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Dedicated to Professor Beloslav Riečan on the occasion of his 70th birthday

ON RIEČAN AND BOSBACH STATES FOR BOUNDED NON-COMMUTATIVE $R\ell$-MONOIDS

ANATOLIJ DVUREČENSKIJ* — JIŘÍ RACHŮNEK**

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ABSTRACT. We generalize the notion of a state for bounded $R\ell$-monoids introducing Riečan and Bosbach states and weak states. We show that for good bounded $R\ell$-monoids all these three notions coincide, and this fact gives the answer to an open problem posed by Georgescu [GEORGESCU, G.: *Bosbach states on fuzzy structures*, Soft Comput. **8** (2004), 217–230] whether there is a Riečan state on a good pseudo BL-algebra which is not a Bosbach state.

1. Introduction

The notion of a state is an analogue of probability measure. Such a notion plays a crucial role in the theory of quantum structures which generalizes the Kolmogorov probabilistic space ([DvPu]). To introduce analogues of probabilities for generalizations of Boolean algebras it is necessary to know what is an event space (it is an appropriate algebra) and its structure and what operation corresponds to addition of disjoint sets. In quantum structures it is orthogonality, that is $x \perp y$ if $x \leq y^-$, where y^- denotes the analogue of negation of y; only in such cases we are able to define connective $x + y$, which denotes the disjunction of mutually orthogonal elements.

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Recently, states were introduced for MV-algebras by [KoCh] and [Mun], and states are averaging the truth-value in the logic corresponding to these algebras. It is interesting to note that the first probabilistic ideas were introduced for MV-algebras only forty years after their introducing by Chang [Cha].

Nowadays there appeared a whole family of generalizations of MV-algebras, also non-commutative ones. Such ones are pseudo MV-algebras introduced by [GeIo], or equivalently GMV-algebras introduced by [Rac], or BL-algebras or pseudo BL-algebras [DGI1], [DGI2], or bounded $R\ell$-monoids [DvRa2] ([DvRa1] for commutative case) known also as bounded integral generalized BL-algebras, see [BCGJT].

If M is a pseudo MV-algebra, then by [Dvu1], M is isomorphic to $\Gamma(G,u) = \{g \in G : 0 \le g \le u\}$, where (G,u) is a unital ℓ-group with strong unit u. We recall that $x \oplus y := (x+y) \wedge u$, $x,y \in \Gamma(G,u)$. Then this allows us to define a partial addition $+$, which is in fact the restriction of the group addition to $\Gamma(G,u)$, that is, $x+y$ is defined in M if $x \le y^- = u - y$, and therefore, a state is simply a positive normalized function s on M which preserves partial addition. That is, $s(u) = 1$ and $s(x+y) = s(x)+s(y)$ whenever $x+y$ is defined in M.

Unfortunately for pseudo BL-algebras or bounded $R\ell$-monoids, there is no analogue of a group representation and it was not clear how to define states for such structures. Georgescu [Geo] find a very nice definition of a state, called a Bosbach state, not using the concept of orthogonal elements. This was also generalized for states on bounded $R\ell$-monoids by the authors in [DvRa2]. Riečan in [Rie] defined a state for BL-algebras, using ideas of orthogonal elements. Georgescu in [Geo] introduced so-called Riečan states for good pseudo BL-algebras, and he showed that every Bosbach state is a Riečan state. He formulated an open problem whether there is a Riečan state which is not Bosbach, [Geo; 2.15].

In the present note we introduce also a weak state, and we show that in fact even for more general structures, good bounded $R\ell$-monoids, all three notions of a state coincide. This is important because we are now able to define again a state as a normalized function preserving orthogonal elements. And it goes back to original Boole's ideas [Boo], who said that to define a probability it is enough to know its behavior for summable ($=$ orthogonal) elements.

2. Bounded $R\ell$-monoids and their pseudo MV-parts

A bounded *$R\ell$-monoid* was introduced in [DvRa2] as an algebra $M = (M; \odot, \vee, \wedge, \rightarrow, \rightsquigarrow, 0, 1)$ of type $\langle 2,2,2,2,2,0,0\rangle$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a monoid (need not be commutative), i.e. $\odot$ is associative with neutral element 1.

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice.

(iii) $x \odot y \leq z$ iff $x \leq y \to z$ iff $y \leq x \rightsquigarrow z$ for any $x, y \in M$.

(iv) M satisfies the identities

$$(x \to y) \odot x = x \wedge y = y \odot (y \rightsquigarrow x).$$

We recall that $\wedge$, $\vee$ and $\odot$ have higher priority than $\to$ or $\rightsquigarrow$, M is a distributive lattice, and $x \leq y$ iff $y^\sim \odot x = 0$.

Such monoids are also known as bounded integral generalized BL-algebras, see [BCGJT], and the operation $\odot$ distributes from the left and from the right over the operations $\vee$ and $\wedge$, see [DvRa2].

We say that a bounded $R\ell$-monoid M is *commutative* if $x \odot y = y \odot x$ for all $x, y \in M$. This is equivalent with the statement that $\to = \rightsquigarrow$.

