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ON COMPLETE LATTICE ORDERED GROUPS WITH TWO GENERATORS II

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§ 7. Singular complete lattice ordered groups with two generators

If A is a direct factor of a lattice ordered group G and if $M \subseteq G$, then we denote

$$M(A) = \{m(A) : m \in M\}.$$

7.1. Lemma. *Assume that a set $M \neq \emptyset$ generates a complete lattice ordered group G and that A is a direct fector of G. Then the set $M(A)$ generates the complete lattice ordered group A.*

Proof. According to the assumption there exists an l-subgroup B of G with $G = A \times B$. Let H_1 be a closed l-subgroup of A such that $M(A) \subseteq H_1$. Let H be the set of all elements $g \in G$ with $g(A) \in H_1$. Clearly $H = H_1 \times B$. Then H is a closed l-subgroup of G and $M \subseteq H$; thus $H = G$. From this it follows $H_1 = A$.

Let G be a lattice ordered group. An element $0 \leqq e \in G$ is said to be a weak unit in G if $e \wedge x > 0$, whenever $0 < x \in G$.

The following assertion is known (cf. [10]).

7.2. Lemma. *Let G be a singular complete lattice ordered group with a weak unit. Let $0 \leqq g \in G$.*

(a) *There exists a (uniquely determined) singular element e in G such that e is a weak unit in G and $e_i \leqq e$ for each singular element e_i of G.*

(b) *There are singular elements e_i $(i \in N)$ such that the set $\{e_i\}$ $(i \in N)$ is disjoint and*

$$g = \bigvee_{i \in N} i e_i$$

holds in G.

The assertion (b) from 7.2 can be generalized as follows:

7.3. Lemma. *Let G be a singular complete lattice ordered group containing a weak unit. Let $g \in G$. Then there are singular elements e_i $(i \in N_0)$ such that*

(a) *the set $\{e_i\}$ $(i \in N_0)$ is disjoint;*

(b) *there exists a singular element $e \in G$ such that e is a weak unit in G and*

$$e = \bigvee_{i \in N_0} e_i .$$

(c) $g^+ = \bigvee_{i \in N} ie_i$, $g^- = \bigvee_{i \in N_y \setminus N} -ie_i$.

Proof. According to 7.2 there exist singular elements e_i $(i \in N)$ in G such that the set $\{e_i\}$ $(i \in N)$ is disjoint and

$$g^+ = \bigvee_{i \in N} ie_i .$$

Analogously there exist singular elements e'_i $(i \in N)$ in G such that the set $\{e'_i\}$ $(i \in I)$ is disjoint and

$$g^- = \bigvee_{i \in N} ie'_i .$$

Also, according to 7.2 there exists $e \in G$ such that e is singular, e is the join of all singular elements of G and e is a weak unit in G. Thus there is $e_0 \in G$ with

$$e_0 = e - \left(\bigvee_{i \in N} e_i\right) \vee \left(\bigvee_{i \in N} e'_i\right).$$

Since $g^+ \wedge g^- = 0$, we have $e_i \wedge e'_j = 0$ for each $i \in N$ and each $j \in N$. Put $e'_i = e_{-i}$ for each $i \in N$. Then the set $\{e_i\}$ $(i \in N_0)$ is disjoint and

$$e = \bigvee_{i \in N_0} e_i ,$$

$$g^- = \bigvee_{i \in N_0 \setminus N} -ie_i .$$

7.4. Lemma. *Let G be a singular complete lattice ordered group with a weak unit. Let e_1, e_2 be singular elements of G. Then $e_1[e_2] = e_1 \wedge e_2$.*

Proof. Let e be as in 7.2. Hence $e_1 \leqq e$, $e_2 \leqq e$. From the definition of a singular element it follows that the interval $[0, e]$ of G is a Boolean algebra. Hence there exists the relative complement e'_2 of e_2 in the interval $[0, e]$. Put $x = e_1 \wedge e'_2$. Then $e_1 = (e_1 \wedge e_2) \vee x$, $x \wedge e_2 = 0$. From this we obtain $e_1 = (e_1 \wedge e_2) + x$, $x[e_2] = 0$, hence

$$e_1[e_2] = (e_1 \wedge e_2)\,[e_2] = e_1 \wedge e_2 .$$

7.5. Lemma. *Let G, g, e, e_i $(i \in N_0)$ have the same meaning as in 7.3. Let $i \in N_0$ and let e' be a singular element in G, $e' \leqq e_i$. Then $g[e'] = ie'$.*

Proof. If $i = 0$, then $|g| \wedge e_i = 0$, hence $|g| \wedge e' = 0$ and thus $g[e'] = 0$. Let $i \in N$. In this case we have $g^- \wedge e_i = 0$, hence $g^- \wedge e' = 0$, from which we infer $g^-[e'] = 0$ and $g[e'] = g^+[e']$. Further, we have

$$g^+ = ie_i \vee \left(\bigvee_{j \in N \setminus \{i\}} je_j\right) = ie_i + \bigvee_{j \in N \setminus \{i\}} je_j .$$

For each $j \in N \setminus \{i\}$ the relation $e_j \wedge e' = 0$ is valid, whence

$$\left(\bigvee_{j \in N \setminus \{i\}} je_j\right) [e'] = 0 .$$

According to 7.4, $ie_i[e'] = ie'$. By summarizing, we obtain

$$g[e'] = ie' .$$

The method for $i \in N_0$, $i < 0$ is analogous.

For a lattice ordered group G we define the completely subdirect product decomposition of G as follows.

