Donald Marxen

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NEIGHBORHOODS OF THE IDENTITY OF THE FREE ABELIAN TOPOLOGICAL GROUPS

DONALD MARXEN*

1. Introduction. In the theory of discrete groups one means of constructing the free abelian group on a set X yields that group as a quotient of the direct product of the free semigroup (on X) with itself. In this paper (§3) the free abelian uniform group $(AG(X), U_G)$ on a uniform space $[X, U]$ will be constructed as a quotient of the direct product of the free uniform semigroup (on $[X, U]$) with itself. We then observe that if X is a completely regular space and U is its largest admissible uniformity, the uniform topology $\tau(U_G)$ determined by U_G is precisely the topology of the free abelian topological group. The explicit nature of our construction allows us to describe in terms of the gage of U, a base for U_G, hence a base for the neighborhood systems relative to $\tau(U_G)$ (§4).

In section 5 it is shown that no sequence of words in $(AG(X), U_G)$ whose lengths increase without bound can have a limit (Abels [1] has shown this to be true for free topological groups). As corollaries we obtain some familiar results regarding the countability axioms and local compactness on free (abelian) topological groups.

Section 2 contains the definitions, notations and results we need concerning uniform semigroups and free uniform semigroups.

We assume familiarity with the notions of free algebraic semigroup [2, Ch. 1] and free (abelian) topological group (in the sense of Markov [8]).

2. Definitions and notations. The material in this section is taken primarily from [9] and [10].

For a set X let $\Delta(X)$ denote the diagonal of $X \times X$, i. e., $\Delta(X) = \{(x, x) : x \in X\}$. If U is a uniformity on X we denote by $[X, U]$ the uniform space determined by X and U, by $\tau(U)$ the uniform topology relative to U and by $E(U)$ the set of all uniformly continuous pseudometrics on $[X, U]$ bounded by 1. For details concerning the theory of uniform spaces the reader is referred to [7, Ch. 6] and [4, Ch. 15].

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2.1. Definition. *A uniform semigroup is a triple* (T, μ, W), *where*

(ui) $[T, W]$ *is a uniform space; and*

(uii) μ *is an associative, uniformly continuous mapping from* $[T, W] \times [T, W]$ *into* $[T, W]$.

If T *is a group with respect to* μ *then* (T, μ, W) *is called a uniform group.*

We usually shorten (T, μ, W) to (T, W).

2.2. [9, Thm. 3]. If (T, W) is a uniform group, then $t \to t^{-1}$ is uniformly continuous, hence $(T, \tau(W))$ is a topological group.

A uniformly continuous, semigroup homomorphism from one uniform semigroup to another is called a uniform homomorphism. A semigroup isomorphism is called a uniform isomorphism if both it and its inverse are uniformly continuous.

A pseudometric f on a semigroup T is called subinvariant if $f(xz, yz) \leq f(x, y)$ and $f(zx, zy) \leq f(x, y)$ for all x, y and z in T. A base B for a uniformity on T is said to be subinvariant if and only if

$$\Delta(T) W \cup W \Delta(T) \subseteq W$$

for each $W \in B$.

The following theorem is a consequence of Theorem 2 [9].

2.3. Theorem. *For a semigroup* T *and a uniformity* W *on* T, *the following are equivalent:*

(*a*) (T, W) *is a uniform semigroup;*

(*b*) W *has a subinvariant base; and*

(*c*) *the gage of* W *has a base consisting of subinvariant pseudometrics.*

2.4. Definition. *Let* $[X, U]$ *be a uniform space. A uniform semigroup* $(S(X), U_S)$ *is called the free uniform semigroup on* $[X, U]$ *if there exists a mapping* $\eta: [X, U] \to (S(X), U_S)$ *such that*

(*si*) η *is a uniform embedding;*

(*sii*) $\eta[X]$ *generates* $S(X)$ *algebraically; and*

(*siii*) *for any uniform semigroup* (T, W) *and uniformly continuous* $\omega: [X, U] \to (T, W)$, *there exists a (unique) uniform homomorphism* $\Omega: (S(X), U_S) \to (T, W)$ *such that* $\Omega \circ \eta = \omega$.

