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UNDERPARAMETRIZATION IN A REGRESSION MODEL WITH CONSTRAINTS II

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ABSTRACT. Mathematical models of many events and processes involve sometimes more unknown parameters than it can be interpreted. Thus a problem arises whether some of them can be neglected without an essential disagreement between experimental data and the reduced model. The aim of the paper is to contribute to a solution of the problem.

Introduction

A utilization of mathematical models of real events and processes is of a great importance in such a case only when all parameters of a model can be interpreted in the language and terms of the science region where the model is used. However, a good fitting of data need sometimes larger numbers of parameters than it can be interpreted. Thus a problem of underparametrization of a model arises.

It can be solved at least in two ways, i.e. either to test a hypothesis that the neglected parameters are zero, or to find a neighbourhood of zero where the nonzero values of the neglected parameters do not cause any essential deterioration of statistical inference. The aim of the paper is to contribute to the second approach.

1. Motivation example

Let in $\mathbb{R}^3$ (three dimensional Euclidean space) points A, P_1, P_2, B be given by coordinates $A \mapsto (x_1, 0, \beta_2+\beta_1 x_1+\gamma x_1^2 = \Theta_1)$, $P_1 \mapsto (x_2, 0, \beta_2+\beta_1 x_2+\gamma x_2^2)$,

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$P_2 \mapsto (x_3, 0, \beta_2+\beta_1 x_3+\gamma x_3^2)$, $B \mapsto (x_4, 0, \beta_2+\beta_1 x_4+\gamma x_4^2 = \Theta_2)$. The coordinates x are known, $x_1 = 4$, $x_2 = 6$, $x_3 = 8$, $x_4 = 10$ and also the heights Θ_1 and Θ_2 are known. The parameters β_1, β_2, γ are unknown. The differences of heights

$$\begin{aligned} h_1 &= \beta_2 + \beta_1 x_2 + \gamma x_2^2 - (\beta_2 + \beta_1 x_1 + \gamma x_1^2)\,, \\ h_2 &= \beta_2 + \beta_1 x_3 + \gamma x_3^2 - (\beta_2 + \beta_1 x_2 + \gamma x_2^2)\,, \\ h_3 &= \beta_2 + \beta_1 x_4 + \gamma x_4^2 - (\beta_2 + \beta_1 x_3 + \gamma x_3^2) \end{aligned}$$

are measured. (In geodesy it is a problem of levelling traverse between two fixed points A and B.)

The model of measurement is

$$\boldsymbol{Y} \sim_3 \left[\begin{pmatrix} x_2 - x_1, & x_2^2 - x_1^2 \\ x_3 - x_2, & x_3^2 - x_2^2 \\ x_4 - x_3, & x_4^2 - x_3^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \gamma \end{pmatrix}, \boldsymbol{\Sigma} \right]$$

and the constraints are

$$\begin{pmatrix} x_1, & x_1^2 \\ x_4, & x_4^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \gamma \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \beta_2 = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}.$$

Here $\boldsymbol{\Sigma}$ is the covariance matrix of the observation vector $\boldsymbol{Y}$. For the sake of simplicity let $\boldsymbol{\Sigma} = \sigma^2 \mathbf{I}$ ($\mathbf{I}$ is identity matrix).

The problem is to estimate the parameters β_1, β_2 and γ and to decide, whether the parameter γ can or cannot be neglected. If it cannot be neglected, then to determine a region around zero, where the nonzero values of γ do not cause any essential deterioration of statistical inference on parameters β_1 and β_2, respectively, based on simplified (underparametrized) model (2).

2. Notation and auxiliary statements

Consider linear regression model with constraints of type II

$$\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left[\begin{pmatrix} \mathbf{X}, & \mathbf{S}, & \mathbf{O} \\ \mathbf{B}_1, & \mathbf{G}, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \gamma \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \right] \tag{1}$$

and its submodel (underparametrized model)

$$\begin{pmatrix} \boldsymbol{Y} \\ -\boldsymbol{b} \end{pmatrix} \sim_{n+q} \left[\begin{pmatrix} \mathbf{X}, & \mathbf{O} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \right], \tag{2}$$

where the covariance matrix $\mathrm{Var}(\boldsymbol{Y}) = \boldsymbol{\Sigma}$ of the observation vector $\boldsymbol{Y}$ is assumed to be known.

Lemma 2.1. *Let the model* (2) *be regular, i.e.:*

$$r(\mathbf{X}_{n,k_1}) = k_1 < n\,, \qquad r(\mathbf{B}_1, \mathbf{B}_2) = q < k_1 + k_2\,, \qquad r(\mathbf{B}_2) = k_2 < q\,,$$

Σ *be positive definite* (*p.d.*). *Then the* BLUE*s* (*best linear unbiased estimators*) *of the parameters* $\boldsymbol{\beta}_1$ *and* $\boldsymbol{\beta}_2$ *are*

$$\hat{\hat{\boldsymbol{\beta}}}_1 = \hat{\boldsymbol{\beta}}_1 - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^{+}(\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b})\,,$$
$$\hat{\hat{\boldsymbol{\beta}}}_2 = -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'(\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b})\,,$$

where $()^+$ *denotes the Moore-Penrose* g*-inverse* ([5]) *of the matrix in the brackets,* $\mathbf{M}_{\mathbf{B}_2} = \mathbf{I} - \mathbf{P}_{\mathbf{B}_2}$, $\mathbf{P}_{\mathbf{B}_2}$ *is the projection matrix* (*in the Euclidean space*) *on the column space* $\mathcal{M}(\mathbf{B}_2) = \{\mathbf{B}_2\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^{k_2}\}$ *of the matrix* $\mathbf{B}_2$, $()^-_{m(\mathbf{U})}$ *denotes the minimum* $\mathbf{U}$*-seminorm* g*-inverse* ([5]) *of the matrix in the brackets* (*the matrix* $\mathbf{U}$ *must be at least positive semidefinite*) *and* $\hat{\boldsymbol{\beta}}_1 = \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y}$, $\mathbf{C} = \mathbf{X}'\Sigma^{-1}\mathbf{X}$ ($\hat{\boldsymbol{\beta}}_1$ *is the* BLUE *in the model without constraints*).

Proof. The minimization of the function $\phi(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_1)'\Sigma^{-1} \times (\boldsymbol{Y} - \mathbf{X}\boldsymbol{\beta}_1)$ under the constraints $\boldsymbol{b} + \mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 = \boldsymbol{O}$ leads to the equation

$$\begin{pmatrix} \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1', & \mathbf{B}_2 \\ \mathbf{B}_2', & \mathbf{O} \end{pmatrix}\begin{pmatrix} \boldsymbol{\lambda} \\ \hat{\hat{\boldsymbol{\beta}}}_2 \end{pmatrix} = -\begin{pmatrix} \boldsymbol{b} + \mathbf{B}_1\hat{\boldsymbol{\beta}}_1 \\ \boldsymbol{O} \end{pmatrix},$$

where $\boldsymbol{\lambda}$ is the vector of the Lagrange multipliers. With respect to Pandora-Box theorem ([5] and [1; p. 105]), we have

$$\begin{pmatrix} \boldsymbol{\lambda} \\ \hat{\hat{\boldsymbol{\beta}}}_2 \end{pmatrix} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix}\begin{pmatrix} -(\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b}) \\ \boldsymbol{O} \end{pmatrix}$$

where

$$\begin{aligned}
\boxed{11} &= (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^{+} \\
&= (\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1} - (\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\times \\
&\quad \times \mathbf{B}_2\left[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\right]^{-1}\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\,, \\
\boxed{12} &= (\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')} \\
&= (\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\left[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\right]^{-1}, \\
\boxed{21} &= \left(\boxed{12}\right)', \\
\boxed{22} &= -\boxed{21}\,\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\boxed{12}\,.
\end{aligned}$$

