{A}(\mathcal{M})\} \subset \bigcup_{\emptyset \neq M \subset N} \left\{A \in \bigcap_{k \in M} \mathcal{A}(\mathcal{M}_k)\colon A \text{ is pairw. indecomp. for } k \in M\right\} \cap \left(\bigcap_{k \in N - M} \mathcal{M}_k\right)$.

(2) *If $\mathcal{M}_k$ are nonatomic for all $k \in N$, then $\mathcal{M}$ is nonatomic as well.*

If $\mathcal{M}_k$ are σ-ideals, then

(3) *if $\mathcal{M}_k$ are atomic for all $k \in N$, then $\mathcal{M}$ is atomic.*

Proof. (1) According to Theorem 3, $\{A - F\colon A \in \mathcal{A}(\mathcal{M})\} \subset \mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right)$. For $\mathcal{A}\left(\bigcap_{k=1}^{\infty} \mathcal{M}_k\right)$ we use Theorem 1 and so we obtain the inclusion (1).

(2) is implied by (1) because if $\mathcal{M}_k$ are nonatomic, then the right-hand side of inclusion is empty.

(3) If all $\mathcal{M}_k$ are atomic, then by Theorem 2 $\bigcap_{k=1}^{\infty} \mathcal{M}_k$ is atomic and according to Theorem 4 $\mathcal{M}$ is atomic too.

4. Applications

The results concerning the atoms of set functions are in the present paper presented abstractly for the families of sets. Namely if ν is a set function with values in the semigroup $(G; +)$, its null system $\mathcal{N} = \{E \in \mathcal{S}: \nu(E) = 0\}$ is a subfamily of $\mathcal{S}$. Then if the null systems of the set functions satisfy the hypothesis of Theorems 1 to 5, these theorems can be applied even to semigroup valued set functions. For these applications see [4, Corollaries 1 to 6]. However, if ν is a set function with values in the extended set of real numbers, then by Remark 1 there holds $\mathcal{A}(\nu) = \mathcal{A}(\mathcal{N})$, so the results concerning the atoms of subfamilies will be generalizations of the results for real valued set functions.

Thus we obtain besides others the following results:

Theorem 1 besides others expresses that for subadditive nonnegative set functions μ_n with the null systems $\mathcal{M}_n$ there holds

$$\mathcal{A}\left(\sum_{n=1}^{\infty} \mu_n\right) =$$

$$= \bigcup_{0 \neq M \subset N} \left[\left\{A \in \bigcap_{k \in M} \mathcal{A}(\mu_k): A \text{ is pairw. indecomp. for } k \in M\right\} \cap \left(\bigcap_{k \in N - M} \mathcal{M}_k\right)\right].$$

Moreover, the above equality holds if instead $\sum_{n=1}^{\infty} \mu_n$ we take an arbitrary μ having a null system equal to $\bigcap_{k=1}^{\infty} \mathcal{M}_k$.

Theorem 2, for example, expresses that the countable sum of nonnegative atomic measures is atomic, too.

Theorem 3 is valid, for example, for set functions μ, ν such that μ is a nonnegative measure dominated by a set function ν satisfying CCC. According to it there exists F μ-null such that $A \in \mathcal{A}(\mu)$ iff $A - F \in \mathcal{A}(\nu)$.

Theorem 4 is valid, for example, for a nonnegative measure dominated by a σ-finite set function ν. According to it, if ν is nonatomic (atomic), so μ is nonatomic (atomic), too.

Theorem 5, (1) (Theorem 5, (3)) is a semigroup valued version of Theorem 1 (Theorem 2). Theorem 5, (1) and (2) is valid for example for semigroup valued set functions μ_n, whose null systems are hereditary and the null system of the set function $\sum_{n=1}^{\infty} \mu_n$ is a σ-ideal.

I point also to the possibility of succesive applications of the results of the present paper to set valued set functions (see, e.g., [1, 7, 9, 12]). If $(G; +)$ is a group and $M: \mathcal{S} \to (2^G - \{\emptyset\})$ is a set valued set function such that $M(\emptyset) = \{0\}$, then we can put as a null system $\mathcal{M} = \{E \in \mathcal{S}: M(E) = \{0\}\}$ and so the results of Theorems 1 to 5 can be applied for set valued set functions.

REFERENCES

[1] ARSTEIN, Z.: Set valued measures. Trans. Amer. Math. Soc. 165, 1972, 103—121.
[2] BERBERIAN, S. K.: Measure and Integration, New York 1965.
[3] CAPEK, P.: Théorèmes de décomposition en théorie de la mesure I, II. Publications du Séminaire d'Analyse de Brest, juin 1976.
[4] CAPEK, P.: Decomposition theorems in measure theory. Math. Slovaca 31, 1981, No. 1, 53—59.
[5] CAPEK, P.: The pathological infinity of measures. Suppl. ai Rendiconti del Circolo Mat. Di Palermo. Série II-6, 1984.
[6] FICKER, V.: On the equivalence of a countable disjoint class of sets of positive measure and a weaker condition than total σ-finiteness of measures. Bull. Austral. Math. Soc. 1, 1969, 237—243.
[7] GODET-THOBIE, C.: Multimesures et multimesures de transitions. Thèse. Montpellier. 1985.
[8] HAHN, H.—ROSENTHAL, A.: Set Functions. The University of New Mexico Press 1948.
[9] HIAI, F.: Radon-Nikodym theorems for set-valued measures. Journal of Miltiv. Analysis 8, 1978, 96—118.
[10] JOHNSON, R. A.: Atomic and nonatomic measures. Proc. Amer. Math. Soc., 25, 1970, 650—655.
[11] SIKORSKI, R.: Boolean Algebras. Springer-Verlag 1969.
[12] DREWNOWSKI, L.: Additive and countably additive correspondences, Commentationes Mathem. XIX 1976.
[13] CAPEK, P.: Abstract comparison of the properties of infinite measures. Acta Math. Univ. Comen. (to appear).

Received October 28, 1986

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АТОМЫ СЧЕТНОЙ СУММЫ ФУНКЦИЙ МНОЖЕСТВ

Peter Capek

Резюме

В работе найдено представление множества атомов неотрицательной функции множства, возникающей как счетная сумма неотрицательных функций множеств.

Аналогичный результат приводится также для мер со значениями в полугруппе. Кроме того, в статье показано, что сумма счетного качества атомических мер является атомической мерой.

Эти и другие результаты получены в абстрактной форме, когда мера заменена понятием σ-идеал, или более общей системой множеств.