Let M be a bounded $R\ell$-monoid. Let us define two unary operations (negations) $^-$ and $^\sim$ on M such that $x^- := x \to 0$ and $x^\sim := x \rightsquigarrow 0$ for any $x \in M$. We have, for all $x \in M$,

(i) $x \leq x^{-\sim}$, $x \leq x^{\sim-}$,
(ii) $x^- = x^{-\sim-}$ and $x^\sim = x^{\sim-\sim}$.

We say that a bounded $R\ell$-monoid M is *good* if $x^{-\sim} = x^{\sim-}$ for any $x \in M$.

For example, every pseudo MV-algebra is good. On the other hand, it is unknown whether any pseudo BL-algebra is good (for definitions see below), [DGI1], [DGI2].

Basic properties of bounded $R\ell$-monoids were exhibited in [DvRa2], here we present some of important properties which will be used in this paper.

LEMMA 2.1. *In any bounded $R\ell$-monoid M we have for each $x, y, z \in M$:*

(1) $x \to (y \to z) = (x \odot y) \to z$.
(2) $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$.
(3) $(x \odot y)^- = x \to y^-$, $(x \odot y)^\sim = y \rightsquigarrow x^\sim$.
(4) $(x \vee y)^- = x^- \wedge y^-$, $(x \vee y)^\sim = x^\sim \wedge y^\sim$.
(5) $(x \wedge y)^- \geq x^- \vee y^-$, $(x \wedge y)^\sim \geq x^\sim \vee y^\sim$.
(6) $(x \vee y)^{-\sim} \geq x^{-\sim} \vee y^{-\sim}$, $(x \vee y)^{\sim-} \geq x^{\sim-} \vee y^{\sim-}$.
(7) $y^- \rightsquigarrow x^- = x^{-\sim} \to y^{-\sim} = x \to y^{-\sim}$, $y^\sim \to x^\sim = x^{\sim-} \rightsquigarrow y^{\sim-}$ $= x \rightsquigarrow y^{\sim-}$.

If, moreover, M is good, then:

(8) $x^{-\sim} \odot y^{-\sim} \leq (x \odot y)^{-\sim}$.
(9) $(x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim}$.
(10) $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$, $(x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim}$.

Other noncommutative structures are pseudo BL-algebras introduced in [DGI1] and [DGI2]. It is possible to show that a bounded $R\ell$-monoid M is a *pseudo BL-algebra* if and only if

$$(x \to y) \vee (y \to x) = 1 = (x \rightsquigarrow y) \vee (y \rightsquigarrow x), \qquad x, y \in M .$$

Pseudo MV-algebras, or equivalently, *GMV-algebras* were introduced in [GeIo] and [Rac], respectively.

We recall that according to [GeIo], a pseudo MV-algebra is an algebra $(M; \oplus, ^{-'}, ^{\sim'}, 0, 1)$ of type $\langle 2, 1, 1, 0, 0\rangle$ such that the following axioms hold for all $x, y, z \in M$ with an additional binary operation $\odot$ defined via

$$y \odot x = (x^{-'} \oplus y^{-'})^{\sim'}$$

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
(A2) $x \oplus 0 = 0 \oplus x = x$;
(A3) $x \oplus 1 = 1 \oplus x = 1$;
(A4) $1^{\sim'} = 0$; $1^{-'} = 0$;
(A5) $(x^{-'} \oplus y^{-'})^{\sim'} = (x^{\sim'} \oplus y^{\sim'})^{-'}$;
(A6) $x \oplus x^{\sim'} \odot y = y \oplus y^{\sim'} \odot x = x \odot y^{-'} \oplus y = y \odot x^{-'} \oplus x$;
(A7) $x \odot (x^{-'} \oplus y) = (x \oplus y^{\sim'}) \odot y$;
(A8) $(x^{-'})^{\sim'} = x$.

We recall that the order on a pseudo MV-algebra M is defined also by $x \leq y$ iff $x^{-'} \oplus y = 1$ (iff $y \oplus x^{\sim'} = 1$), and that $(M, \leq)$ is a bounded distributive lattice such that $x \wedge y = x \odot (x^{-'} \oplus y)$ and $x \vee y = x \oplus (x^{\sim'} \odot y)$, [GeIo; Proposition 1.13]. Moreover, put $x \to y = y \oplus x^{\sim'}$ and $x \rightsquigarrow y = x^{-'} \oplus y$. Then $(M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ becomes a bounded $R\ell$-monoid.[1] It is possible to show by [Rac] that a bounded $R\ell$-monoid M is a pseudo MV-algebra if and only if M satisfies the identities $x^{-\sim} = x = x^{\sim-}$ and $(x^{\sim} \odot y^{\sim})^{-} = (x^{-} \odot y^{-})^{\sim}$ for all $x, y \in M$.

Let $M = (M; \odot, \vee, \wedge, \to, \rightsquigarrow, 0, 1)$ be a bounded $R\ell$-monoid. A *filter* of M is a non-empty subset F of M such that

(i) $x, y \in F \implies x \odot y \in F$,

(ii) $x \in F$, $y \in M$, $x \leq y \implies y \in F$.

A filter F is called *normal* if

(iii) $x \to y \in F \iff x \rightsquigarrow y \in F$ for each $x, y \in M$.

