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AN EXTENSION THEOREM FOR MULTIFUNCTIONS AND A CHARASTERIZATION OF COMPLETE METRIC SPACES

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Let X and Y be topological spaces. If Y is a metrizable space, then Y is topologically complete (see [1], p276) iff each continuous mapping $f: A \to Y$ with A dense in X has a continuous extension to a G_δ-set containing A. To see this just consider the identity mapping $i: Y \to Y$ and view Y as a dense subspace of its completion.

We show that if Y is a metrizable space, then Y is topologically complete if and only if each upper semicontinuous compact-valued multifunction $F: A \to Y$ with A dense in X has an upper semicontinuous compact-valued extension to a G_δ-set containing A.

We shall use the terminology from [1].

Notation. In what follows X, Y denote topological spaces. The closure of a subset M of a topological space X will be denoted by $\bar{M}$.

The intersection of a family $\mathscr{U}$ of sets will be denoted by $\cap \mathscr{U}$.

$\mathscr{P}(Y)$ denotes the collection of all subsets of Y, $C(Y)$ denotes the collection of all nonempty closed subsets of Y. If (Y, d) is a metric space, $(C(Y), \tilde{d})$ denotes the metric space equipped with the Hausdorff metric, i.e. $\tilde{d}(A, B) = \inf\{\varepsilon: A \subset S_\varepsilon[B],\ B \subset S_\varepsilon[A]\}$, where $S_\varepsilon[A] = \cup \{S_\varepsilon[x]: x \in A\}$ and $S_\varepsilon[x] = \{y: d(x, y) < \varepsilon\}$.

$\mathscr{V}(x)$ denotes the set of all open neighbourhoods of x. N denotes a set of all positive integers, R denotes the set of all real numbers.

A family $\mathscr{U}$ of sets has the finite intersection property if the intersection of every finite subfamily is not empty. A centred family is a family of sets having the finite intersection property.

A multifunction F from X to Y is a mapping $F: X \to \mathscr{P}(Y)$. We write $F: X \to Y$ for brevity. We suppose $F(x) \neq \emptyset$ for any $x \in X$.

A multifunction $F: X \to Y$ is upper semicontinuous at $x \in X$ if for every open set V in Y such that $F(x) \subset V$, there exists an open set U in X such that $x \in U$ and $F(U) \subset V$, where $F(U) = \bigcup_{x \in U} F(x)$.

F is upper semicontinuous if F is upper semicontinuous at every $x \in X$.

Let A be a subset of X and F: $A \to Y$ be a multifunction from A to Y. A multifunction F^*: $X \to Y$ is an extension of F if $F^*(x) = F(x)$ for every $x \in A$.

Let (Y, d) be a metric space. Let χ be a functional defined on $\mathscr{P}(Y)$ as follows: $\chi(0) = 0$ and if A is a nonempty subset of Y, then $\chi(A) = \inf\{\varepsilon$: A has a finite ε-dense subset$\}$. In literature χ has been called the Hausdorff measure of noncompactness functional.

Remark. (See [2]) The Hausdorff measure of noncompactness functional has some good properties.

Lemma 1. (See [2]) *The Hausdorff measure of noncompactness functional χ for a metric space (Y, d) acts as follows:*

(a) $\chi(A) = \infty$ *if and only if A is unbounded*

(b) $\chi(A) = 0$ *if and only if A is totally bounded*

(c) *If $A \subset B$, then $\chi(A) \leqslant 2\chi(B)$*

(d) *If A is totally bounded, then for each $\varepsilon > 0$, $\chi(S_\varepsilon[A]) \leqslant \varepsilon$*

(e) $\chi(\bar{A}) = \chi(A)$

(f) *If $\{F_n\}$ is a sequence in $C(Y)$ convergent in the Hausdorff metric to $F \in C(Y)$, then $\lim\limits_{n \to \infty} \chi(F_n) = \chi(F)$.*

Lemma 2. (See [2]) *Let $\{A_n\}$ be a decreasing sequence of nonempty closed sets in a complete metric space (Y, d). The following are equivalent:* (1) $\bigcap\limits_{n=1}^{\infty} A_n$ *is a nonempty compact set, and $\{A_n\}$ is a sequence convergent in the Hausdorff metric to* $\bigcap\limits_{n=1}^{\infty} A_n$. (2) $\lim\limits_{n \to \infty} \chi(A_n) = 0$.

Remark 1. Let (Y, d) be a metric space and F: $A \to Y$ be a multifunction with A dense in X. Put $G = \{x \in X$: the net $\{\chi(F(V \cap A))$: $V \in \mathscr{V}(x)\}$ converges to zero$\}$, where $\mathscr{V}(x)$ denotes the set of all open neighbourhoods of x.