Let $\{A_i\}$ $(i \in I)$ be a set of direct factors of G such that

(a) $A_i \cap A_j = 0$, whenever i and j are distinct elements of I;

(b) for each $g \in G$, $g > 0$ there exists $i \in I$ with $g(A_i) > 0$.

Then G is said to be a completely subdirect product of its l-subgroups G_i.

The notion of the completely subdirect product has been introduced by F. Šik [17] (in a formally different, but equivalent, way). It is not hard to verify that G is a completely subdirect product of its l-subgroups $A_i (i \in I)$ if and only if for each $0 < g \in G$ there are uniquely determined elements $g_i \in A_i$ such that $g = \bigvee_{i \in I} g_i$. (Cf. also [14], §3.)

If G and A_i fulfil the above mentioned conditions, then the mapping f defined by

$$f(x) = \{x(A_i)\}_{i \in I} \text{ for each } x \in G$$

is an isomorphism of G into the direct product $\prod_{i \in I} A_i$.

7.6. Lemma. *Let G be a lattice ordered group. Suppose that*

(a) G is a completely subdirect product of its l-subgroups A_i $(i \in I)$;

(b) G is a completely subdirect product of its l-subgroups B_j $(j \in J)$.

Then G is a completely subdirect product of its l-subgroups $A_i \cap B_j$ $(i \in I, j \in J)$.

Proof. Denote $A_i \cap B_j = C_{ij}$. If $i, i_1 \in I$, $j, j_1 \in J$ and $(i, j) \neq (i_1, j_1)$, then clearly

$$C_{ij} \cap C_{i_1 j_1} = \{0\} .$$

Let $0 < g \in G$. According to (a) there is $i \in I$ with $0 < g(A_i)$. Hence according to (b) there is $j \in J$ such that $(g(A_i))(B_j) > 0$. Thus

$$g(C_{ij}) = g(A_i \cap B_j) = (g(A_i))(B_j) > 0 .$$

7.7. Lemma. *Suppose that a finite set $g_1, g_2, \ldots, g_n$ generates a complete lattice ordered group G. Then the element $h = |g_1| \vee |g_2| \vee \ldots \vee |g_n|$ is a weak unit in G.*

Proof. The set $[h]$ is a closed l-subgroup of G and h is a weak unit in $[h]$. Clearly $g_i \in [h]$ for $i = 1, 2, \ldots, n$. Hence $[h] = G$.

Now suppose that a two-element set $\{g_1, g_2\}$ generates a singular complete lattice ordered group G. According to 7.7, G contains a weak unit and hence by

7.2 there exists a singular element e in G such that e is a weak unit in G. Let e_i $(i \in N_0)$ have the same meaning as in Lemma 7.3 for $g = g_1$ and let the elements e'_i $(i \in N_0)$ have an analogous meaning for $g = g_2$. Denote

$$G_{ij} = [e_i] \cap [e'_j], \quad f_{ij} = e_i \wedge e'_j \qquad (i, j \in N_0).$$

7.8. Lemma. *Let G be as above. Then we have:*

(a) *G is a completely subdirect product of its l-subgroups G_{ij} $(i, j \in N_0)$.*

b) *For each $i, j \in N_0$, either $G_{ij} = \{0\}$ or G_{ij} is isomorphic with N_0.*

(c) *If $i, j \in N_0$, $G_{ij} \neq \{0\}$ and $i \neq 0 \neq j$, then the integers i, j are relatively prime.*

(d) *If $i = j = 0$, then $G_{ij} = \{0\}$. If $i = 0$ and $G_{ij} \neq \{0\}$, then $j \in \{1, -1\}$. If $j = 0$ and $G_{ij} \neq \{0\}$, then $i \in \{1, -1\}$.*

(e) *If $i, j \in N_0$, $G_{ij} \neq \{0\}$, then*

$$g_1(G_{ij}) = if_{ij}, \quad g_2(G_{ij}) = jf_{ij}.$$

Proof. From 7.3 we obtain that for each $0 < g \in G$ there exists $i \in N_0$ with $e_i \wedge g > 0$, and that the set $\{e_i\}$ $(i \in N_0)$ is disjoint. From this it follows that the system of direct factors $[e_i]$ $(i \in N_0)$ fulfils the conditions (a) and (b) in the definition of the completely subdirect product decomposition. Hence G is a completely subdirect product of its l-subgroups $[e_i]$ $(i \in N_0)$. Analogously, G is a completely subdirect product of its l-subgroups $[e'_j]$ $(j \in N_0)$. Hence from 7.6 we obtain that G is a completely subdirect product of its l-subgroups

$$G_{ij} = [e_i] \cap [e'_j] \qquad (i, j \in N_0).$$

Let $i, j \in N_0$. According to 7.1, the set

$$\{g_1(G_{ij}), \quad g_2(G_{ij})\}$$

generates the complete lattice ordered group G_{ij}. From the properties of principal polars it follows $[e_i] \cap [e'_j] = [e_i \wedge e'_j]$. Thus by 7.5 we have

$$g_1(G_{ij}) = i(e_i \wedge e'_j), \quad g_2(G_{ij}) = j(e_i \wedge e'_j).$$

Hence in the case $i = j = 0$ we obtain $G_{ij} = \{0\}$.

Suppose that $G_{ij} \neq \{0\}$. Thus $0 < e_i \wedge e'_j \in G_{ij}$. First let us consider the case $i = 0$. Then the one-element set $\{j(e_i \wedge e'_j)\}$ generates the complete lattice ordered group $G_{ij} \neq \{0\}$ and the element $j(e_i \wedge e'_j)$ is comparable with 0. Thus G_{ij} is the set

$$\{mj(e_i \wedge e'_j)\} \; (m \in N_0).$$

Since $e_i \wedge e'_j \in G_{ij}$, we must have either $j = 1$ or $j = -1$. Moreover, G_{ij} is isomorphic with N_0. The case $j = 0$ is analogous.