Now the statement is obvious. □

LEMMA 2.2. *Let the model* (2) *be regular. Then the estimator of* $\boldsymbol{\beta}_1$ *and* $\boldsymbol{\beta}_2$ *from Lemma* 2.1 *are biased in the model* (1) *and* ($*$ *denotes the model* (1) *and* $**$ *denotes the model* (2), *i.e.* $\widehat{\widehat{\boldsymbol{\beta}_i^{**}}} = \hat{\hat{\boldsymbol{\beta}}}_i$, $i = 1, 2$, *from Lemma* 2.1.)

$$E_*\left(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\right) - \boldsymbol{\beta}_1 = \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+ \times \\ \times(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{G}\boldsymbol{\gamma}),$$

$$E_*\left(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\right) - \boldsymbol{\beta}_2 = -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{G}\boldsymbol{\gamma}).$$

Proof.

$$\begin{aligned} E_*\left(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\right) &= E_*(\hat{\boldsymbol{\beta}}_1) - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+\left[\mathbf{B}_1E_*(\hat{\boldsymbol{\beta}}_1) + \boldsymbol{b}\right] \\ &= \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^-(\mathbf{X}\boldsymbol{\beta}_1 + \mathbf{S}\boldsymbol{\gamma}) - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+ \times \\ &\qquad \times\left[\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{X}\boldsymbol{\beta}_1 + \mathbf{S}\boldsymbol{\gamma}) + \boldsymbol{b}\right] \\ &= \boldsymbol{\beta}_1 + \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+ \times \\ &\qquad \times\left(\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{G}\boldsymbol{\gamma} + \mathbf{B}_2\boldsymbol{\beta}_2 + \boldsymbol{b} + \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{G}\boldsymbol{\gamma}\right) \\ &= \boldsymbol{\beta}_1 + \mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+ \times \\ &\qquad \times(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{G}\boldsymbol{\gamma}). \end{aligned}$$

Analogously

$$\begin{aligned} &E_*\left(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\right) \\ &= -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'\left[\mathbf{B}_1E_*(\hat{\boldsymbol{\beta}}_1) + \boldsymbol{b}\right] \\ &= -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'\left[\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\mathbf{X}\boldsymbol{\beta}_1 + \mathbf{S}\boldsymbol{\gamma}) + \boldsymbol{b}\right] \\ &= -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'(\mathbf{B}_1\boldsymbol{\beta}_1 + \mathbf{B}_2\boldsymbol{\beta}_2 - \mathbf{B}_2\boldsymbol{\beta}_2 + \mathbf{G}\boldsymbol{\gamma} + \boldsymbol{b} - \mathbf{G}\boldsymbol{\gamma} + \mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma}) \\ &= \boldsymbol{\beta}_2 - \left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\right]'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}\mathbf{S}\boldsymbol{\gamma} - \mathbf{G}\boldsymbol{\gamma}). \end{aligned}$$

□

LEMMA 2.3. *The covariance matrices of the estimators from Lemma* 2.1 *are*

$$\mathrm{Var}_{**}\left(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\right) = \mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{B}_1'\left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\right)^+\mathbf{B}_1\mathbf{C}^{-1},$$

$$\mathrm{Var}_{**}\left(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\right) = \left[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2\right]^{-1} - \mathbf{I}.$$

Proof. It is a direct consequence of Lemma 2.1. □

3. Underparametrization

Lemma 3.1. *The* BLUE*s of the parameter* $\binom{\beta_1}{\gamma}$ *and* β_2*, respectively, in the regular model* (1)*, i.e.* $r(\mathbf{X},\mathbf{S}) = k_1 + l < n$, $r(\mathbf{B}_1,\mathbf{G},\mathbf{B}_2) = q < k_1 + l + k_2$, $r(\mathbf{B}_2) = k_2 < q$, $\boldsymbol{\Sigma}$ *p.d., are*

$$\begin{pmatrix}\hat{\hat{\beta}}_1\\ \hat{\hat{\gamma}}\end{pmatrix} = \mathbf{D}^{-1}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y} - \mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\left[\mathbf{M}_{\mathbf{B}_2}(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}\right]^{-}\times$$

$$\times\left[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y} + \boldsymbol{b}\right],$$

$$\mathbf{D} = \begin{pmatrix}\mathbf{X}'\Sigma^{-1}\mathbf{X}, & \mathbf{X}'\Sigma^{-1}\mathbf{S}\\ \mathbf{S}'\Sigma^{-1}\mathbf{X}, & \mathbf{S}'\Sigma^{-1}\mathbf{S}\end{pmatrix},$$

$$\hat{\hat{\beta}}_2 = -\left[(\mathbf{B}_2')^{-}_{m[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}(\mathbf{B}_1,\mathbf{G})']}\right]'\left[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y} + \boldsymbol{b}\right],$$

$$\operatorname{Var}\begin{pmatrix}\hat{\hat{\beta}}_1\\ \hat{\hat{\gamma}}\end{pmatrix} = \mathbf{D}^{-1} - \mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\left\{\mathbf{M}_{\mathbf{B}_2}\left[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\right]\mathbf{M}_{\mathbf{B}_2}\right\}^{-}\times$$

$$\times(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1},$$

$$\operatorname{Var}(\hat{\hat{\beta}}_2) = \left\{\mathbf{B}_2'\left[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\right]^{-}\mathbf{B}_2\right\}^{-1} - \mathbf{I}.$$

P r o o f . It is a direct consequence of Lemma 2.1 and Lemma 2.3. □

Theorem 3.2.

(i) *Let* $h(\beta_1,\gamma,\beta_2) = \boldsymbol{h}_1'\beta_1$. *If* $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{C}\mathbf{M}_{\mathbf{X}'\Sigma^{-1}\mathbf{S}})$ *and* $\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{G}'$, *then the* BLUE *of* $\boldsymbol{h}_1'\beta_1$ *from Lemma* 2.1 *is equal to the* BLUE *of* $\boldsymbol{h}'\beta_1$ *from Lemma* 3.1*, i.e. the parameter* γ *can be neglected in the model.*

(ii) *If* $\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{G}'$, *then the* BLUE *of* β_2 *from Lemma* 2.1 *is equal to the* BLUE *of* β_2 *from Lemma* 3.1*, i.e. the parameter* γ *can be neglected in the model.*

P r o o f .