[1] We recall that for a pseudo MV-algebra $(M; \oplus, ^{-'}, ^{\sim'}, 0, 1)$ (understood as a bounded $R\ell$-monoid) we have $x^{-} =: x \to 0 = x^{\sim'}$ and $x^{\sim} =: x \rightsquigarrow 0 = x^{-'}$ for any $x \in M$.

Recall that normal filters are in a one-to-one correspondence with kernels of homomorphisms of $R\ell$-monoids or with congruences. If F is a normal filter, the corresponding congruence is given by $x \approx_F y$ if $x \to y, y \to x \in F$.

In what follows, we will suppose that a bounded $R\ell$-monoid is good if not stated other.

In every good bounded $R\ell$-monoid M we have

$$(x^{\sim} \odot y^{\sim})^{-} = (x^{-} \odot y^{-})^{\sim}, \qquad x, y \in M, \tag{2.1}$$

see [RaSl].

PROPOSITION 2.2. *Let M be a good bounded $R\ell$-monoid. We define a total binary operation $\oplus$ on M by*

$$x \oplus y := (y^{\sim} \odot x^{\sim})^{-}, \qquad x, y \in M. \tag{2.2}$$

Then, for all $x, y, z \in M$, we have:

(i) $x \oplus y = (y^{-} \odot x^{-})^{\sim}$.
(ii) $\oplus$ *is associative.*
(iii) $x, y \leq x \oplus y$.
(iv) $x \oplus 0 = x^{-\sim} = 0 \oplus x$.
(v) $x \oplus 1 = 1 = 1 \oplus x$.
(vi) $x \oplus y = x^{-} \rightsquigarrow y^{-\sim} = y^{\sim} \to x^{-\sim}$.

Proof. By (2.1) we have $x \oplus y = (y^{\sim} \odot x^{\sim})^{-}$.

Associativity. Check $(x \oplus y) \oplus z = (y^{\sim} \odot x^{\sim})^{-} \oplus z = \big(z^{\sim} \odot (y^{\sim} \odot x^{\sim})^{-\sim}\big)^{-} = z^{\sim} \to (y^{\sim} \cdot x^{\sim})^{-\sim-} = z^{\sim} \to (y^{\sim} \odot x^{\sim})^{-} = z^{\sim} \to (y^{\sim} \to x^{\sim-})$.

On the other hand, $x \oplus (y \oplus z) = x \oplus (z^{\sim} \odot y^{\sim})^{-} = \big((z^{\sim} \odot y^{\sim})^{-\sim} \odot x^{\sim}\big)^{-} = (z^{\sim} \odot y^{\sim})^{-\sim} \to x^{\sim-} = (z^{\sim} \odot y^{\sim})^{-\sim} \to x^{-\sim} = (z^{\sim} \odot y^{\sim}) \to x^{-\sim} = z^{\sim} \to (y^{\sim} \to x^{-\sim})$, where we have used Lemma 2.1(1)(3), which implies the associativity of $\oplus$.

(iii) $x \oplus y = (y^{\sim} \odot x^{\sim})^{-} = y^{\sim} \to x^{\sim-} \geq x^{\sim-} \geq x$ and by (i), $x \oplus y = (y^{-} \odot x^{-})^{\sim} = x^{-} \rightsquigarrow y^{-\sim} \geq y^{-\sim} \geq y$.

(iv) and (v) are evident, and (vi) follows from (3) of Lemma 2.1. □

We set

$$\mathcal{MV}(M) = \{x \in M : \ x^{-\sim} = x^{\sim-} = x\}.$$

Then $0, 1 \in \mathcal{MV}(M)$.

According to [GeLe] or [Geo; Proposition 1.19], if M is a good pseudo BL-algebra, then the subset $\mathcal{MV}(M)$ endowed with $\oplus, ^{-}, ^{\sim}, 0, 1$ is a pseudo MV-algebra. We show that this is also true if M is a good bounded $R\ell$-monoid.

THEOREM 2.3. *Let M be a good bounded $R\ell$-monoid. Then the set $\mathcal{MV}(M) = (\mathcal{MV}(M); \oplus, ^{-'}, ^{\sim'}, 0, 1)$ is a pseudo MV-algebra, where $\oplus$ is defined by* (2.2), *and $^{-'}$ and $^{\sim'}$ are the restriction of $^{\sim}$ and $^{-}$ taken from M. Moreover, the order on $\mathcal{MV}(M)$, defined by $x \leq_{\mathcal{M}} y$ if and only if $y \oplus x^{-} = 1$, agrees with the one on M.*

Proof. It is clear that

(i) $0, 1 \in \mathcal{MV}(M)$,
(ii) $x^{-}, x^{\sim} \in \mathcal{MV}(M)$,
(iii) $x \oplus y \in \mathcal{MV}(M)$,
(iv) $x, y \in \mathcal{MV}(M)$ implies $x \to y, x \rightsquigarrow y \in \mathcal{MV}(M)$ (see (10) of Lemma 2.1).

Define a total binary operation $\odot_{\mathcal{M}}$ and a partial ordering $\leq_{\mathcal{M}}$ on $\mathcal{MV}(M)$ via

$$x \odot_{\mathcal{M}} y = (x \odot y)^{-\sim}, \qquad x, y \in \mathcal{MV}(M),$$

and $x \leq_{\mathcal{M}} y$ iff $x \leq y$.