Further let us assume that $i \neq 0 \neq j$. Let $k \in N$ be the greatest common divisor of the integers i and j. Then the set

$$H_{ij} = \{mk(e_i \wedge e'_j)\}\ (m \in N_0)$$

is a closed l-subgroup of G_{ij} and

$$g_1(G_{ij}) \in H_{ij}, \quad g_2(G_{ij}) \in H_{ij}.$$

Thus $H_{ij} = G_{ij}$. From this it follows that G_{ij} is isomorphic with N_0. Since $e_i \wedge e'_j \in H_{ij}$, we must have $k = 1$. The proof is complete.

Now suppose that a singular complete lattice ordered group is generated by a set $\{g_1, \ldots, g_n\}$. According to 7.7 and 7.2 there exists a singular element e in G such that e is a weak unit in G. Let $k \in \{1, \ldots, n\}$ and let e_{ik} $(i \in N_0)$ have a meaning analogous to that of e_i $(i \in N_0)$ in 7.3 if we put $g = g_k$. For each $i_1, \ldots, i_n \in N_0$ we denote

$$G(i_1, \ldots, i_n) = [e_{i_1,1}] \cap [e_{i_2,2}] \cap \ldots \cap [e_{i_n,n}].$$

Further, we denote by $N(i_1, \ldots, i_n)$ the set of all integers that belong to the set $\{i_1, \ldots, i_n\}$ and are distinct from 0.

By a method analogous to that in the proof of 7.8 we obtain:

7.9. Lemma. *Let G fulfil the above mentioned assumptions. Then:*

(a) *G is a completely subdirect product of its l-subgroups $G(i_1, \ldots, i_n)$ $(i_1, \ldots, i_n \in N_0)$.*

(b) *For each $i_1, \ldots, i_n \in N_0$ either $G(i_1, \ldots, i_n) = \{0\}$ or $G(i_1, \ldots, i_n)$ is isomorphic with N_0.*

(c) *If $i_1 = i_2 = \ldots = i_n = 0$, then $G(i_1, \ldots, i_n) = \{0\}$. If at least one of the numbers $i_1, \ldots, i_n$ is distinct from zero, $G(i_1, \ldots, i_n) \neq \{0\}$ and if $k \in N$ is the greatest common divisor of integers belonging to $N(i_1, \ldots, i_n)$, then $k = 1$.*

(d) *If $i_1, \ldots, i_n \in N_0$, then*

$$g_k(G(i_1, \ldots, i_n)) = i_k(e_{1_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_n})$$

holds for each $k \in \{1, \ldots, n\}$.

7.10. Lemma. *Let G be a complete lattice ordered group. Let $\{G_{ij}\}$ $(i, j \in N_0)$ be a system of l-subgroups of G and let $g_1, g_2 \in G$. Suppose that the conditions* (a)—(d) *from 7.8 are fulfilled. Further suppose that the following condition holds:*

(e′) *If $i, j \in N_0$, $G_{ij} \neq \{0\}$, then*

$$g_1(G_{ij}) = if_{ij}, \quad g_2(G_{ij}) = jf_{ij},$$

where f_{ij} is the least positive element of G_{ij}. Then the set $\{g_1, g_2\}$ generates the complete lattice ordered group G.

Proof. Let H be the intersection of all closed l-subgroups of G containing both g_1 and g_2. Then the set $\{g_1, g_2\}$ generates the complete lattice ordered group H. For $x \in H$ we denote by $[x]_1$ the principal polar in H generated by the element x.

Let φ be the identical mapping on the set H. Then φ is a complete homomorphism of H into G, hence according to 5.14 for each $x, y \in H$ the relation

(*) $$x[y]_1 = x[y]$$

holds.

We have to verify that $H = G$. Let $i, j \in N_0$. If $G_{ij} = \{0\}$, we put $f_{ij} = 0$. If $G_{ij} \neq \{0\}$, then according to (b) there exists $f_{ij} \in G_{ij}$ having the property that f_{ij} covers 0 in G_{ij}. Denote

$$g_1[|jg_1 - ig_2|] = g_1^*.$$

From (*) we obtain $g_1^* \in H$. According to (e′) we have

$$g_1 - g_1^* = if_{ij},$$

hence $if_{ij} \in H$. Analogously we obtain $jf_{ij} \in H$. If $i = 0$ or $j = 0$, then according to (d) we have $f_{ij} \in H$. In the case $i \neq 0 \neq j$, the integers i and j are relatively prime, thus $f_{ij} \in H$ as well.

Let $0 \leqq g \in G$. From the condition (a) it follows

$$g = \bigvee g(G_{ij}) \qquad (i, j \in N_0).$$

According to (b) and (e′) for each $i, j \in N_0$ there exists a non-negative integer k_{ij} such that $g(G_{ij}) = k_{ij}f_{ij}$. Thus

$$g = \bigvee k_{ij}f_{ij} \qquad (i, j \in N_0).$$

Since $f_{ij} \in H$ for each $i, j \in N_0$ and since H is a closed l-subgroup of G, we obtain $g \in H$. From this it follows $G = H$, completing the proof.

Now suppose that a set $\{g_1, g_2\}$ generates a singular complete lattice ordered group G and that a set $\{g_1', g_2'\}$ generates a singular complete lattice ordered group G'. For $i, j \in N$ let f_{ij} have the same meaning as in 7.8. Further let the symbols f'_{ij} have an analogous meaning with respect to G'.

