Mathematica Slovaca

Pavel Goralčík; Václav Koubek

On the right action hierarchy of semigroups

Mathematica Slovaca, Vol. 34 (1984), No. 2, 199--203

Persistent URL: http://dml.cz/dmlcz/130683

Math. Slovaca 34, 1984, No. 2, 199—203

ON THE RIGHT ACTION HIERARCHY OF SEMIGROUPS

P. GORALČÍK, V. KOUBEK

Dedicated to Academician Štefan Schwarz on the occasion of his 70th birthday

A great deal of semigroup structure is revealed in the way a semigroup acts on itself by the inner translations. For example, the notions of ideals, or Green's relations, come out of the study of possible transitions efectuated by the right or left action of a semigroup on itself.

A rough picture of the right action of a semigroup S on itself is provided by its *left division relation* R_S defined by $(x, y) \in R_S$ iff $y = xz$ for some $z \in S$, i.e. iff x *divides* y on the left. The left division relation R_S is always *transitive* and *positive,* in the sense that for every $x \in S$ there is $y \in S$ with $(x, y) \in R_S$. If S is commutative then it is also *updirected:* for every $x, y \in S$ we have $z \in S$ with both $(x, z) \in R_S$ and $(y, z) \in R_S$.

The left division relation may happen to be very poor, for example if S is a left-zero semigroup (i.e. a semigroup satisfying the identity $xy = x$) then R_S is just the diagonal of $S \times S$. On the other hand, if S is right simple (and only in this case) then we have $R_S = S \times S$. Of course, there are intermediate cases.

It may be rather difficult to characterize the relations R which are equal to R_S for some semigroup S. Instead, we propose to compare semigroups with the same underlying set by their division relations. A semigroup S is said to be *majorized* by a semigroup T if S and T have the same underlying set and $R_S \subseteq R_T$.

The majorization defined thus is a preorder (i.e. a reflexive and transitive relation) in the class of all semigroups, which will be referred to as the right action hierarchy of semigroups. A semigroup S is minimal in the right action hierarchy if for any semigroup T majorized by S we have $R_T = R_S$.

In the present note we show that the minimal elements in the right action hierarchy are exactly the inflations of left zero semigroups (easy). Further we show that a semigroup may not majorize an inflation of a left-zero semigroup, but in this case it does majorize an inflation of the product of a left-zero semigroup with the infinite cyclic semigroup (not difficult). Finally, we restrict ourselves to the commutative semigroups and we show that a commutative semigroup either majorizes a zero semigroup or an inflation of the product of a free semilattice with the infinite cyclic semigroup (not quite easy).

We have not found in the literature any reference pertaining to the right action hierarchy of semigroups; probably the notion has not been treated yet. The facts about semigroups we use here are fairly rudimentary.

Recall that a semigroup S is an inflation of a semigroup T if T is an ideal of S such that $S^2 \subseteq T$ and there is an idempotent endomorphism, called an inflation endomorphism, $f\colon S \to S$ with $f(S) = T$. Equivalently, S can be described as a union of a family $\{X_t \mid t \in T\}$ of pairwise disjoint sets such that $X_t \cap T = \{t\}$ and $xy = tu$ iff $x \in X_t$ and $y \in X_u$. The passage from the first description of an inflation to the second one is done by setting $X_t = f^{-1}(t)$ for every $t \in T$. Conversely, the family $\{X_t \mid t \in T\}$ defines an inflation endomorphism $f\colon S \to S$ by $f(x) = t$ for $x \in X_t$.

Theorem 1. (1) *If a semigroup S has a minimal right ideal then it majorizes an inflation of a left-zero semigroup.*

(2) *Every inflation of a left-zero semigroup majorizes only itself, thus is minimal in the right action hierarchy. There are no other minimal elements in the hierarchy.*

(3) *A semigroup S majorizes an inflation of the product of a left-zero semigroup with the infinite cyclic semigroup iff S has no finite right ideal. If this is the case then S is not minimal.*

Proof. (1) Let S have a minimal right ideal aS and let T be a subset of S which meets each right minimal ideal in exactly one point (a transversal of the family of all minimal right ideals of S). For every $s \in S$, the set saS is again a minimal right ideal of S, hence it meets T in one point. Define a mapping $f\colon S \to S$ by $\{f(s)\} = T \cap saS$. For $t \in T$, we have $\{f(t)\} = T \cap taS = T \cap tS = \{t\}$, thus f is an idempotent function and clearly, $f \subseteq R_S$. Defining a new multiplication in S by $x \circ y = f(x)$, T becomes a left-zero ideal of $(S, \circ)$ and f an inflation endomorphism. Note that $f = R_{(S,\circ)}$.

