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Dedicated to Professor Tibor Šalát on the occasion of his 70th birthday

A COMPARISON THEOREM FOR WEIGHTED MEAN AND CESÀRO METHODS

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ABSTRACT. In this paper, we obtain a new inclusion theorem between (C,α), the Cesàro matrix of order α, $0<\alpha<1$, and weighted mean methods $(\overline{N},p)$, generated by certain monotone sequences.

Let $\sum a_n$ be an infinite series with partial sums $\{s_n\}$, $T=(a_{n,k})$ an infinite matrix. Suppose that the sums

$$T_n := \sum_{j=0}^{\infty} a_{nj}s_j \qquad (n=0,1,\dots)$$

exist. If

$$\sum_{n=1}^{\infty} n^{k-1}|T_n - T_{n-1}|^k < \infty\,, \tag{1}$$

then $\sum a_n$ is said to be $|T|_k$ summable, $k\ge 1$.

In a recent paper [3], Sarigöl and Bor obtained some comparison theorems between absolute Cesàro and absolute weighted mean matrices. Specifically, they proved the following two results.

THEOREM SB1. *Let $0<\alpha<1$. Then $|\overline{N},p|_k$ summability $(k\ge 1)$ implies $|C,\alpha|_k$ summability provided that*

$$P_n = O(n^{\alpha}p_n) \qquad as \quad n\to\infty\,. \tag{2}$$

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THEOREM SB2. *Let* $\alpha \geq 1$. *Then* $|\overline{N},p|_k$ *summability* ($k \geq 1$) *implies* $|C,\alpha|_k$ *summability provided that*

$$P_n = O(np_n) \qquad \text{as} \quad n \to \infty. \tag{3}$$

We first note that Theorem SB2 is a consequence of known results. Condition (3) implies that $|\overline{N},p|_k \subseteq |C,1|_k$ from Theorem 1 of Bor [1]. Actually Theorem 1 of [1] has both condition (3) and the condition that $np_n = O(P_n)$ in the hypotheses. However, if one examines that proof and uses (1) as the definition of absolute summability of order k, then the theorem is true using only condition (3). From Flett [2], $|C,1|_k \subseteq |C,\alpha|_k$ for $\alpha \geq 1$, and Theorem SB2 now follows from the transitivity of inclusion.

We also note that there are no nonincreasing sequences satisfying condition (2). For, if $\{p_n\}$ is nonincreasing, then $P_n := \sum_{j=0}^{n} p_j \geq (n+1)p_n$, and $P_n/n^\alpha p_n \geq n^{1-\alpha}$, contradicting (2).

Our result provides the analog of Theorem SB1 for ordinary convergence using nondecreasing sequences satisfying condition (2).

THEOREM. *Let* $0 < \alpha < 1$, $\{p_n\}$ *a nondecreasing sequence satisfying condition* (2). *Then* $(\overline{N},p) \subseteq (C,\alpha)$.

Proof. The entries of $\overline{N}^{-1}$ are $\overline{N}^{-1}_{jj} = P_j/p_j$, $\overline{N}^{-1}_{j+1,j} = -P_j/p_{j+1}$ and $\overline{N}^{-1}_{nj} = 0$, otherwise. With $A = C_\alpha \overline{N}^{-1}$, $E_n^\alpha := \binom{n+\alpha}{\alpha}$,

$$a_{nj} = \begin{cases} \dfrac{P_n}{E_n^\alpha p_n}, & j = n, \\[2ex] \dfrac{1}{E_n^\alpha}\left[E_{n-j}^{\alpha-1}\dfrac{P_j}{p_j} - E_{n-j-1}^{\alpha-1}\dfrac{P_j}{p_{j+1}}\right], & j < n. \end{cases}$$

We shall verify that A satisfies the Silverman-Toeplitz conditions. Since C_α and $\overline{N}$ are both triangles with row sums 1, it follows that A has row sums 1. For each fixed j,

$$a_{nj} \sim \frac{(n-j)^{\alpha-1}}{n^\alpha} + \frac{(n-j-1)^{\alpha-1}}{n^\alpha} \to 0 \qquad \text{as} \quad n \to \infty.$$

$$\sum_{j=0}^{n} |a_{nj}| = \frac{1}{E_n^\alpha}\sum_{j=0}^{n-1}\left|E_{n-j}^{\alpha-1}\frac{P_j}{p_j} - E_{n-j-1}^{\alpha-1}\frac{P_j}{p_{j+1}}\right| + \frac{P_n}{E_n^\alpha p_n}.$$

$$\left| E_{n-j}^{\alpha-1}\frac{P_j}{p_j} - E_{n-j-1}^{\alpha-1}\frac{P_j}{p_{j+1}} \right| = P_j \left| \frac{\Gamma(n-j+\alpha)}{(n-j)!\,\Gamma(\alpha)p_j} - \frac{\Gamma(n-j+\alpha-1)}{(n-j-1)!\,\Gamma(\alpha)p_{j+1}} \right|$$
$$= \frac{P_j\Gamma(n-j+\alpha-1)}{(n-j)!\,\Gamma(\alpha)} \left| \frac{n-j-\alpha-1}{p_j} - \frac{n-j}{p_{j+1}} \right|$$
$$= \frac{P_j|E_{n-j}^{\alpha-2}|}{1-\alpha} \left| \frac{n-j-\alpha-1}{p_j} - \frac{n-j}{p_{j+1}} \right| .$$