(i) The inverse $\mathbf{D}^{-1}$ can be expressed as

$$\mathbf{D}^{-1} = \begin{pmatrix}\mathbf{D}^{1,1}, & \mathbf{D}^{1,2}\\ \mathbf{D}^{2,1}, & \mathbf{D}^{2,2}\end{pmatrix},$$

where

$$\begin{aligned}
\mathbf{D}^{1,1} &= \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\,,\\
\mathbf{D}^{1,2} &= -\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1},\\
\mathbf{D}^{2,1} &= \big(\mathbf{D}^{1,2}\big)',\\
\mathbf{D}^{2,2} &= \big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\operatorname{Var}_*\left[(\boldsymbol{h}_1'\ \boldsymbol{O}')\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}_1^*}}\\ \widehat{\widehat{\boldsymbol{\gamma}^*}}\end{pmatrix}\right]\\
&= \boldsymbol{h}_1'\mathbf{C}^{-1}\boldsymbol{h}_1 - \boldsymbol{h}_1'\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\boldsymbol{h}_1\\
&\quad - \Big\{\boldsymbol{h}_1'\mathbf{C}^{-1}\mathbf{B}_1' + \boldsymbol{h}_1'\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}(\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' - \mathbf{G}')\Big\}\times\\
&\quad \times \Big(\mathbf{M}_{\mathbf{B}_2}\big\{\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + (\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} - \mathbf{G})\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\times\\
&\quad \times (\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' - \mathbf{G}')\big\}\mathbf{M}_{\mathbf{B}_2}\Big)^{+}\times\\
&\quad \times \Big\{\mathbf{B}_1\mathbf{C}^{-1}\boldsymbol{h}_1 + (\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} - \mathbf{G})\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\boldsymbol{h}_1\Big\}\\
&= \boldsymbol{h}_1'\mathbf{C}^{-1}\boldsymbol{h}_1 - \boldsymbol{h}_1'\mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{B}_1\mathbf{C}^{-1}\boldsymbol{h}_1 = \operatorname{Var}_{**}\Big(\boldsymbol{h}_1'\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\Big),
\end{aligned}$$

since $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{C}\mathbf{M}_{\mathbf{X}'\Sigma^{-1}\mathbf{S}}) \iff \boldsymbol{h}_1'\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} = \boldsymbol{O}'$ and $\mathbf{S}'\Sigma^{-1}\mathbf{X}'\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{G}'$.

(ii) If $\mathbf{S}'\Sigma^{-1}\mathbf{X}'\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{G}'$, then

$$\mathbf{B}_2'\left[(\mathbf{B}_1,\mathbf{G})\mathbf{D}^{-1}\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix} + \mathbf{B}_2\mathbf{B}_2'\right]^{-1}\mathbf{B}_2 = \mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2$$

and thus $\operatorname{Var}_*\Big(\widehat{\widehat{\boldsymbol{\beta}_2^*}}\Big) = \operatorname{Var}_{**}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\Big)$.

If two estimators are BLUEs of the same linear function of parameters, then one is equal to another with probability one. Thus the statements are proved. □

In [3] similar problem is solved in model without constraints and without assumption of regularity. However, the condition $\mathcal{M}(\mathbf{X}) \subset \mathcal{M}(\Sigma)$ is assumed.

Even the conditions $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{C}\mathbf{M}_{\mathbf{X}'\Sigma^{-1}\mathbf{S}})$ and $\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' = \mathbf{G}'$, respectively, need not be satisfied, the parameter γ can be sometimes neglected under some a prior information. It will be clarified in the following statements.

Let $\boldsymbol{b}_i = \mathbf{A}_i\gamma$, $i = 1, 2$, where

$$\begin{aligned}
\boldsymbol{b}_1 &= E_*\Big(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\Big) - \boldsymbol{\beta}_1\,,\\
\mathbf{A}_1 &= \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} - \mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} - \mathbf{G})\,,\\
\boldsymbol{b}_2 &= E_*\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\Big) - \boldsymbol{\beta}_2\,,\\
\mathbf{A}_2 &= -\Big[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1')}\Big]'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S} - \mathbf{G})\,,
\end{aligned}$$

$\mathbf{V} = \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^*}} - \widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\Big)$ and $\mathbf{W} = \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^*}}\Big) - \mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\Big)$.

LEMMA 3.3. *Under the condition of regularity it is valid that*

$$\mathcal{M}(\mathbf{A}_1) \subset \mathcal{M}\Big[\mathrm{Var}\Big(\widehat{\widehat{\boldsymbol{\beta}_1^*}} - \widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\Big)\Big]\,.$$

Proof. Let $\mathbf{K} = \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2'$, $\mathbf{U} = \mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1' - \mathbf{G}'$ and $\hat{\boldsymbol{\beta}}_1 = \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y}$. With respect to Lemma 3.1

$$\begin{aligned}
\widehat{\widehat{\boldsymbol{\beta}_1^*}} =& (\mathbf{I},\mathbf{O})\begin{pmatrix}\mathbf{D}^{1,1}, & \mathbf{D}^{1,2}\\ \mathbf{D}^{2,1}, & \mathbf{D}^{2,2}\end{pmatrix}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\Sigma^{-1}\boldsymbol{Y}\\
&- (\mathbf{I},\mathbf{O})\begin{pmatrix}\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\\ -\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\end{pmatrix}\times\\
&\times\Big(\mathbf{M}_{\mathbf{B}_2}\big\{\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\big\}\mathbf{M}_{\mathbf{B}_2}\Big)^+\times\\
&\times\Big\{\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y} + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\times\\
&\times(\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\boldsymbol{Y} - \mathbf{S}'\Sigma^{-1}\boldsymbol{Y}) + \boldsymbol{b}\Big\}\\
=& \hat{\boldsymbol{\beta}}_1 - \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1)\\
&- \Big\{\mathbf{C}^{-1}\mathbf{B}_1' + \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\Big\}\times\\
&\times\Big(\mathbf{M}_{\mathbf{B}_2}\big\{\mathbf{K} + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\big\}\mathbf{M}_{\mathbf{B}_2}\Big)^+\times\\
&\times\Big\{\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b} - \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1)\Big\}\,.
\end{aligned}$$

Further

$$\begin{aligned}
&\Big(\mathbf{M}_{\mathbf{B}_2}\big\{\mathbf{K} + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^{-1}\mathbf{U}\big\}\mathbf{M}_{\mathbf{B}_2}\Big)^+\\
&= (\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+ - (\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{U}'\big\{\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\Sigma\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]\\
&\qquad + \mathbf{U}\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{M}_{\mathbf{B}_2}\mathbf{U}'\big\}^{-1}\mathbf{U}\mathbf{M}_{\mathbf{B}_2}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+
\end{aligned}$$

and thus

$$\begin{aligned}\widehat{\widehat{\beta_1^*}} = {}&\hat{\beta}_1 - \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b})\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1)\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b})\\
&+ \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b}) + \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b})\\
&+ \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1)\\
&+ \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\times\\
&\times \big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1)\\
&- \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1)\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\times\\
&\times \Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1),\end{aligned}$$

what can be rewritten as

$$\begin{aligned}\widehat{\widehat{\beta_1^*}} = {}&\widehat{\widehat{\beta_1^{**}}} + \big[\mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}' - \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\big]\times\\
&\times \Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b})\\
&+ \mathbf{C}^{-1}\mathbf{B}_1'\Big\{\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big) + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{U}\Big\}^+\times\\
&\times \mathbf{U}'\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1) - \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\times\\
&\times \Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\beta}_1)\\
= {}&\widehat{\widehat{\beta_1^{**}}} - \mathbf{A}_1\Big\{\big[\mathbf{S}'(\mathbf{M}_\mathbf{X}\Sigma\mathbf{M}_\mathbf{X})^+\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+(\mathbf{B}_1\hat{\beta}_1 + \boldsymbol{b})\\
&+ \mathbf{C}^{-1}\mathbf{B}_1'\Big(\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+ - \big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^+\times\end{aligned}$$

$$
\begin{aligned}
&\times \mathbf{U}'\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\Big)\times\\
&\times \mathbf{U}'\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1)\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{S}'\Sigma^{-1}\big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1\big)\\
=&\widehat{\widehat{\boldsymbol{\beta}_1^{**}}} - \mathbf{A}_1\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\big(\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b}\big)\\
&+ \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}]^{-1}\mathbf{S}'\Sigma^{-1}\big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1\big)\\
&- \mathbf{C}^{-1}\mathbf{B}_1'\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{S}'\Sigma^{-1}\big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1\big)\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{S}'\Sigma^{-1}\big(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1\big)\,.
\end{aligned}
$$