Then $(\mathcal{MV}(M); \vee_{\mathcal{M}}, \wedge_{\mathcal{M}}, 0, 1)$ is a bounded lattice such that $x \vee_{\mathcal{M}} y = (x \vee y)^{-\sim}$, and $x \wedge_{\mathcal{M}} y = x \wedge y$ (see (4) and (9) of Lemma 2.1), and

$$x \odot_{\mathcal{M}} y = (y^{-} \oplus x^{-})^{\sim}, \qquad x, y \in \mathcal{MV}(M).$$

In addition, we claim that $(\mathcal{MV}(M); \odot_{\mathcal{M}}, \vee_{\mathcal{M}}, \wedge_{\mathcal{M}}, \to_{\mathcal{M}}, \rightsquigarrow_{\mathcal{M}}, 0, 1)$ is a good bounded $R\ell$-monoid, where $\to_{\mathcal{M}}$ and $\rightsquigarrow_{M}$ are the restrictions of $\to$ and $\rightsquigarrow$, respectively, onto $\mathcal{MV}(M)$.

The associativity of $\odot_{\mathcal{M}}$ can be proved from the associativity of $\oplus$. Indeed, $(x \odot_{\mathcal{M}} y) \odot_{\mathcal{M}} z = (y^{-} \oplus x^{-})^{\sim} \odot_{\mathcal{M}} z = (z^{-} \oplus (y^{-} \oplus x^{-})^{\sim -})^{\sim} = (z^{-} \oplus (y^{-} \oplus x^{-}))^{\sim} = ((z^{-} \oplus y^{-}) \oplus x^{-})^{\sim} = ((z^{-} \oplus y^{-})^{\sim -} \oplus x^{-})^{\sim} = x \odot_{\mathcal{M}} (z^{-} \oplus y^{-})^{\sim} = x \odot_{\mathcal{M}} (y \cdot_{\mathcal{M}} z)$.

The other properties of bounded $R\ell$-monoids are now clear, see e.g., $(x \to y) \odot_{\mathcal{M}} y = ((x \to y) \odot y)^{-\sim} = (x \wedge y)^{-\sim} = x \wedge y = x \wedge_{\mathcal{M}} y = y \odot_{\mathcal{M}} (y \rightsquigarrow x)$, etc..

Hence, $\mathcal{MV}(M)$ is a good bounded $R\ell$-monoid, in which we have $x^{-\sim} = x = x^{\sim -}$, and this proves by [RaSl] that $(\mathcal{MV}(M); \oplus, ^{-'}, ^{\sim'}, 0, 1)$ is a pseudo MV-algebra.

We finish with the remark that $y \oplus x^{\sim'} = y \oplus x^{-} = 1$ iff $(x \odot y^{\sim})_{\mathcal{M}} = 0$ iff $x \odot y^{\sim} = 0$, that is, the order on $\mathcal{MV}(M)$ induced by $\oplus$ and the order restricted from M coincide. $\square$

We recall that if M is a pseudo BL-algebra, then on $\mathcal{MV}(M)$ we have $x \vee_{\mathcal{M}} y = x \vee y$. Further, if a bounded $R\ell$-monoid M satisfies the identity $(x \odot y)^{-\sim} = x^{-\sim} \odot y^{-\sim}$, then on $\mathcal{MV}(M)$ the operations $\odot_{\mathcal{M}}$ and $\odot$ coincide. This identity is satisfied e.g. by any good pseudo BL-algebra ([RaSl]) and by any Heyting algebra ([RaSl]) as it is shown below:

Let M be a good pseudo BL-algebra and $x, y \in M$. Then

$$\begin{aligned}
(x \odot y)^{\sim -} &= (x \odot y)^{\sim -} \wedge x^{\sim -} \\
&= x^{\sim -} \odot \left(x^{\sim -} \rightsquigarrow (x \odot y)^{\sim -}\right) \\
&= x^{\sim -} \odot \left(x^{\sim -} \rightsquigarrow (x \odot y)^{- \sim}\right) \\
&= x^{\sim -} \odot \left(x^{\sim -} \rightsquigarrow (x \to y^{-})^{\sim}\right) && \text{(by (3) of Lemma 2.1)} \\
&= x^{\sim -} \odot \left(x^{\sim -} \rightsquigarrow \left(x \to (y^{-})^{- \sim}\right)^{\sim}\right) \\
&= x^{\sim -} \odot \left(x^{\sim -} \rightsquigarrow (x^{\sim -} \to y^{-})^{\sim}\right) && \text{(by (3) of Lemma 2.1)} \\
&= x^{\sim -} \odot \left((x^{\sim -} \to y^{-}) \odot x^{\sim -}\right)^{\sim} && \text{(by (7) of Lemma 2.1)} \\
&= x^{\sim -} \odot (x^{\sim -} \wedge y^{-})^{\sim} \\
&= x^{\sim -} \odot (x^{\sim} \vee y^{- \sim}) && \text{(by [DGI1; Proposition 3.20])} \\
&= (x^{\sim -} \odot x^{\sim}) \vee (x^{\sim -} \odot y^{\sim -}) \\
&= x^{\sim -} \odot y^{\sim -} .
\end{aligned}$$

That means $(x \odot y)^{\sim -} = x^{\sim -} \odot y^{\sim -}$.