7.11. Lemma. *Let G and G' be as above. Let φ be a complete homomorphism of G into G' such that $\varphi(g_1) = g_1'$, $\varphi(g_2) = g_2'$. Then $\varphi(f_{ij}) = f'_{ij}$ holds for each i, $j \in N_0$.*

Proof. Let $i, j \in N_0$. Analogously as in the proof of 7.10 we denote

$$g_1^* = g_1[|jg_1 - ig_2|], \quad g_1'^* = g_1'[|jg_1' - ig_2'|].$$

Then we have

$$\varphi(g_1^*) = g_1'^*,$$
$$g_1 - g_1^* = if_{ij},$$
$$g_1' - g_1'^* = if_{ij},$$

hence

$$\varphi(if_{ij}) = if'_{ij}.$$

Analogously we get

$$\varphi(jf_{ij}) = jf'_{ij}.$$

If $i \neq 0$ or $j \neq 0$, then we infer $\varphi(f_{ij}) = f'_{ij}$. If $i = j = 0$, then $f_{ij} = 0 = f'_{ij}$, thus $\varphi(f_{ij}) = f'_{ij}$ as well.

§ 8. *a*-free complete lattice ordered groups in the class $\mathscr{C}_s$

In this paragraph there will be described the a-free complete lattice ordered group with two a-free generators in the class $\mathscr{C}_s$.

The following lemmas 8.1—8.3 give us a deeper insight into the situation we are investigating and the reasons why we are dealing with the lattice ordered group G_2 in the proof of 8.5.

Let N' be the set of all pairs (i, j) $(i, j \in N_0)$ fulfilling some of the following conditions:

(a) $i \neq 0 \neq j$ and the integers i, j are relatively prime;

(b) $i = 0$ and $j = 1$, or $i = 1$ and $j = 0$.

First we deduce two necessary conditions for an a-free complete lattice ordered group with two a-free generators in $\mathscr{C}_s$.

8.1. Lemma. *Let G be an a free complete lattice ordered group with two a-free generators g_1, g_2 in the class $\mathscr{C}_s$. Then (under the denotation of* 7.8) *we have $f_{ij} > 0$ for each $(i, j) \in N'$.*

Proof. Let $i, j \in N_0$. Then $f_{ij} \geqq 0$. For each $(n, m) \in N'$ we set $G'_{nm} = N_0$. Further we denote

$$G' = \Pi G'_{nm}((n, m) \in N').$$

Choose g'_1, $g'_2 \in G'$, fulfilling

$$g'_1(n, m) = n, \quad g'_2(n, m) = m$$

for each $(n, m) \in N'$. According to 7.10 the set $\{g'_1, g'_2\}$ generates the complete singular lattice ordered group G'. Moreover (under denotations analogous to those in § 7) we have the relations

$$f'_{ij}(i, j) = 1,$$

$f'_{ij}(p, r) = 0$, whenever $(p, r) \in N'$ and $(p, r) \neq (i, j)$;

hence $f'_{ij} > 0$.

According to the assumption there exists a complete homomorphism φ of G into G' such that $\varphi(g_i) = g'_i$ $(i = 1, 2)$. Hence by 7.11,

$$\varphi(f_{ij}) = f'_{ij}.$$

Since $f'_{ij} \neq 0$, we obtain $f_{ij} \neq 0$ and thus $f_{ij} > 0$.

Let G have the same meaning as in 8.1. Put $g_3 = |g_1| \vee |g_2|$. Further, we denote

$$G_0 = \bigcup_{n \in N} [-ng_3, ng_3].$$

Let $o(G)$ be the orthogonal hull of G.

8.2. Lemma. There exists $0 < g \in o(G)$ such that $g \notin G_o$.

Proof. Let N' be as in the proof of 8.1. Further, we shall use the same denotations as in 7.8. The set

$$\{ijf_{ij}\} \quad ((i, j) \in N')$$

is disjoint. If $(i, j) \in N'$, $i > 0$, $j > 0$, then $ijf_{ij} > 0$ according to 8.1. Since $o(G)$ is orthogonally complete, there exists $0 < g \in G$ such that

$$g = \bigvee ijf_{ij} \quad ((i, j) \in N')$$

holds in $o(G)$. Let $n \in N$, $(i, j) \in N'$, $i > n$, $j > n$. Then

$$ng_3(G_{ij}) = n \max\{i, j\} f_{ij} < ijf_{ij} = g(G_{ij}).$$

Thus for each $n \in N$ we have $g \nleqq ng_3$. Therefore $g \notin G_0$.

8.3. Lemma. *Let G, G_0 be as in 8.2. Then $G = G_0$.*

Proof. We have $G_0 \subseteq G$, $G_0 \in \mathscr{C}_s$ and $\{g_1, g_2\} \subseteq G_0$. Hence there exists a complete homomorphism φ of G into G_0 such that $\varphi(g_i) = g_i$ $(i = 1,2)$. Thus according to 5.8, $\varphi(G) = G$. Therefore $G = G_0$.

Let G', g'_1, g'_2 be as in the proof of 8.1. Put $g'_3 = |g'_1| \vee |g'_2|$ and denote

$$G_2 = \bigcup_{n \in N} [-ng'_3, ng'_3].$$

Then G_2 can be also characterized as the set of all elements $g' \in G'$ having the following property: there exists a positive integer $n = n(g')$ such that

$$|g'(i, j)| \leqq n \cdot \max\{|i|, |j|\}$$

for each $(i, j) \in N'$.