(2) If S is an inflation of a left-zero semigroup T then there is a unique inflation endomorphism $f\colon S \to S$ with $f(S) = T$ and we have $R_S = f$. Clearly, when R_S is a function, it cannot contain properly any positive relation, thus even less so any left division relation other than R_S.

If S is not an inflation of a left-zero semigroup then S is not minimal in the right action hierarchy by (1) and (3).

(3) If xS is infinite for every $x \in S$, and only in this case, we can construct in S a subset $X = \{x_{t,n} \mid t \in T,\ n \in N\}$, where T is a large enough left-zero semigroup and N is the additive semigroup of the positive integers, such that

(i) $x_{s,m} = x_{t,n}$ iff $s = t$ and $m = n$,

(ii) for every $t \in T$, $(x_{t,n}, x_{t,n+1}) \in R_S$,

(iii) $X \cap xS \neq \emptyset$ for every $x \in S$.

We start the construction by taking an arbitrary $t \in T$ and an arbitrary $x_{t,1}$ in S. When we have already $x_{t,1}, \ldots, x_{t,n}$, we can always choose $x_{t,n+1} \in S - \{x_{t,1}, \ldots, x_{t,n}\}$

such that $(x_{t,n}, x_{t,n+1}) \in R_S$. When we have the whole sequence $X_t = \{x_{t,n} | n \in N\}$, we form the set $X_t S^{-1} = \{s \in S | X_t \cap sS \neq \emptyset\}$, and we repeate the process by choosing a new sequence X_u in $S - X_t S^{-1}$, if it is not void, form the infinite right ideal $(S - X_t S^{-1}) - X_u S^{-1}$, and so on. We continue until S is exhausted.

We turn X into a semigroup isomorphic to $T \times N$ by defining $x_{s,m} \circ x_{t,n} = x_{s,m+n}$. By (iii), we can choose an idempotent mapping f: $S \to S$ with $f(S) = X$ and $f \cap [(S - X) \times X] \subseteq R_S$. Extending the multiplication in X to the whole S by $x \circ y = f(x) \circ f(y)$, we obtain a semigroup $(S, \circ)$ which is an inflation of $(X, \circ)$ and is majorized by S.

If we put $x'_{t,n} = x_{t,n+1}$ for all $t \in T$ and $n \in N$, we obtain a smaller set $X' = \{x'_{t,n} | t \in T, n \in N\}$ enjoying properties (i)—(iii), thus yielding a semigroup $(X', \sqcap)$ with multiplication $x'_{s,m} \sqcap x'_{t,n} = x'_{s,m+n}$. If we extend this multiplication to the whole S by $x \sqcap y = f'(x) \sqcap f'(y)$, where f' is an inflation endomorphism defined by $f'(s) = x'_{t,n}$ iff $f(s) = x_{t,n}$ for all $s \in S - X'$, we obtain a semigroup $(S, \sqcap)$ strictly majorized by $(S, \circ)$.

Every right ideal of an inflation of the product of a left-zero semigroup with the infinite cyclic semigroup is infinite. If S majorizes T then every right ideal of S contains some right ideal of T.

Theorem 2. (1) *A commutative semigroup S has a minimal ideal iff it majorizes a zero semigroup.*

(2) *The zero semigroups are the only minimal elements in the action hierarchy of the commutative semigroups.*

(3) *A commutative semigroup S has no finite ideal iff it majorizes an inflation of the product of a free semilattice with the infinite cyclic semigroup. If this is the case then S is not minimal.*

Proof. Clearly, the zero multiplication $x \circ y = z$ with z fixed in the minimal ideal of S defines a zero semigroup $(S, \circ)$ majorized by S. Conversely, if S majorizes a zero semigroup $(S, \circ)$ then $z \in xS$ for all $x \in S$, thus S has a minimal ideal.

