Mathematica Slovaca

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Mathematica Slovaca, Vol. 32 (1982), No. 2, 147--154

Persistent URL: http://dml.cz/dmlcz/130644

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ON QUASICONTINUITY OF MULTIFUNCTIONS

TIBOR NEUBRUNN

Various definitions of continuity of multifunctions are given in many papers. They all reduce to the usual continuity if a single valued function is considered.

The aim of this paper is to present two definitions of quasicontinuity of multifunctions, which reduce in case of single valued functions to the usual quasicontinuity and to prove various results for quasicontinuous multifunctions. On the other hand some examples are given, showing that there are some differences between the classical results for quasicontinuous single valued functions and those for multifunctions.

Notations and preliminary results

Given sets X and Y, we denote $S(Y)$ the collection of all nonempty subsets of Y. A function $F: X \to S(Y)$ is called a multifunction. In what follows we say function instead of multifunction. If we consider a function $F: X \to Y$, we refer to it as to a single valued function.

If $F: X \to S(Y)$, then for $A \subset Y$ we denote

$$F^-(A) = \{x: F(x) \cap A \neq \emptyset\} \text{ and } F^+(A) = \{x: F(x) \subset A\}.$$

Now we give two definitions of quasicontinuous multifunctions.

A function $F: X \to S(Y)$ is said to be upper quasicontinuous — briefly u-quasicontinuous (lower quasicontinuous — briefly l-quasicontinuous) at a point $x_0 \in X$ if for any open set V containing $F(x_0)$ (for any point $z \in F(x_0)$ and for any neighbourhood V containing z) and any neighbourhood U of x_0, there exists a nonempty open set $G \subset U$ such that $F(x) \subset V$, $(F(x) \cap V \neq \emptyset)$ for any $x \in G$. If F is u-quasicontinuous (l-quasicontinuous) at any $x \in X$, then it is said to be u-quasicontinuous (l-quasicontinuous).

Note that if a single valued function $f: Y \to Y$ is given, then under the natural interpretation of $f(x)$ as a one point set, both above definitions give the usual definition of a quasicontinuous function as given, e.g., in [4] (compare also the original definition in [2]).

The u-quasicontinuous, as well as the l-quasicontinuous functions may be characterized by means of quasiopen sets similarly as the single valued quasicontinuous functions.

Note that a set $A \subset X$ is said to be *quasiopen* if $A \subset \overline{A^\circ}$ (E° and $\bar{E}$ denote the interior and the closure of the set E respectively).

Proposition 1. *A function F: $X \to S(Y)$ is u-quasicontinuous (l-quasicontinuous) if and only if for any open set G the set $F^+(G)$ ($F^-(G)$) is quasiopen.*

Proof. We prove the case of l-quasicontinuity, the other case is similar. Let F be l-quasicontinuous, $V \subset Y$ an open set. If $F^-(V) \neq \emptyset$, choose $x_0 \in F^-(V)$ and an arbitrary neighbourhood U of x_0. Since $F(x_0) \cap V \neq \emptyset$, we obtain from l-quasicontinuity that a nonempty open set $W \subset U$ exists such that $F(x) \cap V \neq \emptyset$ for any $x \in W$. We have $W \subset (F^-(V))^\circ$, hence $x_0 \in \overline{(F^-(V))^\circ}$. Thus $F^-(V)$ is quasiopen. The case of $F^-(V) = \emptyset$ is obvious.

Now let $F^-(V)$ be quasiopen for any open $V \subset Y$. Let $x_0 \in X$, $y \in F(x_0)$, V any neighbourhood of y and U any neighbourhood of x_0. Since $V \cap F(x_0) \neq \emptyset$, we have $x_0 \in F^-(V)$. Since $F^-(V)$ is quasiopen and U open, we obtain that $F^-(V) \cap U$ is quasiopen. Thus a nonempty open set $W \subset F^-(V) \cap U$ exists. Hence $F(x) \cap V \neq \emptyset$ for any $x \in W$. Thus the l-quasicontinuity at x_0 is proved.