3. States, Bosbach states, and Riečan States

In this section, we recall Bosbach states for (good) bounded $R\ell$-monoids, and we extend the notion of a Riečan state for good bounded $R\ell$-monoids which was introduced by Georgescu [Geo] only for pseudo BL-algebras.[2] For that we define an analogue of addition $+$, which generalizes that known for pseudo MV-algebras. We show that these two notions coincide on good bounded $R\ell$-monoids, which gives us as a by-product the complete answer to a problem posed in [Geo; 2.15] on these two kinds of states.

Assume M is a pseudo MV-algebra. According to [Dvu1], [Dvu2], we define a partial addition $+$ on M: $x + y$ is defined in M iff $x \le y^{-'} = y^{\sim}$, and in this case we set $x + y = x \oplus y$. It is clear that $x + y$ is defined iff $y \odot x = 0$, or iff $y \le x^{\sim'} = x^{-}$ equivalently $y^{-\sim} \le x^{-}$. This operation is associative and $x + 0 = x = 0 + x$ for any $x \in M$. If we take into account the group representation of pseudo MV-algebras, [Dvu1], then this $+$ coincides with the group addition.

Let now M be a good bounded $R\ell$-monoid. We have defined the binary operation $\oplus$ by (2.2). It allows us to define a special partial binary operation

[2]We note that the notion of pseudo BL-algebras introduced in [DGI] and presented by us in this paper is different from that in [Geo]; the way how to identify them is to change $\to$ $(\rightsquigarrow)$ in the one to $\rightsquigarrow$ $(\to)$ in the second, and vice versa.

$+$, defined by $x+y=z$ iff $y^{-\sim} \leq x^{-}$ and $z = x \oplus y$; in such a case, according to [Geo] we write also $x \perp y$. We recall that

(i) $x \perp y$ iff $x^{-\sim} \leq y^{\sim}$;
(ii) $x^{\sim} \perp x$, $x \perp x^{-}$;
(iii) if $x \leq y$, then $x \perp y^{-}$ and $y^{\sim} \perp x$;
(iv) if M is commutative, then $x \perp y$ iff $y \perp x$.

It is worth recalling that if M is a pseudo MV-algebra, then $+$ defined above for pseudo MV-algebras and that defined for pseudo MV-algebras understood as good bounded $R\ell$-monoids coincide.

We define three kinds of states. First, let M be a pseudo MV-algebra. According to [Dvu2], a mapping $s\colon M \to [0,1]$ such that

(i) $s(1) = 1$,
(ii) $s(x+y) = s(x) + s(y)$ whenever $x+y$ is defined in M

is said to be a *state*. The basic properties of states are investigated in [Dvu2]. We have, e.g.

$$s(x \oplus y) + s(x \odot y) = s(x) + s(y)\,, \qquad x, y \in M\,. \tag{3.1}$$

Let now M be a good bounded $R\ell$-monoid. Inspiring by [Geo], we say that a mapping $s\colon M \to [0,1]$ such that $s(1) = 1$ is

(i) a *Bosbach state* if for all $x, y \in M$
 (B1) $s(x) + s(x \to y) = s(y) + s(y \to x)$,
 (B2) $s(x) + s(x \rightsquigarrow y) = s(y) + s(y \rightsquigarrow x)$,
 (B3) $s(0) = 0$,
(ii) a *Riečan state* if $s(x+y) = s(x) + s(y)$ whenever $x+y$ is defined in M.

We note that Bosbach states were defined in any bounded $R\ell$-monoid in the same way as here, see [DvRa2]. We recall that a state and a Riečan state on a pseudo MV-algebra coincide. The main properties of Bosbach states and Riečan ones are exhibited for pseudo BL-algebras in [Geo; Proposition 2.7] and [Geo; Proposition 2.13], respectively. The following statement was proved for pseudo BL-algebras [Geo; Proposition 2.13].

PROPOSITION 3.1. *Let s be a Riečan state on a good bounded $R\ell$-monoid M. Then for all $x, y \in M$ we have*

(i) $s(x^{-}) = s(x^{\sim}) = 1 - s(x)$;
(ii) $s(0) = 0$;
(iii) $s(x^{-\sim}) = s(x^{\sim -}) = s(x)$;
(iv) *if* $x \leq y$, *then* $s(y) - s(x) = 1 - s(x \oplus y^{-}) = 1 - s(y^{\sim} \oplus x)$;
(v) *if* $x \leq y$, *then* $s(x) \leq s(y)$;
(vi) $s(x \vee y) + s(x \wedge y) = s(x) + s(y)$.

P r o o f.

(i) $s(x) + s(x^-) = s(x + x^-) = s\big((x^{-\sim} \odot x^{\sim})^-\big) = 1$.

(ii) (v) can be proved analogically as those in [Geo; Proposition 3.1].