(2) The zero semigroups are clearly minimal relative to the majorization. By (1) and (3), there are no other minimal commutative semigroups.

(3) Assume that S has no finite ideal. Take a well ordered set $(M, \leq)$ with $\mathrm{card}(M) > \mathrm{card}(S)$ and extend the well order to the set $K(M)$ of all non-void finite subsets of M, by setting

$$A < B \Leftrightarrow \max[(A - B) \cup (B - A)] \in B$$

for $A, B \in K(M)$. If $\varkappa$ is the ordinal corresponding to $(K(M), \leq)$ then we can label the members of $K(M)$ by the ordinals less than $\varkappa$ in such a way that $K(M) = \{N_\alpha | \alpha < \varkappa\}$ and $N_\alpha \leq N_\beta$ iff $\alpha \leq \beta$. Under the set union, $K(M)$ is a free semilattice. We make the ordinal $\varkappa$ itself into a free join semilattice $(\varkappa, \vee)$ by defining $\alpha \vee \beta = \gamma$ iff $N_\alpha \cup N_\beta = N_\gamma$ for $\alpha, \beta \in \varkappa$.

Let $\nu \in \varkappa$, $\nu \neq 0$. If $|N_\nu| = 1$ then clearly the sets N_α with $\alpha < \nu$ are all non-void finite subsets of $\{a \in M \mid a < \min N_\nu\}$ thus they form a free subsemilattice of $(K(M), \cup)$ and $(\nu, \vee)$ is a free subsemilattice of $(\varkappa, \vee)$. If $|N_\nu| > 1$ then $N_\nu - \{\min N_\nu\}$ is non-void, thus equal to N_μ for some ordinal μ. We show that if $\nu \neq \mu + 1$ then the sets N_α with $\mu < \alpha < \nu$ form a free subsemilattice of $(K(M), \cup)$ isomorphic to the semilattice $(\pi, \vee)$, where π is the ordinal with $N_\pi = \{\min N_\nu\}$. The isomorphism is established by the assignment $\beta \mapsto N_\beta \cup N_\mu$ for $\beta \in \pi$. Indeed, the assignment is injective and respects unions. To show that the assignment is onto $\{N_\alpha \mid \mu < \alpha < \nu\}$, consider

$$a = \max [(N_\alpha - N_\nu) \cup (N_\nu - N_\alpha)] \quad \text{and} \quad b = \max [(N_\alpha - N_\mu) \cup (N_\mu - N_\alpha)]$$

for some α with $\mu < \alpha < \nu$. We have $a \in N_\nu - N_\alpha$, $b \in N_\alpha - N_\mu$, hence $a \neq b$. If it would be $a < b$ then $b \notin (N_\alpha - N_\nu) \cup (N_\nu - N_\alpha)$, hence $b \in N_\nu$, but then $b \in N_\nu - N_\mu$ would lead to $b = \min N_\nu \leq a$, a contradiction. So we have $b < a$. Moreover, $a = \min N_\nu$, for otherwise it would be $a \in N_\mu - N_\alpha$, whence $a \leq b$, a contradiction. We conclude that for β such that $N_\beta = (N_\alpha - N_\mu) \cup (N_\mu - N_\alpha)$ we have $\beta < \pi$ and $N_\alpha = N_\beta \cup N_\mu$.

We next define a cofinal (relative to R_S) subset $X \subseteq S$ by the following transfinite induction:

I. Select in S a sequence $X_0 = \{x_{0,i} \mid i \in N\}$, with all elements pairwise distinct and such that $(x_{0,i}, x_{0,i+1}) \in R_S$ for all $i \in N$, and set $\tilde{X}_0 = \{x \in S \mid X_0 \cap xS \neq \emptyset\}$.