If in 2.4 'uniform semigroup' is replaced everywhere by 'abelian uniform group' we then have the definition of the free abelian uniform group on $[X, U]$.

The remainder of this section contains notations and results, pertaining to the construction in [10, §3] of the free uniform semigroup, which will be used in §3.

Let $[X, U]$ be a uniform space and $S(X)$ be the free algebraic semigroup on the set X. For $f \in E(U)$ and for nonnegative reals $\delta_1, \ldots, \delta_m$ let $\langle f; \delta_1, \ldots, \delta_m \rangle$ denote the set

$$\{(x_1 \ldots x_m, y_1 \ldots y_m) : f(x_i, y_i) \leq \delta_i, i = 1, \ldots, m\}.$$

Let D denote the set of diadic rationals in the interval $[0,1]$ and let A be the family of functions $\alpha: D \to [0,1]$ satisfying

(ri) $\alpha(r)=0$ if and only if $r=0$;

(rii) $q \leqslant r$ implies $\alpha(q) \leqslant \alpha(r)$; and

(riii) $q+r \leqslant 1$ implies $\alpha(q)+\alpha(r) \leqslant \alpha(q+r)$.

Finally, for $f \in E(U)$, $\alpha \in A$ and $n \in N$ set

$$f[n, \alpha] = \cup \{\langle f; \alpha(r_1), \ldots, \alpha(r_m)\rangle : m \in N, \sum_1^m r_i = 2^{1-n}\}.$$

2.5. *If $g \geqslant f$, $k \geqslant n$ and $\gamma \leqslant \alpha$, then $g[k, \gamma] \subseteq f[n, \alpha]$.*

2.6. *For each f, n, and α, $f[n+1, \alpha]^\circ f[n+1, \alpha] \subseteq f[n, \alpha]$.*

2.7. *For each f, n, and α, $\Delta(S(X))(a, b) \cup (a, b)\Delta(S(X)) \subseteq f[n, \alpha]$ if and only if $(a, b) \in f[n, \alpha]$.*

It follows from 2.5—2.7 that the collection $\{f[n, \alpha] : f \in E(U),\ n \in N$ and $\alpha \in A\}$ is a subinvariant base for a uniformity U_S on $S(X)$. In fact, $(S(X), U_S)$ is the free uniform semigroup on $[X, U]$. The mapping property (siii) of 2.4 is a consequence of 2.8 below.

2.8. [10, 3.3.]. *Let ω be a uniformly continuous mapping from a pseudometric space $[X, f]$ to a pseudometric space $[T, g]$. Then there exists an $\alpha \in A$ satisfying the additional property*

$$(riv) \quad g(\omega(x), \omega(y)) \leqslant r, \quad \text{whenever} \quad f(x, y) \leqslant \alpha(r).$$

3. The free abelian uniform and topological groups. Let $w = w_1 \ldots w_m$ be a word in the free semigroup $S(X)$ on the set X. By a permutation of w we mean a word $u = u_1 \ldots u_m$ in $S(X)$ such that for some permutation s of $\{1,2, \ldots, m\}$, $w_i = u_{\sigma(i)}$ for $i = 1, 2, \ldots, m$. The set of all permutations of w will be denoted by $P(w)$.

Define the congruence relation F on $S(X) \times S(X)$ according to $((a, b), (c, d)) \in F$ if and only if $bc \in P(ad)$ and let $AG(X)$ be the collection of F-equivalence classes in $S(X) \times S(X)$. Letting ϱ denote the natural homomorphism from $S(X) \times S(X)$ onto $AG(X)$, it can easily be shown that

$$\varrho((a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_k)) \to \sum_1^n a_i - \sum_1^k b_i$$

is an isomorphism between $AG(X)$ and $\Sigma\{Z_x : x \in X\}$, whence $AG(X)$ is the free abelian group on the set X.