(vi) $s(x \vee y) + s(x \wedge y) = s\big((x \vee y)^{-\sim}\big) + s\big((x \wedge y)^{-\sim}\big) = s\big(x^{-\sim} \vee_{\mathcal{M}} y^{-\sim}\big) + s\big(x^{-\sim} \wedge_{\mathcal{M}} y^{-\sim}\big) = s(x^{-\sim}) + s(y^{-\sim}) = s(x) + s(y)$ when we have restricted s onto $\mathcal{MV}(M)$. $\square$

In [Geo; Proposition 2.8] it is proved that if M is a pseudo MV-algebra, then a mapping s on M is a state iff s is a Bosbach state. Moreover, every Bosbach state on a good pseudo BL-algebra is a Riečan state, [Geo; Proposition 2.14], and there was formulated an open problem whether there exists a Riečan state which is not a Bosbach one [Geo; Open problem 2.15]. We show that at least for the case of good bounded $R\ell$-monoids these two notions of states coincide.

Let s be a state (of any kind), we define the kernel $\mathrm{Ker}(s)$ by

$$\mathrm{Ker}(s) = \big\{x \in M : \ s(x) = 1\big\}.$$

If s is a Bosbach state, then $\mathrm{Ker}(s)$ is a normal filter, $M/\mathrm{Ker}(s)$ is an MV-algebra, and $\hat{s}$ defined on $M/\mathrm{Ker}(s)$ by $\hat{s}(\hat{x}) = s(x)$, is a state, where $\hat{x}$ is a coset in $M/\mathrm{Ker}(s)$ determined by the element $x \in M$ [DvRa2; Theorem 4.6].

PROPOSITION 3.2. *Let M be a good bounded $R\ell$-monoid. Any Bosbach state on M is a Riečan state.*

P r o o f. Let s be a Bosbach state on M and let $x \perp y$, i.e., $y^{-\sim} \le x^-$. Then by [DvRa2], we have $s(y^{-\sim}) + s(y^{-\sim} \to x^-) = s(x^-) + s(x^- \to y^{-\sim})$ and $s(y) + 1 = 1 - s(x) + s(x^- \to y^{-\sim})$, i.e., $s(x^- \to y^{-\sim}) = s(x) + s(y)$. Therefore, $s(x + y) = s\big((y^- \odot x^-)^{\sim}\big) = s(x^- \rightsquigarrow y^{-\sim}) = s(x^- \to y^{-\sim}) = s(x) + s(y)$. $\square$

In what follows, we assume that M is a good bounded $R\ell$-monoid.

PROPOSITION 3.3. *Let s be a Riečan state on a good bounded $R\ell$-monoid M. Then the kernel* $\mathrm{Ker}(s)$ *is a normal filter on M, and the mapping $\hat{s}$ defined on $\hat{M} = M/\mathrm{Ker}(s)$ by $\hat{s}(\hat{x}) = s(x)$ $(x \in M)$ is a Riečan state on $\hat{M}$, where $\hat{x}$ is the equivalence class determined by $x \in M$.*

P r o o f.

Claim 1. $\mathrm{Ker}(s)$ *is a normal filter of M.*

It is clear that $1 \in \mathrm{Ker}(s)$. Assume $x, y \in \mathrm{Ker}(s)$. We show that $x \odot y \in \mathrm{Ker}(s)$.

First take into account that $x^{-\sim}$ and $y^{-\sim}$ belong to the pseudo MV-algebra $\mathcal{MV}(M)$. Therefore, by (3.1) and Proposition 3.1(iii) we have

$$\begin{aligned}
s(x^{-\sim} \oplus y^{-\sim}) + s(x^{-\sim} \odot_{\mathcal{M}} y^{-\sim}) &= s(x^{-\sim}) + s(y^{-\sim}),\\
s(x \oplus y) + s(x^{-\sim} \odot_{\mathcal{M}} y^{-\sim}) &= s(x) + s(y),\\
s(x^{-\sim} \odot_{\mathcal{M}} y^{-\sim}) &= 1,\\
s\big((x^{-\sim} \odot y^{-\sim})^{-\sim}\big) &= 1,\\
s(x^{-\sim} \odot y^{-\sim}) &= 1.
\end{aligned}$$

On the other hand, by Proposition 3.1(iii) and Lemma 2.1(3), we have $1 = s(x^{-\sim} \odot y^{-\sim}) = 1 - s\big((x^{-\sim} \odot y^{-\sim})^{-}\big) = 1 - s(x^{-\sim} \to y^{-\sim-}) = 1 - s\big(x^{-\sim} \to (y^{-})^{\sim-}\big) = 1 - s\big(x \to (y^{-})^{\sim-}\big) = 1 - s(x \to y^{-}) = 1 - s\big((x \odot y)^{-}\big) = s(x \odot y)$, which proves that $\mathrm{Ker}(s)$ is a filter.

Since by (10) of Lemma 2.1 $(x \to y)^{-\sim} = x^{-\sim} \to y^{-\sim}$ and $(x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim}$, we have that if $s(x \to y) = 1$, then $1 = s(x \to y) = s\big((x \to y)^{-\sim}\big) = s(x^{-\sim} \to y^{-\sim})$. Since $x^{-\sim}, y^{-\sim} \in \mathcal{MV}(M)$ and by Theorem 2.3, $\mathcal{MV}(M)$ is a pseudo MV-algebra, the restriction of s onto $\mathcal{MV}(M)$ is a state, then its kernel is always a normal filter, [Dvu2]. Therefore, $s(x^{-\sim} \rightsquigarrow y^{-\sim}) = 1$ that is $s(x \rightsquigarrow y) = 1$. The converse implication is similar.