It is easy to verify that $G = \left\{x \in X\right.$: for any $n \in N$ there exists $V \in \mathscr{V}(x)$ such that $\left.\chi(F(V \cap A)) \leqslant \frac{1}{n}\right\}$, which mens that G is a G_δ-set in X.

If F: $A \to Y$ is a compact-valued upper semicontinuous multifunction, then $A \subset G$. Let $x \in A$ and $n \in N$. Since $F(x)$ is compact by Lemma 1 (d) $\chi\left(S_{\frac{1}{2n}}[F(x)]\right) \leqslant \frac{1}{2n}$. The upper semicontinuity of F at x implies there exists a set $V \in \mathscr{V}(x)$ such that $F(V \cap A) \subset S_{\frac{1}{2n}}[F(x)]$. Then by Lemma 1 (c) $\chi(F(V \cap A)) \leqslant \frac{1}{n}$. The inclusion $A \subset G$ is proved.

Theorem 1. *Let Y be a complete metric space. Let F: $A \to Y$ be an upper semicontinuous closed-valued multifunction, where A is dense in X. Let the net $\{\chi(F(V \cap A))$: $V \in \mathscr{V}(x)\}$ converge to zero for any $x \in X \setminus A$. There exists an upper semicontinuous extension F^* of F defined on X.*

Proof. Put $F^*(x) = F(x)$ for $x \in A$. Now let $x \in X \setminus A$ and $\mathscr{V}(x)$ be the set of all open neighbourhoods of x. First we show that $\cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\} \neq \emptyset$. There exists a decreasing sequence $\{V_n\}$ of open sets from $\mathscr{V}(x)$ such that $\chi(F(V_n \cap A))) \leqslant 1/n$. By Lemma 1 (e) and Lemma 2 $\bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)}$ is a nonempty compact set and $\{\overline{F(V_n \cap A)}\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)}$.

Let $V \in \mathscr{V}(x)$. Then $\{\overline{F(V \cap V_n \cap A)}\}_{n=1}^{\infty}$ is a decreasing sequence of closed sets such that the sequence $\{\chi(\overline{F(V \cap V_n \cap A)})\}_{n=1}^{\infty}$ converges to zero and thus by Lemma 2 $\bigcap_{n=1}^{\infty} \overline{F(V \cap V_n \cap A)}$ is a nonempty compact set and $\{\overline{F(V \cap V_n \cap A)}\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} \overline{F(V \cap V_n \cap A)}$.

A family $\mathscr{O} = \{\overline{F(V \cap A)} \cap \left(\bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)}\right)$: $V \in \mathscr{V}(x)\}$ is a centred family of nonempty compact sets. Then $\emptyset \neq \cap \mathscr{O} \subset \cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\}$. It is easy to verify that $\cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\} = \cap \mathscr{O}$.

Since $\cap \mathscr{O}$ is a compact set, the set $\cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\}$ is also a compact set. Put $F^*(x) = \cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\}$ for $x \in X \setminus A$.

We show that F^* is upper semicontinuous. Let $x \in A$. Let U be an open set in Y such that $F^*(x) \subset U$. Since $F^*(x) = F(x)$ is a closed set in Y and Y is a normal space, there exists an open set U_1 such that $F^*(x) \subset U_1 \subset \overline{U_1} \subset U$. The upper semicontinuity of F at x implies, there is an open neighbourhood V of x such that $F(V \cap A) \subset U_1$. Let $z \in V \setminus A$. Then $F^*(z) = \cap \{\overline{F(G \cap A)}$: $G \in \mathscr{V}(z)\} \subset \overline{F(V \cap A)} \subset \overline{U_1} \subset U$. The upper semicontinuity of F^* at $x \in A$ is proved.

Now let $x \in X \setminus A$. It is sufficient to prove that for any $\varepsilon > 0$ there exists a neighbourhood V of x such that $F^*(V) \subset S_\varepsilon[F^*(x)]$ ($F^*(x)$ is a compact set for any $x \in X \setminus A$).