For a single valued function f: $X \to Y$ the somewhat continuity was introduced in [1]. *A function is called somewhat continuous if for any open $V \subset Y$ for which $f^-(V) \neq \emptyset$ we have $(f^-(V))^\circ \neq \emptyset$.*

In a natural way we introduce two notions of somewhat continuity for multivalued functions.

A function F: $X \to S(Y)$ is said to be u-somewhat continuous (l-somewhat continuous) if for any open set $V \subset Y$ for which $F^+(V) \neq \emptyset (F^-(V) \neq \emptyset)$ we have $(F^+(V))^\circ \neq \emptyset (F^-(V))^\circ \neq \emptyset$. The following proposition relates the u-quasicontinuity to u-somewhat continuity and l-quasicontinuity to l-somewhat continuity. Since an analogical result with an analogical proof is known for single valued functions (see[5]), we omit the proof.

Proposition 2. *A function F: $X \to S(Y)$ is u-quasicontinuous (l-quasicontinuous) if and only if there exists a basis $\mathcal{B}$ of open sets in X such that the restriction $F | U$ is u-somewhat continuous (l-somewhat continuous) for any $U \in \mathcal{B}$.*

Somewhat and quasicontinuity of multifunctions in product spaces

The classical result of Kempisty (see [2]) asserts that a real function of two real variables, which is separately quasicontinuous, is quasicontinous as a function of two variables. In [4] and [5] it was generalized for more general topological spaces. In [6] some related results for somewhat continuous functions were obtained. In this part we give some results of somewhat continuity and quasicontinuity of multifunctions, which generalize the above mentioned. On the other hand we give examples showing that a straightforward generalization result of Kempisty is not possible either for u-quasicontinuity or for l-quasicontinuity. But the results may be considered as a natural generalization of a theorem from [2] concerning the quasisemicontinuity of a real function. To obtain it one has to associate with a real function f a multivalued function $x \to (-\infty, f(x)\rangle$.

Theorem 1. *Let X be a Baire space, Y second-countable and Z a normal topological space. Let F: $X \times Y \to S(Z)$ be a function assuming as values only closed sets. Let the sections F_x be u-somewhat continuous for every $x \in X$ and the sections F^y u-somewhat continuous and l-quasicontinuous for every $y \in Y$. Then F is u-somewhat continuous.*

Proof. Suppose F not to be u-somewhat continuous. Then there exists an open set $H \subset Z$ such that

$$F^+(H) \neq \emptyset \quad \text{and} \quad (F^+(H))^\circ = \emptyset.$$

Hence the set of those points (x', y') for which

$$F(x', y') \cap H' \neq \emptyset \tag{1}$$

is dense in $X \times Y$. (E' denotes the complement of the set E.)

Let $(x_0, y_0) \in F^+(H)$. Since $F(x_0, y_0)$ is a closed set and Z a normal space, there exists an open set H_1 such that

$$F(x_0, y_0) \subset H_1 \subset \bar{H}_1 \subset H.$$

The function F^{y_0} is u-somewhat continuous and $(F^{y_0})^+(H_1) \neq \emptyset$, hence an open set G exists such that

$$F^{y_0}(x) \subset H_1, \quad \text{for any} \quad x \in G.$$

Denote by $\{V_n\}_{n=1}^{\infty}$ the countable base of Y and put for $n = 1, 2, \ldots$

$$A_n = \{x\colon x \in G,\ V_n \subset F_x^+(H_1)\}.$$

We have

$$G = \bigcup_{n=1}^{\infty} A_n \tag{2}$$

The inclusion $\bigcup_{n=1}^{\infty} A_n \subset G$ is trivial. Now let $x \in G$. Since $F_x(y_0) = F^{y_0}(x) \subset H_1$, we have $F_x^+(H_1) \neq \emptyset$. Hence by u-somewhat continuity of F_x we obtain

$$(F_x^+(H_1))^\circ \neq \emptyset.$$

From the last it follows that a number n exists such that

$$V_n \subset (F_x^+(H_1))^\circ \subset F_x^+(H_1)$$

proving that $x \in A_n$. Thus (2) holds.