8.4. Theorem. *The set $\{g'_1, g'_2\}$ is a set of a-free generators of the complete lattice ordered group G_2 in $\mathscr{C}_s$.*

Proof. Obviously $G_2 \in \mathscr{C}_s$. According to 7.10, the set $\{g'_1, g'_2\}$ generates the complete lattice ordered group G_2. Let H be a complete singular lattice ordered group and let $g_1, g_2 \in H$. We denote by G the intersection of all closed l-subgroups of H containing both g_1 and g_2. Then the set $\{g_1, g_2\}$ generates the complete lattice ordered group G. Thus we can use for G the denotations from §7.

We have to show that there exists a complete homomorphism φ of G_2 into G

such that $\varphi(g'_i)=g_i$ $(i=1, 2)$. According to 7.11 it suffices to consider only such mappings φ of G_2 into G that fulfil the relation

$$\varphi(f'_{ij})=f_{ij} \text{ for each } (i, j)\in N'.$$

Let $g'\in G_2$. For each $(i, j)\in N'$ there is an integer c_{ij} such that

$$g'(G'_{ij})=c_{ij}f'_{ij}.$$

From the fact that G is a completely subdirect product of linearly ordered groups G_{ij} $((i, j)\in N')$ it follows that $o(G)$ is the (complete) direct product of linearly ordered groups G_{ij} $(i, j)\in N')$. Thus in $o(G)$ there exists a (uniquely determined) element g such that

$$g(G_{ij})=c_{ij}f_{ij}$$

holds for each $(i, j)\in N'$. Consider the mapping φ of G_2 into $o(G)$ that is defined by

$$\varphi(g')=g$$

(under the above denotations). Since all (not only finite) joins and intersections in a completely subdirect product of lattice ordered groups are performed component -wise, the mapping φ is a complete homomorphism of G_2 into $o(G)$. Now from the fact that G is a convex l-subgroup of $o(G)$ it follows: if $\varphi(G_2)\subseteq G$, then φ is a complete homomorphism of G_2 into G.

From the definition of φ and from 7.8 we obtain

$$\varphi(g'_1)=g_1, \quad \varphi(g'_2)=g_2.$$

Let $g'\in G_2$, $\varphi(g')=g$. According to the definition of G_2 there exists a positive integer n such that

$$|g'|\leqq n(|g'_1|\vee|g'_2|).$$

Since φ is a homomorphism of G_2 into $o(G)$, we obtain

$$|g|\leqq n(|g_1|\vee|g_2|).$$

Because $n(|g_1|\vee|g_2|)\in G$ and since G is convex in $o(G)$ we infer that $g\in G$. This completes the proof.

Let $n>2$ be a fixed integer. Let N'_n be the set of all n-tuples $(i_1, \ldots, i_n)$ of integers that fulfil the following conditions:

(a) at least one of the integers $i_1, \ldots, i_n$ is distinct from 0;

(b) if $k\in N$ is the greatest common divisor of the nonzero integers belonging to the set $\{i_1, \ldots, i_n\}$, then $k=1$.

For each n-tuple $(i_1, \ldots, i_n)\in N'_n$ we put

$$G(i_1, \ldots, i_n)=N_0;$$

further we set

$$G'_n = \Pi G(i_1, \ldots, i_n) \qquad ((i_1, \ldots, i_n) \in N'_n).$$

For each $k \in \{1, \ldots, n\}$ we define an element $g'_k \in G'_n$ by the relations

$$g'_k(i_1, \ldots, i_n) = i_k \text{ for each } (i_1, \ldots, i_n) \in N'_n.$$

Denote $g' = |g'_1| \vee \ldots \vee |g'_n|$,

$$G_n = \prod_{m \in N} [-mg', mg'].$$

8.4'. Theorem. *The set $\{g'_1, g'_2, \ldots, g'_n\}$ is a set of a-free generators of the complete lattice ordered group G_n in $\mathscr{C}_s$.*

The idea of the proof is the same as in 8.4; the denotations would be more complicated. We omit the details.

8.5. Lemma. *Let G be a complete lattice ordered group. Assume that G is a completely subdirect product of its lattice ordered subgroups G_i $(i \in I)$ and that G is orthogonally complete. Then G is a direct product of its l-subgroups G_i $(i \in I)$.*

Proof. Let H be the direct product of lattice ordered groups G_i $(i \in I)$. Without loss of generality we can suppose that G is an l-subgroup of H. Let $0 \leqq h \in H$. Then

$$h = \bigvee_{i \in I} h(G_i)$$

holds in H and $\{h(G_i)\}_{i \in I}$ is a disjoint subset of G. Thus there is $g \in G$ such that the relation

$$g = \sup \{h(G_i)\}_{i \in I}$$

is valid in G.

Thus $h \leqq g$. Since G is a convex subset of $o(G) = H$, we obtain $h \in G$; therefore $h = g$. Hence $H = G$.

8.6. Lemma. *Assume that a set $\{g_1, g_2)$ generates a complete singular lattice ordered group G. Suppose that G is orthogonally complete. Then (under the same denotations as in* §7*) G is a direct product of its l-subgroups G_{ij} $((i, j) \in N')$.*

This assertion follows from 7.8 and 8.5.