Let $\varepsilon > 0$. $F^*(x) = \cap \left\{\overline{F(U \cap A)} \cap \left(\bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)}\right): U \in \mathscr{V}(x)\right\} \subset S_{\varepsilon/2}[F^*(x)]$,

where $\{V_n\}$ is a decreasing sequence of neighbourhoods of x such that the sequence $\{\chi(\overline{F(V_n \cap A)})\}_{n=1}^{\infty}$ converges to zero. Put $B = \bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)}$. The compactness of B implies that $\overline{F(U \cap A)} \cap B$ is a compact set for any $U \in \mathscr{V}(x)$. Since $\cap \{\overline{F(U \cap A)} \cap B\colon U \in \mathscr{V}(x)\}$ is a subset of the open set $S_{\varepsilon 2}[F^*(x)]$ by [1] 5E there exist sets $U_1, U_2, \ldots, U_n \in \mathscr{V}(x)$ such that $\bigcap_{i=1}^{n} \overline{F(U_i \cap A)} \cap B \subset S_{\varepsilon 2}[F^*(x)]$. Put $G = \bigcap_{i=1}^{n} U_i$. Then $G \in \mathscr{V}(x)$ and $\overline{F(G \cap A)} \cap B = \overline{F(G \cap A)} \cap \left(\bigcap_{n=1}^{\infty} \overline{F(V_n \cap A)} \right) \subset S_{\varepsilon 2}[F^*(x)]$.

By Lemma 2 $\{\overline{F(G \cap V_n \cap A)}\}_{n=1}^{\infty}$ is a sequence convergent in the Hausdorff metric to $\bigcap_{n=1}^{\infty} \overline{F(G \cap V_n \cap A)}$. Thus there exists M such that for any $m \geqslant M$ $\overline{F(G \cap V_m \cap A)} \subset S_{\varepsilon 2}\left[\bigcap_{n=1}^{\infty} \overline{F(G \cap V_n \cap A)}\right] \subset S_{\varepsilon 2}\left[\bigcap_{i=1}^{\infty} \overline{F(G \cap A)} \cap \cap \overline{F(V_i \cap A)}\right] \subset S_{\varepsilon 2}[S_{\varepsilon 2}[F^*(x)]] \subset S_{\varepsilon}[F^*(x)]$. That implies $F^*(z) \subset S_{\varepsilon}[F^*(x)]$ for any $z \in G \cap V_M$. The upper semicontinuity of F^* is proved.

Theorem 2. *A metric space Y is complete if and only if each upper semicontinuous closed-valued multifunction $F\colon A \to Y$ with A dense in X and such that ford any $x \in X \setminus A$ the net $\{\chi(F(V \cap A))\colon V \in \mathscr{V}(x)\}$ converges to zero, has an upper semicontinuous extension to X.*

Proof. The necessity is obvious from Theorem 1.

Suppose that a metric space Y is not omplete. Then there exists a Cauchy sequence $\{y_n\}$ such that no point in Y is a cluster point of $\{y_n\}_{n=1}^{\infty}$. We can suppose that $y_i \neq y_j$ $i \neq j$. Put $X = \{y_1, \ldots, y_n, \ldots\}$. Let $\mathscr{T}$ consist of $\emptyset$ and of the sets $\{y_1, y_n, y_{n+1}, \ldots\}$, $n = 1, 2, \ldots$, $\mathscr{T}$ is a topology on X. Put $A = \{y_2, y_3, \ldots\}$. It is easy to verify that A is dense in X. Define a multifunction $F\colon A \to Y$ as follows: $F(y_n) = \{y_n, y_{n+1}, \ldots\}$ $n = 2, 3, \ldots$. Then F is a closed-valued upper semicontinuous multifunction on A.

Since $\{y_n\}$ is a cauchy sequence in Y, the net $\{\chi(F(V \cap A))\colon V \in \mathscr{V}(y_1)\}$ converges to zero. There exists no upper semicontinuous extension F^* of F defined on X.

Suppose that F^* is an upper semicontinuous extension of F defined on X. The upper semicontinuity of F^* at y_n for $n = 2, 3$, implies $F^*(y_1)$, contains no point from the set $\{y_2, y_3, \ldots\}$. Let $i > 1$ be such that $y_i \in F^*(y_1)$. Let $n > 1$. There exists an open set U in Y such that $F^*(y_n) = F(y_n) \subset U$ and $y_i \notin U$. Thus $f^*(y_1) \cap (X \setminus U) \neq 0$, which is a contradiction with the upper semicontinuity of F^* at y_n. $F^*(y_1) \cap (\{y_2, y_3, \ldots\}) = \emptyset$, that means $V = Y \setminus \{y_2, \ldots, y_n\}$ is open in Y such that $F^*(y_1) \subset V$ and $F^*(y_n) \cap V = \emptyset$ for any $n = 2, 3, \ldots$ However, that is a contradiction with the upper semicontinuity of F^* at y_1.