Since $\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b}$ and $\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1$ are uncorrelated and $\mathrm{Var}\big(\mathbf{B}_1\hat{\boldsymbol{\beta}}_1 + \boldsymbol{b}\big) = \mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'$ and $\mathrm{Var}\big[\mathbf{S}'\Sigma^{-1}(\boldsymbol{Y} - \mathbf{X}\hat{\boldsymbol{\beta}}_1)\big] = \mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}$, we can write

$$
\begin{aligned}
&\mathrm{Var}\left(\widehat{\widehat{\boldsymbol{\beta}_1^{*}}} - \widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\right)\\
=&\mathbf{A}_1\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\times\\
&\times \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{K}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\times\\
&\times \Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\mathbf{A}_1'\\
&+ \Big(\mathbf{C}^{-1}\mathbf{B}_1'\Big\{\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+} + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{U}\Big\}^{+}\times\\
&\times \mathbf{U}'\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\\
&- \mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\Big)\mathbf{S}'\big(\mathbf{M_X}\Sigma\mathbf{M_X}\big)^{+}\mathbf{S}\times\\
&\times \Big(\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{U}\Big\{\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+} + \mathbf{U}'\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big]^{-1}\mathbf{U}\Big\}^{+}\mathbf{B}_1\mathbf{C}^{-1}\\
&- \Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\Big)\\
=&\mathbf{A}_1\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\times\\
&\times \mathbf{K}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\{\big[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^{+}\mathbf{S}\big] + \mathbf{U}\big(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2}\big)^{+}\mathbf{U}'\Big\}^{-1}\mathbf{A}_1'
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{A}_1\Big\{[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}]+\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'\Big\}^{-1}\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}\times\\
&\times\Big\{[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}]+\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'\Big\}^{-1}\mathbf{A}_1'\\
=&\,\mathbf{A}_1\Big\{[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}]+\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'\Big\}^{-1}\mathbf{A}_1',
\end{aligned}
$$

since $\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{K}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'=\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'$. Thus

$$
\begin{aligned}
&\mathcal{M}[\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}-\mathbf{C}^{-1}\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}-\mathbf{G})]=\mathcal{M}(\mathbf{A}_1)\\
&\subset\mathcal{M}\Big(\mathbf{A}_1\Big\{[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}]+\mathbf{U}(\mathbf{M}_{\mathbf{B}_2}\mathbf{K}\mathbf{M}_{\mathbf{B}_2})^+\mathbf{U}'\Big\}^{-1}\mathbf{A}_1'\Big)\\
&=\mathcal{M}\Big[\operatorname{Var}\Big(\widehat{\widehat{\beta_1^*}}-\widehat{\widehat{\beta_1^{**}}}\Big)\Big].
\end{aligned}
$$

□

Lemma 3.4. *Under the condition of regularity it is valid that*

$$\mathcal{M}(\mathbf{A}_2)\subset\mathcal{M}(\mathbf{W}).$$

Proof.

$$
\begin{aligned}
&\operatorname{Var}\Big(\widehat{\widehat{\beta_2^*}}\Big)-\operatorname{Var}\Big(\widehat{\widehat{\beta_2^{**}}}\Big)\\
&=\Big(\mathbf{B}_2'\big\{(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{X}'\Sigma^{-1}\mathbf{S}-\mathbf{G})[\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}]^{-1}(\mathbf{S}'\Sigma^{-1}\mathbf{X}\mathbf{C}^{-1}\mathbf{B}_1'-\mathbf{G}')\\
&\quad+\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'+\mathbf{B}_2\mathbf{B}_2'\big\}^{-1}\mathbf{B}_2\Big)^{-1}-\mathbf{I}-[\mathbf{B}_2'(\mathbf{B}_1\mathbf{C}^{-1}\mathbf{B}_1'+\mathbf{B}_2\mathbf{B}_2')^{-1}\mathbf{B}_2]^{-1}+\mathbf{I}\\
&=\Big[\mathbf{B}_2'\Big(\mathbf{K}^{-1}-\mathbf{K}^{-1}\mathbf{U}'\{\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}+\mathbf{U}\mathbf{K}^{-1}\mathbf{U}'\}\mathbf{U}\mathbf{K}^{-1}\Big)^{-1}\mathbf{B}_2\Big]^{-1}\\
&\quad-(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\\
&=(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}+(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{U}'\times\\
&\quad\times\Big\{\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{S}+\mathbf{U}\mathbf{K}^{-1}\mathbf{U}'-\mathbf{U}\mathbf{K}^{-1}\mathbf{B}_2(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{U}'\Big\}^{-1}\times\\
&\quad\times\mathbf{U}\mathbf{K}^{-1}\mathbf{B}_2(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}-(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\\
&=(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{U}'\Big\{\mathbf{S}'(\mathbf{M_X}\Sigma\mathbf{M_X})^+\mathbf{U}'\Big\}^{-1}\mathbf{U}\mathbf{K}^{-1}\mathbf{B}_2(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}.
\end{aligned}
$$

Obviously

$$\mathcal{M}(\mathbf{A}_2)=\mathcal{M}[(\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{B}_2)^{-1}\mathbf{B}_2'\mathbf{K}^{-1}\mathbf{U}']\subset\mathcal{M}\Big[\operatorname{Var}\Big(\widehat{\widehat{\beta_2^*}}\Big)-\operatorname{Var}\Big(\widehat{\widehat{\beta_2^{**}}}\Big)\Big].$$

□

THEOREM 3.5.

(i)

$$\gamma \in \{\boldsymbol{g} : \ \boldsymbol{g}'\mathbf{A}_1'\mathbf{V}^-\mathbf{A}_1\boldsymbol{g} \le c^2\} \\ \implies (\forall \boldsymbol{h} \in \mathbb{R}^{k_1}) \left(|\boldsymbol{h}'\boldsymbol{b}_1| \le c\sqrt{\mathrm{Var}\Big[\boldsymbol{h}'\big(\widehat{\widehat{\beta_1^*}} - \widehat{\widehat{\beta_1^{**}}}\big)\Big]} \right).$$

(ii)

$$\gamma \in \{\boldsymbol{g} : \ \boldsymbol{g}'\mathbf{A}_2'\mathbf{W}^-\mathbf{A}_2\boldsymbol{g} \le 1\} \\ \implies (\forall \boldsymbol{h} \in \mathbb{R}^{k_2})\Big(\mathrm{Var}\left(\boldsymbol{h}'\widehat{\widehat{\beta_2^{**}}}\right) + (\boldsymbol{h}'\boldsymbol{b}_2)^2 \le \mathrm{Var}\left(\boldsymbol{h}'\widehat{\widehat{\beta_2^*}}\right)\Big).$$

P r o o f.