Claim 2. *If $x \approx y$, then $s(x) = s(y)$.*

Suppose now $x \approx y$, that is $s(x \to y) = 1 = s(y \to x)$. By the above, we have $s(x^{-\sim} \to y^{-\sim}) = s(y^{-\sim} \to x^{-\sim})$, i.e., $x^{-\sim} \approx y^{-\sim}$. Since the elements $x^{-\sim}$ and $y^{-\sim}$ belong to the pseudo MV-algebra $\mathcal{MV}(M)$, we know that for states on pseudo MV-algebras we have then $s(x^{-\sim}) = s(y^{-\sim})$, i.e., $s(x) = s(y)$.

Claim 3. *$\hat{s}$ is a Riečan state on $\hat{M}$.*

First we recall that if $\hat{a} \le \hat{b}$, then there is an element $a_1 \in \hat{a}$ such that $a_1 \le b$. Indeed, it is sufficient to take $a_1 = a \wedge b$.

Assume now $\hat{x} \perp \hat{y}$, that is $\hat{x}^{-\sim} \le \hat{y}^{\sim}$ and take $x_1 \in \hat{x}^{-\sim}$ such that $x_1 \le y^{\sim}$. Hence $x_1^{-\sim} \le y^{\sim}$. Therefore, $\hat{s}(\hat{x} + \hat{y}) = \hat{s}\big((\hat{y}^{\sim} \odot \hat{x}^{\sim})^{-}\big) = \hat{s}\big(((y^{\sim} \odot x^{\sim})^{-})\hat{}\big) = s\big((y^{\sim} \odot x^{\sim})^{-}\big) = s(x \oplus y) = s(x^{-\sim} \oplus y^{-\sim}) = \hat{s}(\hat{x}_1 + \hat{y}^{-\sim}) = s(x_1 + y^{-\sim}) = s(x_1) + s(y^{-\sim}) = s(x) + s(y) = \hat{s}(\hat{x}) + \hat{s}(\hat{y})$. □

We denote by $\mathcal{S}(M)$ and $\mathcal{RS}(M)$ the set of all Bosbach states and Riečan states on M, respectively. Both sets are convex (which in the commutative case of M are always non-empty), that is, if $s_1, s_2 \in \mathcal{S}(M)$ and $\lambda \in [0,1]$, then $s = \lambda s_1 + (1-\lambda)s_2 \in \mathcal{S}(M)$. Similar for the second set. We denote by $\mathrm{Ext}_{\mathcal{S}}(M)$ and $\mathrm{Ext}_{\mathcal{RS}}(M)$ the sets of extremal Bosbach and extremal Riečan states on M, respectively.

We recall that a mapping s from a bounded $R\ell$-monoid M into the standard BL-algebra $[0,1]$ is called a *state-morphism* if, for any $x, y \in M$,

(i) $s(x \to y) = s(x \rightsquigarrow y) = s(x) \to s(y)$,
(ii) $s(x \wedge y) = \min\{s(x), s(y)\}$,
(iii) $s(1) = 1$ and $s(0) = 0$.

We know that a Bosbach state s on M is extremal iff s is a state-morphism iff $\mathrm{Ker}(s)$ is a maximal filter which is normal, [DvRa2; Proposition 4.8].

PROPOSITION 3.4. *Every extremal Bosbach state on a good bounded $R\ell$-monoid is an extremal Riečan state.*

Proof. Let s be an extremal Bosbach state and let $s = \lambda s_1 + (1-\lambda)s_2$ where $s_1, s_2 \in \mathcal{RS}(M)$ and $0 < \lambda < 1$. Then $\mathrm{Ker}(s) \subseteq \mathrm{Ker}(s_1) \cap \mathrm{Ker}(s_2)$, and the maximality of the filter $\mathrm{Ker}(s)$ implies $\mathrm{Ker}(s) = \mathrm{Ker}(s_1) = \mathrm{Ker}(s_2)$. Then $M/\mathrm{Ker}(s)$ is an MV-algebra ([DvRa2; Theorem 4.6]), and $\hat{s}$, $\hat{s}_1$ and $\hat{s}_2$ are Riečan states on $M/\mathrm{Ker}(s)$, Proposition 3.3. Moreover, they are even states on the MV-algebra $M/\mathrm{Ker}(s)$. Indeed, if $\hat{x} \leq \hat{y}^{-'}$, then $\hat{x}^{-'\sim'} = \hat{x} \leq \hat{y}^{-'}$, which means $\hat{x} \perp \hat{y}$ and hence $\hat{s}_i(\hat{x}+\hat{y}) = \hat{s}_i(\hat{x}) + \hat{s}_i(\hat{y})$ for $i = 1,2$. Since $M/\mathrm{Ker}(s)$ is linear, it admits a unique state. Therefore $\hat{s} = \hat{s}_1 = \hat{s}_2$ and consequently $s = s_1 = s_2$ proving $s \in \mathrm{Ext}_{\mathcal{RS}}(M)$. □