8.7. Theorem. *Let G', g'_1 and g'_2 be as in* 8.1. *The set $\{g'_1, g'_2\}$ is a set of b-free generators of the complete lattice ordered group G' in the class $\mathscr{C}_s \cap \mathscr{C}_0$.*

Proof. We have $G' \in \mathscr{C}_s \cap \mathscr{C}_0$ and according to 7.10, the set $\{g'_1, g'_2\}$ generates the complete lattice ordered group G'. We use the same denotations as in 8.4 with the distinction that now we have $g \in G'$, $G \in \mathscr{C}_s \cap \mathscr{C}_0$ (and hence $G = o(G)$). Then the mapping φ is a complete homomorphism of G' into G. Let $g^* \in G$. According to 7.8 there are integers d_{ij} $((i, j) \in N')$ such that

$$g^*(G_{ij}) = d_{ij}f_{ij}$$

is valid for each $(i, j) \in N'$. There exists $g^{*\prime} \in G'$ fulfilling

$$g^{*\prime}(G'_{ij}) = d_{ij}f'_{ij}$$

for each $(i, j) \in N'$. From the definition of φ we obtain $\varphi(g^{*\prime}) = g^*$, whence $\varphi(G') = G$, which completes the proof.

From 8.7 and 2.2 it follows

8.8. Corollary. *Let G', g'_1, g'_2 have the same meaning as in 8.7. The set $\{g'_1, g'_2\}$ is a set of a-free generators of the complete lattice ordered group G' in the class $\mathscr{C}_s \cap \mathscr{C}_0$.*

By an analogous method we obtain:

8.8'. Theorem. *Let G'_n, $g'_1, \ldots, g'_n$ be as in 8.4'. The set $\{g'_1, \ldots, g'_n\}$ is a set of b-free generators (and a set of a-free generators) of the complete lattice ordered group G'_n in the class $\mathscr{C}_s \cap \mathscr{C}_0$.*

Assume that a set $\{g_1, g_2\}$ generates a complete lattice ordered group G such that $G \in \mathscr{C}_s \cap \mathscr{C}_0$. Then according to 8.8 there exists a complete homomorphism of G' into G. Hence there exists a complete congruence relation ϱ on G' such that G'/ϱ is isomorphic with G.

Let M be the set of all $(i, j) \in N'$ with $\varphi(f'_{ij}) \neq 0$. From the fact that ϱ is a complete congruence relation we infer that G'/ϱ is isomorphic with

(*)
$$\prod_{(i,j) \in M} G'_{ij}.$$

Since each G'_{ij} $((i, j) \in N')$ is isomorphic with N_0, the lattice ordered group (*) is determined up to isomorphisms by the power of the set M. Since any of the cardinalities $1, 2, \ldots, \aleph_0$ can occur as the power of M (cf. Lemma 7.8), we obtain the following result:

8.9. Proposition. *Let M_0 be the set of all nonisomorphic types of complete lattice ordered groups that are generated by a two-element set and belong to $\mathscr{C}_s \cap \mathscr{C}_0$. Then* $\operatorname{card} M_0 = \aleph_0$.

§9. Completely distributive lattice ordered groups

A lattice L is said to be completely distributive if it fulfils the following condition (d_1) and the condition (d_2) dual to (d_1).

(d_1) Let $\{x_{t,s}\}_{t \in T, s \in S} \subseteq L$. Assume that there are elements $u, v \in L$ such that

$$u = \bigvee_{t \in T} \bigwedge_{s \in S} x_{t,s}$$

$$v = \bigwedge_{\varphi \in S^T} \bigvee_{t \in T} x_{t,\varphi(t)}.$$

Then $u = v$.

A lattice ordered group G is called completely distributive if the corresponding lattice $(G; \wedge, \vee)$ is completely distributive.

It is not hard to verify that a lattice ordered group G is completely distributive if and only if, for each $0<g \in G$, the interval $[0, g]$ is completely distributive. If $[0, g]$ fails to be completely distributive and $0<g_1 \leqq g$, then the interval $[0, g_1]$ is not completely distributive. Each linearly ordered set is completely distributive. From this we infer that the following lemma is valid:

9.1. Lemma. *Let G be a lattice ordered group. Suppose that for each $0<g \in G$ there exists $g_1 \in G$ such that*

(a) $0<g_1 \leqq g$,

(b) *the interval $[0, g_1]$ of G is a chain.*

Then G is completely distributive.

It can be shown by examples that a complete singular lattice ordered group need not be completely distributive.

From 7.8 and 9.1 it follows:

9.2. Theorem. *Let G be a complete singular lattice ordered group. If G is generated by a two-element set, then G is completely distributive.*

From 9.2, 8.4 and 8.7 we obtain:

9.2.1. Corollary. *Let G_2 be as in 8.4. Then G_2 is an a-free complete lattice ordered group with two a-free generators in the class $\mathscr{C}_s \cap \mathscr{C}_d$.*

9.2.2. Corollary. *Let G' be as in 8.7. Then G' is a b-free complete lattice ordered group with two b-free generators in the class $\mathscr{C}_s \cap \mathscr{C}_d \cap \mathscr{C}_0$.*

We need the following well-known result (cf. [19]):

9.3. Theorem. *Let G be a complete lattice ordered group. Assume that G is completely distributive. Then G is a completely subdirect product of linearly ordered groups.*

Now assume that a set $\{g_1, g_2\}$ generates a complete lattice ordered group G and that G is completely distributive. According to (T) from § 1, G can be written as

$$G = A \times B,$$

where $A \in \mathscr{C}_s$, $B \in \mathscr{C}_v$. From the complete distributivity of G it follows that both A and B are completely distributive. According to 7.1, the set $\{g_1(A), g_2(A)\}$ generates the complete lattice ordered group A. Hence either $A = \{0\}$, or the structure of A is described by Lemma 7.8 (if we take A, $g_1(A)$ and $g_2(A)$ instead of G, g_1 and g_2).