(i) With respect to Lemma 3.3, $\mathcal{M}(\mathbf{A}_1) \subset \mathcal{M}(\mathbf{V})$ and therefore regarding a nonessential generalization of the S c h e f f é theorem ([6]) we obtain

$$(\forall \boldsymbol{h} \in \mathbb{R}^{k_1})\Big(|\boldsymbol{h}'\boldsymbol{b}_1| = |\boldsymbol{h}'\mathbf{A}_1\gamma| \le \sqrt{\boldsymbol{h}'\mathbf{V}\boldsymbol{h}}\Big) \iff \gamma'\mathbf{A}_1'\mathbf{V}^-\mathbf{A}_1\gamma \le 1.$$

(ii) With respect to Lemma 3.4, $\mathcal{M}(\mathbf{A}_2) \subset \mathcal{M}(\mathbf{W})$. Analogously as in (i)

$$(\forall \boldsymbol{h} \in \mathbb{R}^{k_2})\Big(|\boldsymbol{h}'\boldsymbol{b}_2| = |\boldsymbol{h}'\mathbf{A}_2\gamma| \le c\sqrt{\boldsymbol{h}'\mathbf{W}\boldsymbol{h}}\Big) \iff \gamma'\mathbf{A}_2'\mathbf{W}^-\mathbf{A}_2\gamma \le c^2.$$

□

4. Generalization

In this section, conditions of regularity are not assumed. Then the notation ULSM (universal linear statistical model) will be used. ULSMII means ULSM with constraints II.

LEMMA 4.1.

(i) *A function* $h(\beta_1, \gamma, \beta_2) = \boldsymbol{h}_1'\beta_1$ *is unbiasedly estimable in* (2) *if and only if* $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})$.

(ii) *A function* $h(\beta_1, \gamma, \beta_2) = \boldsymbol{h}_2'\beta_2$ *is unbiasedly estimable in* (2) *if and only if* $\boldsymbol{h}_2 \in \mathcal{M}(\mathbf{B}_2'\mathbf{M}_{\mathbf{B}_1\mathbf{M}_{\mathbf{X}'}})$.

P r o o f. Both statements are a direct consequence of the relationships

$$\boldsymbol{h}_1'\beta_1 \text{ is unbiasedly estimable} \iff \begin{pmatrix} \boldsymbol{h}_1 \\ \boldsymbol{O} \end{pmatrix} \in \mathcal{M}\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{O}, & \mathbf{B}_2' \end{pmatrix},$$

$$\boldsymbol{h}_2'\beta_2 \text{ is unbiasedly estimable} \iff \begin{pmatrix} \boldsymbol{O} \\ \boldsymbol{h}_2 \end{pmatrix} \in \mathcal{M}\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{O}, & \mathbf{B}_2' \end{pmatrix}.$$

As far as (ii) is concerned, it is to be remarked:

$$(\forall \boldsymbol{t} \in \mathbb{R}^q)(\exists \boldsymbol{u} \in \mathbb{R}^n)(\mathbf{X}'\boldsymbol{u} + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_1\mathbf{M}_{\mathbf{X}'}}\boldsymbol{t} = \boldsymbol{O}) \Longleftarrow$$

$$\begin{aligned}\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_1\mathbf{M}_{\mathbf{X}'}} &= \mathbf{B}_1' - [(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X}) - \mathbf{X}'\mathbf{X}](\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{B}_1' \\ &\quad + \mathbf{B}_1'\mathbf{B}_1(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{X}'[\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{X}']^-\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{B}_1' \\ &= \mathbf{X}'\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{B}_1' + \mathbf{X}'[\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{X}']^-\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{B}_1' \\ &\quad - \mathbf{X}'\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{X}'[\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{X}']^-\mathbf{X}(\mathbf{B}_1'\mathbf{B}_1 + \mathbf{X}'\mathbf{X})\mathbf{B}_1',\end{aligned}$$

thus $\mathcal{M}(\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_1\mathbf{M}_{\mathbf{X}'}}) \subset \mathcal{M}(\mathbf{X}')$. □

LEMMA 4.2.

(i) *An unbiasedly estimable function* $\boldsymbol{h}_1'\boldsymbol{\beta}_1$ (*i.e.* $\boldsymbol{h}_1 \in \mathcal{M}(\mathbf{X}', \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})$) *in the model* (2) *is biased in the model* (1) *and*

$$E_*\left(\widehat{\widehat{\boldsymbol{h}_1'\boldsymbol{\beta}_1^{**}}}\right) - \boldsymbol{h}_1'\boldsymbol{\beta}_1 = \boldsymbol{h}_1'\left\{\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]'\mathbf{S} + (\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1)^-_{m(\mathbf{W})}\mathbf{G}\right\}\boldsymbol{\gamma}.$$

(ii) *An unbiasedly estimable function* $\boldsymbol{h}_2'\boldsymbol{\beta}_2$ (*i.e.* $\boldsymbol{h}_2 \in \mathcal{M}(\mathbf{B}_2'\mathbf{M}_{\mathbf{B}_1\mathbf{M}_{\mathbf{X}'}})$) *in the model* (2) *is biased in the model* (1) *and*

$$\begin{aligned}&E_*\left(\widehat{\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2^{**}}}\right) - \boldsymbol{h}_2'\boldsymbol{\beta}_2 \\ &\quad = \boldsymbol{h}_2'\left\{-\left[(\mathbf{B}_2')^-_{m(\mathbf{V})}\right]'\mathbf{B}_1\mathbf{W}^+\mathbf{X}'\left(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\right)^-\mathbf{S} + \left[(\mathbf{B}_2')^-_{m(\mathbf{V})}\right]'\mathbf{G}\right\}\boldsymbol{\gamma},\end{aligned}$$

where

$$\mathbf{V} = \mathbf{B}_1\mathbf{W}^+\mathbf{B}_1' + \mathbf{B}_2\mathbf{B}_2', \qquad \mathbf{W} = \mathbf{X}\left(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\right)^+ + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1.$$

Proof. Since (cf. [2; Theorem 4.6]) $\begin{pmatrix}\boldsymbol{\Sigma}, \mathbf{O}\\ \mathbf{O}, \mathbf{O}\end{pmatrix}$-seminorm g-inverse of the matrix $\begin{pmatrix}\mathbf{X}', \mathbf{B}_1'\\ \mathbf{O}, \mathbf{B}_2'\end{pmatrix}$ is given by the relation

$$\left[\begin{pmatrix}\mathbf{X}', & \mathbf{B}_1'\\ \mathbf{O}, & \mathbf{B}_2'\end{pmatrix}^-_{m\begin{pmatrix}\boldsymbol{\Sigma}, \mathbf{O}\\ \mathbf{O}, \mathbf{O}\end{pmatrix}}\right]' = \begin{pmatrix}\boxed{1}, & \boxed{2}\\ \boxed{3}, & \boxed{4}\end{pmatrix},$$

where

$$\begin{aligned}\boxed{1} &= \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\boldsymbol{\Sigma})}\right]', \\ \boxed{2} &= \mathbf{W}^+\mathbf{B}_1'(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2})^+, \\ \boxed{3} &= -\left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\right]'\mathbf{B}_1\mathbf{W}^+\mathbf{X}'\left(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\right)^+, \\ \boxed{4} &= \left[(\mathbf{B}_2')^-_{m(\mathbf{B}_1\mathbf{W}^+\mathbf{B}_1')}\right]',\end{aligned}$$

and $\mathbf{W} = \mathbf{X}'\left(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\right)^+ + \mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1$, we have