Now we prove that any of the sets A_n is nowhere dense in G. Let $W\subset G$ be a nonempty open set. According to (1) there exists a point (x', y'), $x'\in W$ for which $F(x', y')\cap H'\neq\emptyset$. Let $z\in F(x', y')\cap H'$. Choose a neighbourhood $\tilde{V}$ of z such that $\tilde{V}\cap H_1=\emptyset$. Owing to the l-quasicontinuity of $F^{y'}$ at x' we have that a nonempty set $\tilde{W}\subset W$ exists such that $F(x, y')\cap\tilde{V}\neq\emptyset$ for any $x\in\tilde{W}$. Since $y'\in V_n$, we have $x\notin A_n$ for any $x\in\tilde{W}$. Thus $\tilde{W}\cap A_n=\emptyset$, proving that the set A_n is nowhere dense. It follows from (2) that G is of the first cathegory, in contradiction to to the assumption that X is a Baire space. The theorem is proved.

Remark 1. It is easily seen from the proof that the assumption that F_x are u-somewhat continuous for all $x\in X$ may be weakened. It is sufficient to suppose that F_x are u-somewhat continuous with the exception of a set of the first category.

Theorem 2. *(Theorem on product u-quasicontinuity.) Let X be a Baire space. Y locally second-countable and Z normal. Let F: $X\to S(Y)$ be closed-valued with u-quasicontinuous x-sections for every $x\in X$ and let for every $y\in Y$ the y-sections be both u-quasicontinuous and l-quasicontinuous. Then F is u-quasicontinuous.*

Proof. The collection of all sets $U\times V$, where U, V are open in X and Y respectively, is a base of open sets in $X\times Y$. Since Y is locally second-countable, we may suppose, with no loss of generality, that V is second-countable. From the assumptions of the theorem, we have immediately that F_x is u-quasicontinuous and hence u-somewhat continuous on any fixed V, *for every* x belonging to a fixed U, and similarly F^y is both u-quasicontinuous and l-quasicontinuous on U for every $y\in V$. Thus by Theorem 2 we have that $F/U\times V$ is u-somewhat continuous. Since it is true for arbitrary $U\times V$, we obtain from Proposition 1 that F is u-quasicontinuous.

Neither Theorem 1, no Theorem 2 may be proved if we omit the assumption that F^y is l-quasicontinuous. Thus a straightforward analogy of the Kempisty Theorem for single valued functions is not true for multifunctions.

Example 1. Let us consider R^2 with the usual topology. Let S be a countable subset of R^2, dense in R^2 and such that on any horizontal and any vertical line there is at most one point of the set S. Such set may be constructed as follows.

Denote by $\{B_n\}_{n=1}^{\infty}$ the sequence of all mutually distinct spheres with rational centres and rational diameters. Choose $(p_1, q_1)\in B_1$. Suppose that for given $n\geqq 1$ the points (p_i, q_i) were constructed for $i=1, 2, \ldots, n$ such that on every horizontal and every vertical line there is at most one of them. Now take $(p_{n+1}, q_{n+1})\in B_{n+1}$ such that $p_{n+1}\neq p_i$, $q_{n+1}\neq q_i$ for $i=1, 2, \ldots, n$. Then evidently on every horizontal and every vertical line there lies at most one of the points (p_i, q_i) $(i=1, 2, \ldots, n)$. The set of values of such a constructed sequence $\{(p_n, q_n\}_{n=1}^{\infty}$ may be taken for S.