Let us consider the structure of the lattice ordered group B. Since $B \in \mathscr{C}_v$, we can define a multiplication of elements of B by reals in such a way that B turns out to be a vector lattice. By 9.3, B is a completely subdirect product of complete linearly ordered groups B_i $(i \in I)$.In what follows we are dealing only with the nontrivial case $B \neq \{0\}$. Then we may suppose that $B_i \neq \{0\}$ for each $i \in I$. Moreover, each B_i is a vector lattice, thus B_i is a linearly ordered group isomorphic with R. For each $i \in I$ we have

$$(g_1(B))\,(B_i) = g_1(B \cap B_i) = g_1(B_i)$$

and an analogous equality holds for g_2. The set $\{g_1(B_i), g_2(B_i)\}$ generates the complete lattice ordered group B_i. If $g_1(B_i) = 0$ or $g_2(B_i) = 0$, then the complete lattice ordered group B_i is generated by a one-element set, whence either $B_i = \{0\}$ or B_i is isomorphic with N_0, which is a contradiction. Thus $g_1(B_i) \neq 0 \neq g_2(B_i)$ for each $i \in I$. Since B_i is a vector lattice, there exists a real $x(i) \neq 0$ with

$$g_2(B_i) = x(i) g_1(B_i).$$

Assume that $x(i)$ is rational, i.e., we can write

$$x_i = \frac{m_i}{n_i}, \qquad m_i, n_i \in N_0, n_i > 0,$$

where m_i and n_i are relatively prime. There exists $h \in B_i$ *with*

$$n_i h = g_1(B_i).$$

The set $H_i' = \{mh\}_{m \in N_0}$ is a closed l-subgroup of B_i, $g_1(B_i)$, $g(B_i) \in H_i'$. Hence $H_i' = H_i$. But H_i' is isomorphic with N_0, which is a contradiction. Therefore all $x(i)$ are irrational.

9.4. Lemma. *Let B_i $(i \in I)$ have the same meaning as above. Let $i, j \in I$, $i \neq j$. Then $x(i) \neq x(j)$.*

Proof. Assume that $x(i) = x(j)$. We shall show that then for each $g \in B$ the following assertion is valid:

(∗) If $y \in R$ and $g(B_i) = y g_1(B_i)$, then $g(B_j) = y g_1(B_j)$.

Let $g \in G$. There are uniquely determined reals $y(g, i)$, $y(g, j)$ with

$$g(B_i) = y(g, i) g_1(B_i), \quad g(B_j) = y(g, j) g_1(B_j).$$

Let A_α $(\alpha \leqq \alpha_0)$ have the same meaning as in 5.8 with the distinction that $A_0 = \{g_1(B), g_2(B)\}$ and that we now have B instead of G. According to the assumption, (∗) holds for each $g \in A_0$. Since all joins and meets in B are perfomed componentwise, by a transfinite induction we obtain that (∗) is valid for each $g \in A_{\alpha_0} = B$. Now put

$$g = g_1(B_i) + g_2(B_j).$$

Then we have

$$g(B_i) = g_1(B_i), \quad g(B_j) = g_2(B_j),$$

hence $y(g, i)=1, y(g, j)=x(j)$. We have verified above that $s(j)$ is irrational, thus $y(g, i)\neq y(g, j)$. In view of (*) we have a contradiction. Hence $x(i)\neq x(j)$.

Let R' be the set of all irrationals and let N' be as in the previous paragraphs. From 7.8 and 9.4 we obtain:

9.5. Lemma. *Suppose that a set $\{g_1, g_2\}$ generates a complete lattice ordered group G. Assume that G is completely distributive. Then there exist lattice ordered groups $A_{ij}((i, j)\in N')$, $B_k(k\in R')$ having the following properties:*

(a) *G is a completely subdirect product of lattice ordered groups $A_{ij}((i, j)\in N')$, $B_x(x\in R')$.*

(b) *If $(i, j)\in N'$, then either $A_{ij}=\{0\}$ or A_{ij} is isomorphic with N_0.*

(c) *If $x\in R'$, then either $B_x=\{0\}$ or B_x is isomorphic with R.*

(d) *If $(i, j)\in N'$ and $A_{ij}\neq\{0\}$, then $g_1(A_{ij})=if_{ij}$, $g_2(A_{ij})=jf_{ij}$, where f_{ij} is a strong unit in A_{ij}.*

(e) *If $x\in R'$ and $B_x\neq\{0\}$, then $0\neq g_2(B_x)=xg_1(B_x)$.*

9.6. Lemma. *Let G be a complete lattice ordered group, $g_1, g_2\in G$. Assume that there are lattice ordered groups A_{ij} $((i, j)\in N')$, B_x $(x\in R')$ such that the conditions* (a)—(e) *from 9.5 are fulfilled. Then the set $\{g_1, g_2\}$ generates the complete lattice ordered group G and G is completely distributive.*

Proof. From 9.1 it follows that G is completely distributive. Let $x\in R'$, $B_x\neq\{0\}$. We denote by B'_x the intersection of all closed l-subgroups of B_x containing both elements $g_1(B_x)$ and $g_2(B_x)$. Then the set $\{g_1(B_x), g_2(B_x)\}$ generates the complete lattice ordered group B'_x. By (e), $B'_x\neq\{0\}$ and hence B'_x is isomorphic either with N_0 or with R. Again from (e) we obtain that the first case is impossible. Therefore $B'_x=B_x$. Thus we have verified that the set $\{g_1(B_x), g_2(B_x)\}$ generates the complete lattice ordered group B_x.