(i)

$$E_*\left(\widehat{\widehat{h_1'\beta_1^{**}}}\right)$$

$$= (u', t'M_{B_2})\begin{pmatrix} X, & O \\ B_1, & B_2 \end{pmatrix}\begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix}\left[\begin{pmatrix} X, & O \\ B_1, & B_2 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} S \\ G \end{pmatrix}\gamma\right]$$

$$= h_1'\beta_1 + h_1'\left(\boxed{1}S + \boxed{2}G\right)\gamma,$$

since $u'X + t'M_{B_2}B_1 = h_1'$.

(ii) Analogously

$$E_*\left(\widehat{\widehat{h_2'\beta_1^{**}}}\right)$$

$$= \left(u', t'M_{B_1M_{X'}}\right)\begin{pmatrix} X, & O \\ B_1, & B_2 \end{pmatrix}\begin{pmatrix} \boxed{1}, & \boxed{2} \\ \boxed{3}, & \boxed{4} \end{pmatrix}\left[\begin{pmatrix} X, & O \\ B_1, & B_2 \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} S \\ G \end{pmatrix}\gamma\right]$$

$$= h_2'\beta_2 + h_2'\left(\boxed{3}S + \boxed{4}G\right)\gamma,$$

since $u'X + t'M_{B_1M_{X'}}B_1 = O'$ and $t'M_{B_1M_{X'}}B_2 = h_2'$. Now the proof can be easily finished. □

Remark 4.3. Let two ULSMs (without constraints)

$$Y \sim_n \left[(X, S)\begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \Sigma\right], \qquad \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^{k_1+l} \tag{3}$$

and

$$Y \sim_n (X\beta, \Sigma), \qquad \beta \in \mathbb{R}^{k_1} \tag{4}$$

be under consideration. Then the variance of the BLUE of the function $t'M_SX\beta$ in the model (3) is

$$\operatorname{Var}_*\left(\widehat{t'M_SX\beta^*}\right) = t'M_SX\left[X'(M_STM_S)^+X\right]^+X'M_St - t'M_SXX'M_St,$$

where $T = \Sigma + XX'$ and in the model (4)

$$\operatorname{Var}_{**}\left(\widehat{t'M_SX\beta^{**}}\right) = t'M_SX(X'T^+X')^+X'M_St - t'M_SXX'M_St.$$

The inequality

$$\operatorname{Var}_*\left(\widehat{t'M_SX\beta^*}\right) \geq \operatorname{Var}_{**}\left(\widehat{t'M_SX\beta^{**}}\right)$$

is implied by the inequalities (in the Loevner sense)

$$(M_STM_S)^+ = (T + SS')^+ - (T + SS')^+S\left[S'(T + SS')^+S\right]^+S'(T + SS')^+ \leq_L T^+$$
$$\Longrightarrow \left[X'(M_STM_S)^+X\right]^+ \geq_L (X'T^+X)^+.$$

Therefore analogous inequalities are valid in ULSMIIs (1) and (2), i.e.

$$(\forall \boldsymbol{h}_1 \in \mathcal{M}_1)\Big(\mathrm{Var}_*\big(\widehat{\widehat{\boldsymbol{h}_1'\boldsymbol{\beta}_1^*}}\big) \geq \mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{h}_1'\boldsymbol{\beta}_1^{**}}}\big)\Big),$$

$$(\forall \boldsymbol{h}_2 \in \mathcal{M}_2)\Big(\mathrm{Var}_*\big(\widehat{\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2^*}}\big) \geq \mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{h}_2'\boldsymbol{\beta}_2^{**}}}\big)\Big).$$

Here $\mathcal{M}_1 = \mathcal{M}\left((\mathbf{X}', \mathbf{B}_1')\mathbf{M}_{\begin{pmatrix}\mathbf{S}, & \mathbf{O}\\ \mathbf{G}, & \mathbf{B}_2\end{pmatrix}}\right)$ and $\mathcal{M}_2 = \mathcal{M}\left((\mathbf{O}, \mathbf{B}_2')\mathbf{M}_{\begin{pmatrix}\mathbf{X}, & \mathbf{S}\\ \mathbf{B}_1, & \mathbf{G}\end{pmatrix}}\right)$ characterize the unbiasedly estimable functions of $\boldsymbol{\beta}_1$ ($\mathcal{M}_1$) and the unbiasedly estimable functions of $\boldsymbol{\beta}_2$ ($\mathcal{M}_2$).

Let the following notation

$$\mathbf{W}_* = \begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\left[\boldsymbol{\Sigma} + (\mathbf{X}, \mathbf{S})\mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\right]^-(\mathbf{X}, \mathbf{S}) + \begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}(\mathbf{B}_1, \mathbf{G}),$$

$$\mathbf{V}_* = (\mathbf{B}_1, \mathbf{G})\mathbf{W}_*^-\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix} + \mathbf{B}_2\mathbf{B}_2',$$

$$\mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\big) = \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^+\mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}} - \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}},$$

$$\mathrm{Var}_*\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}_1^*}}\\ \widehat{\widehat{\boldsymbol{\gamma}^*}}\end{pmatrix} = \mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\left[\boldsymbol{\Sigma} + (\mathbf{X}, \mathbf{S})\mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\right]^- \times$$
$$\times (\mathbf{X}, \mathbf{S})\mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}} - \mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}},$$

$$\mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\big)$$
$$= \big[(\mathbf{B}_2')_{m(\mathbf{V})}\big]'\mathbf{B}_1\mathbf{W}^+\mathbf{X}'\big(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^-\boldsymbol{\Sigma}\big(\boldsymbol{\Sigma} + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\big)^- \times$$
$$\times \mathbf{X}\mathbf{W}^+\mathbf{B}_1'(\mathbf{B}_2')_{m(\mathbf{V})}^-,$$

$$\mathrm{Var}_*\big(\widehat{\widehat{\boldsymbol{\beta}_2^*}}\big)$$
$$= \big[(\mathbf{B}_2')_{m(\mathbf{V}_*)}\big]'(\mathbf{B}_1, \mathbf{G})\mathbf{W}_*^+\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\left[\boldsymbol{\Sigma} + (\mathbf{X}, \mathbf{S})\mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\right]^- \times$$
$$\times \boldsymbol{\Sigma}\left[\boldsymbol{\Sigma} + (\mathbf{X}, \mathbf{S})\mathbf{M}_{\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}\mathbf{M}_{\mathbf{B}_2}}\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\right]^-(\mathbf{X}, \mathbf{S})\mathbf{W}_*^+\begin{pmatrix}\mathbf{B}_1'\\ \mathbf{G}'\end{pmatrix}(\mathbf{B}_2')_{m(\mathbf{V}_*)}^-,$$