Theorem 3. *Let Y be a metric space. Y is topologically complete if and only if* *each upper semicontinuous compact-valued multifunction F: $A \to Y$ with A dense in X has upper semicontinuous compact-valued extension to a G_δ-set containing A.*

Proof. Suppose that a metric space (Y, d) is topologically complete. That means, there exists a complete metric ϱ in Y topologically equivalent to d.

Put $G = \{x \in X$: the net $\{\chi(F(V \cap A)$: $V \in \mathscr{V}(x)\}$ converges to zero$\}$. By Remark 1 G is a G_δ-set and $A \subset G$.

Define F^* as in the proof of Theorem 1., that means $F^*(x) = \cap \{\overline{F(V \cap A)}$: $V \in \mathscr{V}(x)\}$ for $x \in G \setminus A$ and $F^*(x) = F(x)$ for $x \in A$. Then F^* is an upper semicontinuous compact-valued multifunction. (see the proof of Theorem 1.)

Suppose that the metric space (Y, d) is not topologically complete. We show that there exist a topological space X and an upper semicontinuous compact-valued multifunction F from a dense set in X to Y, which has no upper semicontinuous compact-valued extension to a G_δ-set in X.

Let $(\tilde{Y}, \tilde{d})$ be a completion of (Y, d). Put $X = (\tilde{Y}, \tilde{d})$. Then Y is a dense subset of X, which is not a G_δ-set in X. (Suppose that Y is a G_δ-set in X. Then by [4] p. 49 Y is topologically complete.)

Consider the identity mapping i: $Y \to Y$. There exists no upper semicontinuous compact-valued extension of i to a G_δ-set in X containing Y.

Suppose that there exists an upper semicontinuous compact-valued extension i^* of i to a G_δ-set L containing Y. Let $y \in L \setminus Y$. There exists a sequence $\{y_n\}$ of points of Y which is convergent to y. The sequence $\{y_n\}$ has no cluster point in Y, that means every subsequence of $\{y_n\}$ is a closed set in Y.

There exists $N_1 \in N$ such that for any $n \geqslant N_1$ $y_n \in i^*(y)$. Otherwise there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \notin i^*(y)$ for any $k \in N$. The upper semicontinuity of i^* at y implies there exists an open set V such that $y \in V$ and $i^*(z) \subset$ - $Y \setminus \{y_{n_1}, y_{n_2}, \ldots, y_{n_k}, \ldots\}$ for any $z \in V$. But there exists $N_2 \in N$ such that for any $k \geqslant N_2$ $y_{n_k} \in V$, which is a contradiction.

Thus $i^*(y) \supset \{y_n, y_{n+1}, \ldots\}$ where $n \geqslant N_1$, that means $i^*(y)$ is not compact, which is a contradiction.

The following example shows that the assumption on values of the multifunction in Theorems 1 and 3 is essential.

Example 1. Put $Y = R$ with the usual topology. Put $X = \{1, 1/2, \ldots, 1/n, \ldots, 0\}$. Let $\mathscr{G}$ be a family consisting of $\emptyset$ and of the sets of the form $\{0, 1/n, 1/n+1, \ldots\}$ for $n = 1, 2, \ldots$ Then $\mathscr{G}$ is a topology on X. Put $A = \{1, 1/2, 1/3, \ldots, 1/n, \ldots\}$. Then A is dense in X in the topology $\mathscr{G}$ and only the G_δ-set containing A is the set X. Define F: $A \to Y$ in this way: $F(1/n) = (-1/n, 0)$. Then F is upper semicontinuous on A. It is easy to verify that the net $\{\chi(F(V \cap A))$: $V \in \mathscr{V}(0)\}$ converges to zero, where $\mathscr{V}(0)$ is the set of all open neighbourhoods of 0. There is no upper semicontinuous extension of F defined on X. (Suppose that there exists an upper semicontinuous extension F^* of F defined on X. The upper semicontinuity of F^* at points of A implies $F^*(0) \subset \bigcap_{n=1}^{\infty} (-1/n, 0) = \emptyset$. That is a contradiction.)

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ТЕОРЕМА О ПРОДОЛЖЕНИИ ДЛЯ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ И ХАРАКТЕРИЗАЦИЯ ПОЛНЫХ МЕТРИЧЕСКИХ ПРОСТРАНСТВ

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Резюме

Пусть Y — метризуемое пространство. Доказывается: пространство Y топологически полно, тогда и только тогда, когда каждое сверху напрерывное многозначное отображение Φ: $A \to Y$ с бикомпактными значениями, где A плотное множество в X, имеет сверху непрерывное продолжение на Γ_δ-множество Γ, причем $\Gamma \supset A$.