Define F: $R^2\to S(R^1)$ as follows:

$$F(x, y)=\begin{cases}\{0\}, & \text{if } (x, y)\notin S,\\ \{0, 1, 2, \ldots, n\}, & \text{if } (x, y)=(p_n, q_n).\end{cases}$$

The sections F_x, F^y are u-quasicontinuous for every x and every y, respectively. Let us check it for F_x. If $x \neq p_n$, where $n=1, 2, \ldots$, then $F_x(y)=\{0\}$ for every real number y and the u-quasicontinuity of F_x is evident. If for some n $x=p_n$, then $F_x(y)$ is equal $\{0\}$, with the exception of exactly one point $y=q_n$, where $F_x(q_n) = F(p_n, q_n) = \{0, 1, 2, \ldots, n\}$. But the u-quasicontinuity in this case is obvious too. Analogical reasoning shows that F^y is u-quasicontinuous for any real y.

The function F is not u-quasicontinuous at any $(x, y) \in R^2$. Because if $(x_0, y_0) \in R^2$, we have $F(x_0, y_0) = \{0, 1, 2, \ldots, n\}$, where $n \geqq 0$. Choose the open interval $(-1, n+1)$. We have $(-1, n+1) \supset F(x_0, y_0)$. Now let U be any neighbourhood of (x_0, y_0) and $W \subset U$ any nonempty set. Evidently W contains a point (p_k, q_k), where $k>n$. Hence $F(p_k, q_k) = \{0, 1, 2, \ldots, k\} \not\subset (-1, n+1)$. Thus F is not u-quasicontinuous at (x_0, y_0). In an analogical way we may prove that F is not even u-somewhat continuous.

Theorem 3. *Let X be a Baire space, Y second-countable and Z a regular topological space. Let $F: X \times Y \to S(Z)$ be a multifunction such that F_x is l-somewhat continuous for every $x \in X$, F^y is both l-somewhat continuous and u-quasicontinuous for every $y \in Y$. Then F is l-somewhat continuous.*

Proof. Suppose F not to be l-somewhat continuous. Then there exists an open set $H \subset Z$ such that

$$F^-(H) \neq \emptyset \text{ and } (F^-(H))^\circ = \emptyset.$$

Hence the set of all (x', y') for which $F(x', y') \cap H = \emptyset$ is dense in $X \times Y$. Let $(x_0, y_0) \in F^-(H)$ and let $z \in F(x_0, y_0) \cap H$. Choose a neighbourhood H_1 of the point z such that

$$H_1 \subset \bar{H}_1 \subset H \tag{3}$$

From the fact that $z \in F^{y_0}(x_0) \cap H_1$ we have $(F^{y_0})^{-1}(H_1) \neq \emptyset$. Hence from the l-somewhat continuity of F^{y_0} a nonempty open set G exists such that

$$F^{y_0}(x) \cap H_1 \neq \emptyset \text{ for every } x \in G.$$

Let $\{V_n\}_{n=1}^{\infty}$ be a countable base of Y. We can proceed analogicaly to the proof of Theorem 1, but we put

$$A_n = \{x: x \in G, F(x, y) \cap H_1 \neq \emptyset \text{ for any } y \in V_n\}.$$

Using the l-somewhat continuity of the sections F_x we prove similarly as in the proof of Theorem 1 that $G = \bigcup_{n=1}^{\infty} A_n$.

Now to obtain a contradiction, we have to prove that A_n is nowhere dense for $n=1, 2, \ldots$

If W is a nonempty open set, $W \subset G$, take in $W \times V_n$ a point (x', y') such that $F(x', y') \subset H'$. According to (3) the set $\bar{H}_1'$ is an open set containing H'. Hence

$F(x', y') \subset \bar{H}_1'$. The u-quasicontinuity of $F^{y'}$ at x' implies that a set $\tilde{W} \subset W$, which is nonempty and open, exists and $F(x, y') \subset \bar{H}_1'$ for any $x \in \tilde{W}$. Thus

$$F(x, y') = F^{y'}(x) \cap H_1 = \emptyset .$$

Since $y' \in V_n$, we have $x \notin A_n$. Hence $\tilde{W} \cap A_n = \emptyset$, proving that A_n is nowhere dense. The proof is finished.