From the hypotheses on $\{p_n\}$,

$$\left| \frac{n-j-\alpha-1}{p_j} - \frac{n-j}{p_{j+1}} \right| \le (n-j)\left| \frac{1}{p_j} - \frac{1}{p_{j+1}} \right| + \frac{1+\alpha}{p_j} .$$

$$\frac{1}{E_n^\alpha}\sum_{j=0}^{n-1}\frac{(1+\alpha)P_j|E_{n-j}^{\alpha-2}|}{(1-\alpha)p_j} \sim \frac{1}{n^\alpha}\sum_{j=0}^{n-1} j^\alpha (n-j)^{\alpha-2} < \sum_{j=0}^{n-1}(n-j)^{\alpha-2} = O(1) .$$

$$\frac{1}{(1-\alpha)E_n^\alpha}\sum_{j=0}^{n-1} P_j|E_{n-j}^{\alpha-2}| \left| \frac{1}{p_j} - \frac{1}{p_{j+1}} \right| (n-j)$$
$$\sim \frac{1}{n^\alpha}\sum_{j=0}^{n-1}(n-j)^{\alpha-1}\left(\frac{1}{p_j} - \frac{1}{p_{j+1}}\right) P_j$$
$$= \frac{1}{n^\alpha}\sum_{j=0}^{n-1}\frac{(n-j)^{\alpha-1}P_j}{p_j} - \frac{1}{n^\alpha}\sum_{j=0}^{n-1}\frac{(n-j)^{\alpha-1}P_j}{p_{j+1}}$$
$$= \frac{n^{\alpha-1}P_0}{n^\alpha p_0} + \frac{1}{n^\alpha}\sum_{j=1}^{n-1}\frac{[(n-j)^{\alpha-1}P_j - (n-j+1)^{\alpha-1}]P_{j-1}}{p_j} - P_{n-1}n^\alpha p_n$$
$$= O(1) + \frac{1}{n^\alpha}\sum_{j=1}^{n-1}\frac{P_{j-1}}{p_j}[(n-j)^{\alpha-1} - (n-j+1)^{\alpha-1}] + \frac{1}{n^\alpha}\sum_{j=1}^{n-1}(n-j)^{\alpha-1}$$
$$= O(1) + \frac{O(1)}{n^\alpha}\sum_{j=1}^{n-1} j^\alpha[(n-j)^{\alpha-1} - (n-j+1)^{\alpha-1}]$$
$$< O(1)\left[1 + \sum_{j=1}^{n-1}[(n-j)^{\alpha-1} - (n-j+1)^{\alpha-1}]\right] = O(1) .$$

□

Remark. Condition (2) is not satisfied for nondecreasing sequences of the form $(n+1)^\alpha$ for $\alpha > 0$. For sequences of the form a^n, $a > 1$, the matrix method $(\overline{N}, p)$ is equivalent to convergence, so that the Theorem is trivially true. However, there do exist nondecreasing sequences which satisfy (2), and for which the corresponding matrix method is not equivalent to convergence. For example, define $\{p_n\}$ by $p_0 = 1$ and $p_n = e^{n^\alpha}/n^{1-\alpha}$ for $n > 0$.

A reasonable conjecture is the following: Let $0 < \alpha < 1$. If $\{p_n\}$ is either

(a) nonincreasing, or

(b) is nondecreasing and satisfies

$$\frac{n^\alpha p_n}{P_n}, \tag{4}$$

then $(C, \alpha) \subseteq (\overline{N}, p)$.

If condition (a) is satisfied, then it is known that $(C, 1) \subseteq (\overline{N}, p)$. The result then follows by the transitivity of inclusion.

Suppose that condition (b) is satisfied.

Since (C, α) is a Hausdorff matrix with nonzero diagonal entries, the inverse matrix is also a Hausdorff matrix with nonzero entries of the form $\binom{n}{k}\Delta^{n-k}\mu_k$, where $\mu_k = E_k^\alpha$, $\Delta\mu_k = \mu_k - \mu_{k+1}$, $\Delta^n\mu_k = \Delta(\Delta^{n-1}\mu_k)$. A straightforward calculation verifies that

$$\Delta^{n-k}\mu_k = \frac{-\alpha(1-\alpha)\dots(n-k-1-\alpha)\Gamma(k+\alpha+1)}{\Gamma(\alpha+1)n!}.$$

Hence

$$\binom{n}{k}\Delta^{n-k}\mu_k = E_k^\alpha E_{n-k}^{-\alpha-1}.$$

With $B = \overline{N}C_\alpha^{-1}$,

$$b_{nk} = \begin{cases} \dfrac{E_k^\alpha}{P_n} \displaystyle\sum_{j=k+1}^{n} p_j E_{j-k}^{\alpha-1} + \dfrac{p_k E_k^\alpha}{P_n}, & k < n, \\ \dfrac{p_n E_n^\alpha}{P_n}, & k = n. \end{cases}$$

B has row sums equal to 1. For k fixed, using (4),

$$|b_{nk}| \sim \frac{1}{P_n}\left(\sum_{j=k+1}^{n} p_j (j-k)^{-\alpha-1}\right) + o(1) = \frac{p_n O(1)}{P_n} + o(1) = o(1).$$

For $k < n$, $b_{nk} = (E_k^\alpha/P_n) f(k)$, where

$$f(k) := \sum_{j=k+1}^{n} p_j E_{j-k}^{-\alpha-1} + p_k = \sum_{i=1}^{n-k} p_{i+k} E_i^{-\alpha-1} + p_k.$$

To verify the conjecture, it would be sufficient to show that $f(k)$ is of fixed sign for all k sufficiently large.

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