PROPOSITION 3.5. *Let s be a Riečan state on a good bounded $R\ell$-monoid M. Then $x \approx y$ if and only if $s(x \wedge y) = s(x \vee y)$. In particular, if $x \leq y$, then $x \approx y$ if and only if $s(x) = s(y)$.*

Proof. Assume first $x \leq y$ and $s(x) = s(y)$. Then $x^{-\sim} \leq y^{-\sim}$ and $s(x^{-\sim}) = s(y^{-\sim})$. The state s restricted to the MV-part of M, $\mathcal{MV}(M)$, gives $s(x^{-\sim} \to y^{-\sim}) = 1 = s(y^{-\sim} \to x^{-\sim}) = s((y \to x)^{-\sim}) = s(y \to x)$ when we have used (10) of Lemma 2.1. Hence $x \approx y$.

Let now x and y be arbitrary elements of M, then by the first part of the present proof, $x \wedge y \approx x \vee y$. On the other hand $s(x) = s(y)$. Therefore, $x \approx x \wedge y \approx y$. □

PROPOSITION 3.6. *Let s be a Riečan state on a good bounded $R\ell$-monoid M. Then $M/\mathrm{Ker}(s)$ is an MV-algebra and $\hat{s}$ is a Bosbach state.*

Proof. Since $s(x) = s(x^{-\sim})$ and $x \leq x^{-\sim}$, we have by Proposition 3.5 that $x \approx x^{-\sim}$ and consequently $\hat{x} = \hat{x}^{-\sim}$, which proves $M/\mathrm{Ker}(s)$ is an MV-algebra. Due to the property that $\hat{x} \leq \hat{y}^{-'}$, we have $\hat{x}^{-'\sim'} \leq \hat{y}^{-'}$, which proves that $\hat{s}$ is a state on $M/\mathrm{Ker}(s)$, consequently a Bosbach state. □

PROPOSITION 3.7. *If s is an extremal Riečan state on a good bounded $R\ell$-monoid M, then it is an extremal Bosbach state.*

Proof. Assume s is an extremal Riečan state on M. We assert that $\hat{s}$ is an extremal Bosbach state on the MV-algebra $M/\operatorname{Ker}(s)$. Indeed, assume $\hat{s} = \lambda\mu_1 + (1-\lambda)\mu_2$, where μ_1, μ_2 are states on $M/\operatorname{Ker}(s)$ and $0 < \lambda < 1$. Then s_i defined by $s_i(x) := \mu_i(\hat{x})$, $x \in M$, are Bosbach states on M and consequently Riečan states. It is evident that $s = \lambda s_1 + (1-\lambda)s_2$ which implies $s = s_1 = s_2$ and $\hat{s} = \mu_1 = \mu_2$. This proves $\hat{s}$ is an extremal Bosbach state on $M/\operatorname{Ker}(s)$. The above characterization of extremal Bosbach states as state-morphisms implies that s is a state-morphism, consequently s is an extremal Bosbach state. □

We now present the main result giving as a by-product the answer to Georgescu's open problem [Geo; 2.15].

THEOREM 3.8. *Every Riečan state on a good bounded $R\ell$-monoid is a Bosbach state.*

Proof. If we define the weak topology on $\mathcal{RS}(M)$ by saying that a net $\{m_\alpha\} \to m$ if $\lim\limits_\alpha m_\alpha(a) = m(a)$ for any $a \in M$, then $\mathcal{RS}(M)$ is a compact Hausdorff topological space. Due to the Krein-Mil'man theorem, [Goo; Theorem 5.17], every Riečan state on M is a weak limit of convex combinations of extremal Riečan states. The similar property holds also for $\mathcal{S}(M)$. Due to Proposition 3.4 and Proposition 3.7, we have $\operatorname{Ext}_{\mathcal{RS}}(M) = \operatorname{Ext}_{\mathcal{S}}(M)$. Therefore, $\mathcal{RS}(M) = \mathcal{S}(M)$. □

4. Weak States

In the appendix we define a weak state on a good bounded $R\ell$-monoid, and we show that it coincides with a Riečan state.

Let M be a good bounded $R\ell$-monoid. We define a new partial addition $+_w$ on M: $x +_w y$ is defined by $x +_w y = z$ iff $y \le x^-$ and $z = x \oplus y$; in such a case we write also $x \perp_w y$.

A *weak state* on M is a mapping $s\colon M \to [0,1]$ such that

(i) $s(1) = 1$,

(ii) $s(x +_w y) = s(x) + s(y)$ whenever $x +_w y$ is defined in M.

THEOREM 4.1. *Every weak state on a good bounded $R\ell$-monoid is a Riečan state and vice versa.*

Proof. Suppose that $x \perp_w y$, then $y \le x^-$ i.e., $y^{-\sim} \le x^-$ and $x \perp y$. Conversely, let $x \perp y$, then $y \le y^{-\sim} \le x^-$ and $x \perp_w y$. Therefore, if s is a weak state and $x \perp y$, then $s(x+y) = s(x +_w y) = s(x) + s(y)$, and s is a Riečan state. Conversely, if s is a Riečan state and if $x \perp_w y$, then $s(x +_w y) = s(x+y) = s(x) + s(y)$, and s is a weak state. □

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