In the same way as Theorem 2 was proved by means of Theorem 1, we obtain from the preceding result the following

Theorem 4. (Theorem on l-continuity in the product.) Let *X be a Baire space, Y locally second-countable and Z regular. Let for every $x \in X$ the sections F_x be l-quasicontinuous and for every $y \in Y$ the sections F^y are both u-quasicontinuous and l-quasicontinuous. Then F is l-quasicontinuous.*

Remark 2. The somewhat continuity in Theorem 3 as well as the quasicontinuity of the sections F_x in Theorem 4 may be weakemed in a similar way as it was done for the u-somewhat continuity in Theorem 1. (See Remark 1).

Since the case of a single valued function implies that u-quasicontinuity and l-quasicontinuity coincide with the usual quasicontinuity, we obtain from Theorem 4 a general variant of the Kempisty Theorem for single valued functions.

Corollary. *(See [5]). Let X be a Baire space, Y second countable and Z regular. Let f: $X \times Y \to Z$ be a single valued function with f_x quasicontinuous for every $x \in X$ and with f^y quasicontinuous for every $y \in Y$. Then f is quasicontinuous.*

Remark 3. An analogical Corollary may be obtained from Theorem 3, guaranteeing the somewhat continuity of the point functions f: $X \times Y \to Z$ under the assumptions of the somewhat continuity of their x-sections and the quasicontinuity of the y-sections (See [6]).

The assumption of u-quasicontinuity of F^y in theorems 3 and 4 may not be omitted.

Example 2. Let $S = \{(p_n, q_n): n = 1, 2, \ldots\}$ have the same meaning as in Example 1. Define F: $R^2 \to S(R^1)$ as follows:

$$F(x, y) = \begin{cases} \{n\}, \text{ if } (x, y) = (p_n, q_n) \ (n = 1, 2, \ldots), \\ \{1, 2, \ldots\}, \text{ if } (x, y) \neq (p_n, q_n) \ (n = 1, 2, \ldots). \end{cases}$$

F_x and F^y are l-quasicontinuous for every $x \in R^1$ and $y \in R^1$. It is sufficient to prove it for F_x, because the case for F^y is symetrical. If $x \neq p_n$ $(n = 1, 2, \ldots)$, then the l-quasicontinuity is evident, because $F_x(y) = \{1, 2, \ldots\}$ for all $y \in Y$. If $x = p_n$ for some n, then

$$F_x(y) = \begin{cases} \{n\}, \text{ if } y = q_n, \\ \{1, 2, \ldots\}, \text{ if } y \neq q_n . \end{cases}$$

But l-quasicontinuity of a function of this type may also be immediately verified.

F is not l-quasicontinuous at any point (x_0, y_0). In fact there exists at least one positive integer n_0 such that $n_0 \in F(x_0, y_0)$. The interval (n_0-1, n_0+1) is a neighbourhood of n_0. If we choose a neighbourhood G of (x_0, y_0) and any nonempty open set $W \subset G$, then evidently W contains a point $(p_n, q_n) \neq (p_{n_0}, q_{n_0})$. We have

$$F(p_n, q_n) \cap (n_0-1, n_0+1) = \{n\} \cap (n_0-1, n_0+1) = \emptyset,$$

hence F is not l-quasicontinuous at (x_0, y_0).

F is even not l-somewhat continuous, because if we take for any n the set $F^-((-n-1, n+1))$, it is a nonempty set which does not contain a dense set $\{(p_k, q_k), k \neq n$ $(k=1, 2, \ldots.)$ Thus $(F^-((-n-1, n+1)))^\circ = \emptyset$.