$$\mathbf{R}_1 = (\mathbf{I}, \mathbf{O})\,\mathrm{Var}_*\begin{pmatrix}\widehat{\widehat{\boldsymbol{\beta}_1^*}}\\ \widehat{\widehat{\boldsymbol{\gamma}^*}}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{O}\end{pmatrix} - \mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{\beta}_1^{**}}}\big),$$

$$\mathbf{R}_2 = \mathrm{Var}_*\big(\widehat{\widehat{\boldsymbol{\beta}_2^*}}\big) - \mathrm{Var}_{**}\big(\widehat{\widehat{\boldsymbol{\beta}_2^{**}}}\big)$$

be used. Let

$$\mathbf{F}_1 = \mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\left[(\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}')^-_{m(\Sigma)}\right]'\mathbf{S} + \left(\mathbf{M}_{\mathbf{B}_2}\mathbf{B}_1\right)^-_{m(\mathbf{W})}\mathbf{G}\,,$$
$$\mathbf{F}_2 = -\left[(\mathbf{B}_2')^-_{m(\mathbf{V})}\right]'\mathbf{B}_1\mathbf{W}^+\mathbf{X}'\left(\Sigma + \mathbf{X}\mathbf{M}_{\mathbf{B}_1'\mathbf{M}_{\mathbf{B}_2}}\mathbf{X}'\right)^-\mathbf{S} + \left[(\mathbf{B}_2')^-_{m(\mathbf{V})}\right]'\mathbf{G}$$

(cf. Lemma 4.2). Then, analogously as in Theorem 3.5, if $\mathcal{M}(\mathbf{F}_i) \subset \mathcal{M}(\mathbf{R}_i)$, $i = 1,2$, then for all $i = 1,2$

$$(\forall \boldsymbol{h}_i \in \mathcal{M}_i)\left(\left|E_{**}\left(\widehat{\widehat{\boldsymbol{h}_i'\boldsymbol{\beta}_i^{**}}}\right) - \boldsymbol{h}_i'\boldsymbol{\beta}_i\right| \le c\sqrt{\operatorname{Var}_*\left(\widehat{\widehat{\boldsymbol{h}_i'\boldsymbol{\beta}_i^*}}\right) - \operatorname{Var}_{**}\left(\widehat{\widehat{\boldsymbol{h}_i'\boldsymbol{\beta}_i^{**}}}\right)}\right)$$
$$\iff \gamma'\mathbf{F}_i'\mathbf{R}_i^-\mathbf{F}_i\gamma \le c^2\,.$$

Sometimes there occurs such a situation that either a function $\boldsymbol{h}_1'\boldsymbol{\beta}_1$ or a function $\boldsymbol{h}_2'\boldsymbol{\beta}_2$ can be unbiasedly estimated in (2) and the estimator is the same as in (1) (cf. Theorem 3.2).

To find a subspace of functions which can be unbiasedly estimated by BLUE in the model (2) and this estimator is simultaneously the BLUE in the model (1) is rather difficult. Until now only the following result is known to the author.

THEOREM 4.4. *Let* ULSMII

$$\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix} \sim_{n+q} \left[\begin{pmatrix}\mathbf{X}, & \mathbf{O}, & \mathbf{S}\\ \mathbf{B}_1, & \mathbf{B}_2, & \mathbf{G}\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\\ \gamma\end{pmatrix}, \begin{pmatrix}\Sigma, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\right]$$

and

$$\begin{pmatrix}\mathbf{Y}\\ -\boldsymbol{b}\end{pmatrix} \sim_{n+q} \left[\begin{pmatrix}\mathbf{X}, & \mathbf{O}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1\\ \boldsymbol{\beta}_2\end{pmatrix}, \begin{pmatrix}\Sigma, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\right]$$

be considered. Let

$$\mathcal{M}\begin{pmatrix}\mathbf{S}\\ \mathbf{G}\end{pmatrix} \subset \mathcal{M}\begin{pmatrix}\mathbf{T}, & \mathbf{X}\mathbf{S}'\\ \mathbf{S}\mathbf{X}', & \mathbf{S}\mathbf{S}'\end{pmatrix} \qquad (\mathbf{T} = \Sigma + \mathbf{X}\mathbf{X}').$$

Then BLUE *of the function* $h(\boldsymbol{\beta}_1,\boldsymbol{\beta}_2) = \boldsymbol{h}_1'\boldsymbol{\beta}_1 + \boldsymbol{h}_2'\boldsymbol{\beta}_2$, *where*

$$\begin{pmatrix}\boldsymbol{h}_1\\ \boldsymbol{h}_2\end{pmatrix} = \begin{pmatrix}\mathbf{X}', & \mathbf{B}_1'\\ \mathbf{O}, & \mathbf{B}_2'\end{pmatrix}\mathbf{M}_{(\mathbf{S}',\mathbf{G}')'}\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{t}\end{pmatrix}, \qquad \boldsymbol{s} \in \mathbb{R}^n\,, \quad \boldsymbol{t} \in \mathbb{R}^q\,,$$

in both models are identical if and only if

$$\begin{pmatrix}\boldsymbol{h}_1\\ \boldsymbol{h}_2\end{pmatrix} \in \mathcal{M}\left[\begin{pmatrix}\mathbf{A}_{1,1}, & \mathbf{A}_{1,2}\\ \mathbf{A}_{2,12}, & \mathbf{A}_{2,2}\end{pmatrix}\mathbf{M}_{\mathbf{K}}\right]$$

where

$$\mathbf{D} = \mathbf{X}'\mathbf{T}^-\mathbf{X}\,,$$

$$\mathbf{A}_{1,1} = \mathbf{D} + (\mathbf{D}\mathbf{S}' - \mathbf{B}_1')\big[\mathbf{S}(\mathbf{I}-\mathbf{D})\mathbf{S}'\big]^-(\mathbf{S}\mathbf{D} - \mathbf{B}_1)\,,$$

$$\mathbf{A}_{1,2} = -(\mathbf{D}\mathbf{S}' - \mathbf{B}_1')\big[\mathbf{S}(\mathbf{I}-\mathbf{D})\mathbf{S}'\big]^-(\mathbf{S}\mathbf{D} - \mathbf{B}_1)\mathbf{B}_2 = \mathbf{A}_{2,1}'\,,$$

$$\mathbf{A}_{2,2} = \mathbf{B}_2'\big[\mathbf{S}(\mathbf{I}-\mathbf{D})\mathbf{S}'\big]^-\mathbf{B}_2\,,$$

$$\mathbf{K} = \begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{O}, & \mathbf{B}_2' \end{pmatrix} \begin{pmatrix} \mathbf{T}, & \mathbf{X}\mathbf{S}' \\ \mathbf{S}\mathbf{X}', & \mathbf{S}\mathbf{S}' \end{pmatrix}^- \begin{pmatrix} \mathbf{S} \\ \mathbf{G} \end{pmatrix}.$$

P r o o f . It is a direct consequence of the following consideration.