II. If sequences X_α are already defined for all α, $\alpha < \beta$, and we have $S - \bigcup_{\alpha < \beta} \tilde{X}_\alpha \neq \emptyset$, then (since R_S is updirected) we can select a sequence $X_\beta = \{x_{\beta,i} \mid i \in N\}$ in $S - \bigcup_{\alpha < \beta} \tilde{X}_\alpha$, with all elements pairwise distinct and such that $(x_{\beta,i}, x_{\beta,i+1}) \in R_S$ and $(x_{\alpha,i}, x_{\beta,i}) \in R_S$ for all $i \in N$ and all α such that $N_\alpha \leq N_\beta$, $N_\alpha \neq N_\beta$, and set $\tilde{X}_\beta = \left\{x \in S - \bigcup_{\alpha < \beta} \tilde{X}_\alpha \mid X_\beta \cap xS \neq \emptyset\right\}$. Clearly, for some ν, $\nu < \varkappa$, this construction will end up with $S = \bigcup_{\alpha < \nu} \tilde{X}_\alpha$. If $|N_\nu| = 1$ we put $X = \bigcup_{\alpha < \nu} X_\alpha$. If $|N_\nu| > 1$ and $N_\nu - \{\min N_\nu\} = N_\mu$, then if $\nu = \mu + 1$ we put $X = X_\mu$ and $\tilde{X} = \tilde{X}_\mu$, if $\nu \neq \mu + 1$ we put $X = \bigcup_{\mu < \alpha < \nu} X_\alpha$ and $\tilde{X} = \bigcup_{\mu < \alpha < \nu} \tilde{X}_\alpha$.

We show that X is cofinal in S. Clearly, X is cofinal in $\tilde{X}$. Let $s \in S - X$. Then for an arbitrary $x \in X$, say $x \in X_\alpha$, there is (by updirectedness) some $z \in S$, say $z \in \tilde{X}_\beta$, with (s, z), $(x, z) \in R_S$. It cannot be $\beta < \alpha$, because by the construction we would have $x \in \tilde{X}_\beta$, thus $z \in \tilde{X}$. We have $(z, t) \in R_S$ for some $t \in X$, hence by the transitivity of R_S we have $(s, t) \in R_S$.

To close with, we make X into the product of a free semilattice with the additive semigroup $(N, +)$ of non-negative integers by defining $x_{\alpha,i} \circ x_{\beta,j} = x_{\alpha \vee \beta, i+j}$. Since

$X \cap sS \neq \emptyset$ for every $s \in S$, we can find an idempotent mapping f: $S \to S$ with $f(S) = X$ and $f \cap [(S - X) \times X] \subseteq R_S$, and inflate $(X, \circ)$ by f to $(S, \circ)$ which obviously is majorized by S.

All ideals of an inflation of the product of a free semilattice with $(N, +)$ are infinite, therefore a commutative semigroup S which majorizes it cannot have a finite ideal.

Finally, S is shown not to be minimal in a similar way as in (3) of Theorem 1. The proof is complete.

Received July 25, 1983

Matematicko-fyzikálna fakulta
University Karlovy
Sokolovská 83
186 00 Praha 8

ИЕРАРХИЯ ПРАВЫХ ДЕЙСТВИЙ НА ПОЛУГРУППАХ

P. Goralčík—V. Koubek

Резюме

Каждой полугруппе S сопоставляется бинарное отношение левой делимости $R_S = \{(x, xy) | x, y \in S\}$. Сравнивая полугруппы по их левой делимости, мы говорим, что полугруппа S мажорирует полугруппу T, если $y \in S$ и T одинаковые элементы и $R_T \subseteq R_S$. Изучается отношение мажорирования в классе всех полугрупп и в классе всех коммутативных полугрупп. В частности, описываются полугруппы минимальные по мажорированию как инфляции полугрупп левых нулей. Доказывается, что каждая полугруппа S, у которой все правые идеалы бесконечны, мажорирует некоторую инфляцию произведения некоторой полугруппы левых нулей и бесконечной циклической полугруппы. Если, более того, S коммутативна, то она также мажорирует некоторую инфляцию произведения некоторой свободной полурешетки и бесконечной циклической полугруппы.