A remark on the topology for u-quasicontinuity

Given two topological spaces X and Y, one has the possibility to define a topology on $S(Y)$ and to consider the quasicontinuity of $F: X \to S(Y)$ as the quasicontinuity in the usual sense, taking the values of F as points of the topological space $S(Y)$.

Several topologies were considered on various subcollections of $S(Y)$, to define various types of continuites of multivalued functions. Some of them were defined on the subcollection $F(Y) \subset S(Y)$ of all closed sets of Y, the others on the subcollection $K(Y)$ of all compact subsets of Y (see e.g. [3] p. 392—393). In [7] a topology on $S(Y)$ is defined in such a way that the base of all open sets is the collection of all families $S(G) \subset S(Y)$ where G is any open set and $S(G)$ denotes the family of all nonempty subsets of G. By means of such a topology there is defined a type of continuity (in [7] it is called weak continuity). The following proposition shows that our notion of u-quasicontinuity may be described in this topology.

Proposition 3. *A function $F: X \to S(Y)$ is u-quasicontinuous if and only if it is quasicontinuous as a single valued mapping into the topological space $S(Y)$ with the topology defined by means of the base $S(G)$, where $S(G)$ is the collection of all nonempty subsets of G and G is any open set in Y.*

Proof. Let F be u-quasicontinuous at $x_0 \in X$. Let $S(G)$ be element of the base containing $F(x_0)$ and U any neighbourhood of x_0. Then $F(x_0) \subset G$ and by u-quasicontinuity there exists a monempty open set $W \subset U$ such that $F(x) \subset G$ for every $x \in W$. The later means that $F(x) \in S(G)$. Thus the quasicontinuity at x_0 is proved.

Now, let F be quasicontinuous at x_0. If G is an open set containing $F(x_0)$ and U a neighbourhood of x_0, then there is a nonempty open set $W \subset U$ with $F(x) \in S(G)$ for every $x \in W$. This means $F(x) \subset G$ for every $x \in W$ and the u-quasicontinuity is proved.

Of course the somewhat continuity of the mapping $F: X \to S(Y)$ considered in the usual sense as the somewhat continuity defined by means of the topology on $S(Y)$ gives the u-somewhat continuity. Thus from the above consederations and from the known results for single valued functions we may obtain the parts of Proposition 1 and Proposition 2 concerning the u-quasicontinuous functions.

Note. As the author has found out the notion of quasicontinuity for multifunctions was for the first time introduced in [8] in connection with different results.

REFERENCES

[1] GENTRY, K. R. and HOYLE, H. B.: Somewhat continuous functions. Czech. Math. J. 96, 1971, 5—12.

[2] KEMPISTY, S.: Sur les fonctions quasicontinues. Fund. Math. 19, 1932, 184—197.

[3] KURATOWSKI, K.—MOSTOWSKI, A.: Set theory with an introduction to descriptive set theory. PWN, Warszawa, 1976.

[4] MARTIN, N. F. G.: Quasicontinuous functions on product spaces. Duke Math. J. 28, 1961, 39—44.

[5] NEUBRUNN, T.: A generalized continuity and product spaces. Math. Slovaca, 26, 1976, 97—99.

[6] NEUBRUNN, T.: Generalized continuity and separate continuity. Math. Slovaca, 27, 1977, 307—314.

[7] STROTHER, W. L.: Continuous multi-valued functions. Boll. da Soc. de Mat. de Sao Paolo, 1958, 87—120.

[8] POPA, V.: Asupra unei decompuneri a cvasicontinuitatii multifunctilor. St. Cerc. Mat. 27, 1975, 323—328.

Received March 24, 1980

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О КВАЗИНЕПРЕРЫВНОСТИ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ

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Резюме

Исследуются отображения, определенные на произведении двух топологических пространств X, Y, принимающие как значения непустые множества топологического пространства Z. Определены два типа квазинепрерывности таких отображений. Главным результатом работы являются теоремы, установливающие связь между отдельной квазинепрерывностью и квазинепрерывностью этих отображений.