Let two ULSMs

$$\boldsymbol{Y} \sim \left[(\mathbf{X},\mathbf{S})\begin{pmatrix}\beta\\ \gamma\end{pmatrix}, \Sigma\right] \tag{$*$}$$

and

$$\boldsymbol{Y} \sim (\mathbf{X}\beta, \Sigma) \tag{$**$}$$

be considered and let $\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\Sigma + \mathbf{X}\mathbf{X}')$. Then

$$\begin{aligned}
&\operatorname{Var}_*\left[\widehat{(\mathbf{X},\mathbf{S})\begin{pmatrix}\beta\\ \gamma\end{pmatrix}}^*\right] \\
&= (\mathbf{X},\mathbf{S})\left[\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix}\mathbf{T}^-(\mathbf{X},\mathbf{S})\right]^-\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix} - (\mathbf{X}\mathbf{X}' + \mathbf{S}\mathbf{S}') \\
&= (\mathbf{X},\mathbf{S})\begin{pmatrix}\mathbf{D}, & \mathbf{X}'\mathbf{T}^-\mathbf{S}\\ \mathbf{S}'\mathbf{T}^-\mathbf{X}, & \mathbf{S}'\mathbf{T}^-\mathbf{S}\end{pmatrix}^-\begin{pmatrix}\mathbf{X}'\\ \mathbf{S}'\end{pmatrix} - (\mathbf{X}\mathbf{X}' + \mathbf{S}\mathbf{S}') \\
&= \mathbf{X}\mathbf{D}^-\mathbf{X}' - \mathbf{X}\mathbf{X}' + (\mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{T}^- - \mathbf{I})\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\mathbf{T}\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^- \times \\
&\qquad\qquad \times \mathbf{S}'(\mathbf{T}^-\mathbf{X}\mathbf{X}\mathbf{D}^-\mathbf{X}' - \mathbf{I}) - \mathbf{S}\mathbf{S}'
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\ & \operatorname{Var}_*\big(\widehat{\mathbf{M}_{\mathbf{S}}\mathbf{X}\beta}^*\big) = \operatorname{Var}_{**}\big(\widehat{\mathbf{M}_{\mathbf{S}}\mathbf{X}\beta}^{**}\big) \\
&\qquad + \mathbf{M}_{\mathbf{S}}\mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\mathbf{S}\big[\mathbf{S}'(\mathbf{M}_{\mathbf{X}}\mathbf{T}\mathbf{M}_{\mathbf{X}})^+\mathbf{S}\big]^-\mathbf{S}'\mathbf{T}^-\mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{M}_{\mathbf{S}} \\
\Longrightarrow\ & \operatorname{Var}_*\big(\widehat{\mathbf{M}_{\mathbf{S}}\mathbf{X}\beta}^*\big) = \operatorname{Var}_{**}\big(\widehat{\mathbf{M}_{\mathbf{S}}\mathbf{X}\beta}^{**}\big) \\
\Longleftrightarrow\ & \boldsymbol{q}'\mathbf{M}_{\mathbf{S}}\mathbf{X}\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\mathbf{S} = \boldsymbol{O}' \\
\Longleftrightarrow\ & \mathbf{X}'\mathbf{M}_{\mathbf{S}}\boldsymbol{q} \in \mathcal{M}\big(\mathbf{D}\mathbf{M}_{\mathbf{X}'\mathbf{T}^-\mathbf{S}}\big)\,.
\end{aligned}$$

Now $\mathbf{X}$, $\mathbf{S}$, β, γ, Σ are substituted by the following scheme

$$\mathbf{X} \mapsto \begin{pmatrix}\mathbf{X}, & \mathbf{O}\\ \mathbf{B}_1, & \mathbf{B}_2\end{pmatrix}, \qquad \mathbf{S} \mapsto \begin{pmatrix}\mathbf{S}\\ \mathbf{G}\end{pmatrix}, \qquad \gamma \mapsto \gamma, \qquad \Sigma \to \begin{pmatrix}\Sigma, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}$$

and the equality

$$\begin{pmatrix} \mathbf{X}', & \mathbf{B}_1' \\ \mathbf{O}, & \mathbf{B}_2' \end{pmatrix} \begin{pmatrix} \mathbf{T}, & \mathbf{XS}' \\ \mathbf{SX}', & \mathbf{SS}' \end{pmatrix}^{-} \begin{pmatrix} \mathbf{X}, & \mathbf{O} \\ \mathbf{B}_1, & \mathbf{B}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{E}_{1,1}, & \mathbf{E}_{1,2} \\ \mathbf{E}_{2,1}, & \mathbf{E}_{2,2} \end{pmatrix},$$

$$\mathbf{E}_{1,1} = \mathbf{D}^- + \mathbf{D}^-\mathbf{X}'\mathbf{T}^-\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\mathbf{T}\mathbf{M_X})^+\mathbf{S}\big]^-\mathbf{S}'\mathbf{T}^-\mathbf{X}\mathbf{D}^-,$$

$$\mathbf{E}_{1,2} = -\mathbf{D}^-\mathbf{X}'\mathbf{T}^-\mathbf{S}\big[\mathbf{S}'(\mathbf{M_X}\mathbf{T}\mathbf{M_X})^+\mathbf{S}\big]^- = \mathbf{E}_{2,1}'$$

$$\mathbf{E}_{2,2} = \big[\mathbf{S}'(\mathbf{M_X}\mathbf{T}\mathbf{M_X})^+\mathbf{S}\big]^-,$$

are used and the proof can be easily finished. □

5. Numerical example

In 1. Motivation example, we have

$$\mathbf{X} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix} 20 \\ 28 \\ 36 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 4 \\ 10 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} 16 \\ 100 \end{pmatrix}, \quad \mathbf{B}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Further

$$\operatorname{Var}_{**}\left(\widehat{\widehat{\beta_1^{**}}}\right) = \operatorname{Var}_{**}\left(\widehat{\widehat{\beta_2^{**}}}\right) = 0,$$

$$\operatorname{Var}_*\left(\widehat{\widehat{\beta_1^*}}\right) = \sigma^2 1.479, \quad \operatorname{Var}_*\left(\widehat{\widehat{\beta_2^*}}\right) = \sigma^2 12.5, \quad \operatorname{Var}_*\left(\widehat{\widehat{\gamma^*}}\right) = \sigma^2 0.0078,$$

$$A_1 = 14, \quad A_2 = -40.$$

In this case there does not exist a function of the parameters which can be unbiasedly estimated by the same estimators in both models; i.e. the parameter γ cannot be neglected. However, the relationships

$$|\gamma| \le \sigma 0.086861 \implies \left(\forall h_1 \in \mathbb{R}^1\right)\left(|h_1 b_1| \le \sqrt{\operatorname{Var}_*\left(\widehat{\widehat{h_1\beta_1^*}}\right)}\right),$$

$$|\gamma| \le \sigma 0.088388 \implies \left(\forall h_2 \in \mathbb{R}^1\right)\left(|h_2 b_2| \le \sqrt{\operatorname{Var}_*\left(\widehat{\widehat{h_2\beta_2^*}}\right)}\right)$$

are valid. It is to be remarked that $\sigma 0.088388 = \sqrt{\operatorname{Var}\left(\widehat{\widehat{\gamma^*}}\right)}$.

An admissible region around zero for the parameter γ is rather small in this case. However, in another example, where, e.g. $\mathbf{X}'\mathbf{\Sigma}^{-1}\mathbf{S} = \mathbf{O}$, it can be the whole parametric space. Thus the approach used in this paper gives another view on the problem of underestimation than testing hypothesis $\gamma = 0$ does.

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