Let $(i, j)\in N'$. If $A_{ij}=\{0\}$, we put $f_{ij}=0$. If $A_{ij}\neq\{0\}$, then let f_{ij} be as in (d). Let H be a closed l-subgroup of G, $g_1, g_2\in H$. Analogously as in 7.10 we can verify that $f_{ij}\in H$ for each $(i, j)\in N'$.

Let $0<g\in G$. Let A and B be as in (T) (cf. § 1). There are non-negative integers c_{ij} $((i, j)\in N')$ such that

$$g(A)=\bigvee c_{ij}f_{ij} \qquad ((i, j)\in N')$$

holds in G. From this we obtain $g(A)\in H$. Similarly we obtain $g(A)\in H$ for each $0>g\in G$. Therefore $g(A)\in H$ for each $g\in G$. In particular, $g_i(A)\in H$ $(i=1, 2)$. From this it follows

$$g_i(B)=g_i-g_i(A)\in H \qquad (i=1, 2).$$

Let $x\in R'$. Since B is a vector lattice, there exists $xg_1(B)$ in B. Put

$$g_1(B)[|g_2(B)-xg_1(B)|]=g_1^*.$$

By a reasoning analogous to that in 7.10 we can verify that

$$g_1^* \in H$$

is valid. Then from (d) and (e) we infer that

$$g_1(B) - g_1^* = g_1(B_x),$$

hence $g_1(B_x) \in H$. Analogously we denote

$$g_2(B)[|g_2(B) - xg_1(B)|] = g_2^*;$$

then $g_2^* \in H$ and we have

$$g_2(B) - g_2^* = g_2(B_x),$$

thus $g_2(B_x) \in H$. From $\{g_1(B_x), g_2(B_x)\} \subseteq H$ and from the fact that the set $\{g_1(B_x), g_2(B_x)\}$ generates the complete lattice ordered group B_x we obtain $B_x \subseteq H$.

Again, let $0 \leqq g \in G$. Then

$$g(B) = \bigvee_{x \in R'} g(B_x).$$

Since $g(B_x) \in B_x$, we have $g(B_x') \in H$ for each $x \in R'$ and thus $g \in H$. From this it follows $H = G$, completing the proof.

For each $(i, j) \in N'$ let A'_{ij} be a lattice ordered group isomorphic with N_0, and for each $x \in R'$ let B'_x be a lattice ordered group isomorphic with R. Further let G'_d be the direct product of lattice ordered groups A'_{ij}, B'_x $((i, j) \in N', x \in R')$. In each of the lattice ordered groups A'_{ij} there exists a strong unit f'_{ij}. Let g'_1 and g'_2 be elements of G'_d such that

$$g'_1(A_{ij}) = if'_{ij}, \quad g'_2(A_{ij}) = jf'_{ij} \quad \text{for each} \quad (i, j) \in N',$$

$$0 \neq g'_2(B'_x) = xg'_1(B'_x) \quad \text{for each} \quad x \in R'.$$

Put $g_3 = |g'_1| \vee |g'_2|$ and

$$G_2^d = \bigcup_{n \in N} [-ng_3, ng_3].$$

9.7. Theorem. *The set $\{g'_1, g'_2\}$ is a set of a-free generators of the complete lattice ordered group G_2^d in the class $\mathscr{C}_d$.*

Proof. The method is analogous to that in 8.4. According to 9.1, G_2^d is completely distributive. From 9.6 it follows that the set $\{g'_1, g'_2\}$ generates the complete lattice ordered group G_2^d.

Let H be a complete lattice ordered group, $g_1, g_2 \in H$. Assume that H is completely distributive. Let G be the closed l-subgroup of H generated by the set $\{g_1, g_2\}$. Then G is completely distributive. Hence the structure of G is described by 9.5 and we can use the denotations from 9.5. Let $g' \in G_2^d$. There exist integers c_{ij} $((i, j) \in N')$ and reals $y(x)$ $(x \in R')$ such that

$$g'(A'_{ij}) = c_{ij}f'_{ij}, \quad g'(B'_x) = y(x)g_1(B'_x).$$

The orthogonal hull $o(G)$ of G is the direct product of lattice ordered groups A_{ij} $((i, j) \in N')$, $B_x (x \in R')$. Let us define a mapping φ of G_2^d into $o(G)$ such that $\varphi(g') = g$, where g is defined by

$$g(A_{ij}) = c_{ij}f_{ij}, \quad g(B_x) = y(x)g_1(B_x)$$

for each $(i, j) \in N'$ and each $x \in R'$. Then φ is a complete homomorphism of G_2^d into $o(G)$ and $\varphi(g'_1) = g_1$, $\varphi(g'_2) = g_2$. By steps analogous to those in the proof of 8.4 we can verify that $\varphi(G_2^d) \subseteq G$, completing the proof.

Similarly as in 8.7 we obtain:

9.8. Theorem. *The set* $\{g'_1, g'_2\}$ *is a set of b-free generators (and, at the same time, a set of a-free generators) of the complete lattice ordered group* G'_d *in the class* $\mathscr{C}_d \cap \mathscr{C}_0$.

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О ПОЛНЫХ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУППАХ С ДВУМЯ ОБРАЗУЮЩИМИ II

Мария Якубикова

Резюме

Понятие а-свободной полной структурно упорядоченной группы было введено в части I этой статьи. В части II исследованы а-свободные полные структурно упорядоченные группы с двумя свободными образующими в классе всех сингулярных полных структурно упорядоченных групп и в классе всех структурно упорядоченных групп, которые являются полными и вполне дистрибутивными.