Let $[X, U]$ be a uniform space and B^2 be the collection of all sets

$$f^2[n, \alpha] = \{((a, b), (c, d)) : (a, c) \in f[n, \alpha] \quad \text{and} \quad (b, d) \in f[n, \alpha]\}$$

$(f \in E(U),\ n \in N,\ \alpha \in A)$. Then B^2 is a base for the product uniformity U_S^2 on $S(X) \times S(X)$. Finally, let U_G denote the image filter $\{(\varrho \times \varrho)[V] : V \in U_S^2\}$. It will now be shown that U_G is a uniformity on $AG(X)$ and that $(AG(X), U_G)$ is the free abelian uniform group on $[X, U]$.

3.1. *The filter U_G is a uniformity on $AG(X)$.*

Proof. It will suffice to show that $\{(\varrho\times\varrho)[V]: V\in B^2\}$ is a base for a uniformity.

(i) That each $(\varrho\times\varrho)[f^2[n,\alpha]]$ contains $\Delta(AG(X))$ is clear.

(ii) Given $f^2[n,\alpha]$ and $g^2[m,\beta]$ in B^2, set $h=\sup\{f,g\}$, $k=\max\{n,m\}$ and $\gamma=\inf\{\alpha,\beta\}$. It is easy to prove that $\gamma\in A$. Moreover, since $h[k,\gamma]\subset f[n,\alpha]\cap g[m,\beta]$ (2.5), it follows that $(\varrho\times\varrho)[h^2[k,\gamma]]\subset(\varrho\times\varrho)[f^2[n,\alpha]]\cap(\varrho\times\varrho)[g^2[m,\beta]]$.

(iii) Let $U=(\varrho\times\varrho)[f^2[n,\alpha]]$ and $V=(\varrho\times\varrho)[f^2[n+1,\alpha]]$. It will be shown that $V\circ V\subseteq U$. Suppose $(\varrho(a,b),\varrho(c,d))$ and $(\varrho(s,t),\varrho(u,v))$ are elements of V and $\varrho(c,d)=\varrho(s,t)$. We can assume that

$$\{(a,c),(b,d),(s,u),(t,v)\}\subseteq f[n+1,\alpha],$$

hence, according to 2.7, that

$$\{(asd,csd),(csd,cud),(bct,dct),(dct,dcv)\}\subseteq f[n+1,\alpha].$$

This implies that $((asd,bct),(cud,dcv))\in f^2[n,\alpha]$. Furthermore, $\varrho(cud,dcv)=\varrho(u,v)$ and, since $ds\in P(ct)$, $\varrho(a,b)=\varrho(asd,bct)$. Thus $(\varrho(a,b),\varrho(u,v))\in U$.

3.2. Theorem. *For a uniform space $[X, U]$, $(AG(X), U_G)$ is the free uniform group on* [X, U].

Proof. The collection B^2 is a subinvariant base for U_S^2, therefore $\{(\varrho\times\varrho)[V]: V\in B^2\}$ is a subinvariant base for U_G. According to 2.3, $(AG(X), U_G)$ is a uniform group.

Let x be fixed in X and define $\eta: X\to AG(X)$ by $\eta(x)=\varrho(xx, x)$. The mapping η is clearly a uniformly continuous injection and $\eta[X]$ generates $AG(X)$ algebraically. The uniform continuity of the inverse of η will now be shown. For $f\in E(U)$ and $\varepsilon>0$ set $V=\{(x,y): f(x,y)\leqslant\varepsilon\}$. Choosing $\alpha\in A$ such that $\alpha(1)\leqslant\varepsilon$, set

$$U=(\varrho\times\varrho)[f^2[2,\alpha]]\cap(\eta[X]\times\eta[X]).$$

If $(\eta(x),\eta(y))\in U$ there exists $((a,b),(c,d))\in f^2[2,\alpha]$ such that $\varrho(a,b)=\varrho(xx, x)$ and $\varrho(c,d)=\varrho(yx, x)$, whence $bx\in P(a)$ and $dy\in P(c)$. Therefore $(bx,(dy)')\in f[2,\alpha]$ for some $(dy)'\in P(dy)$. Since (d,b) is also in $f[2,\alpha]$, $(dbx, b(dy)')\in f[1,\alpha]$. It follows from 4.3 that $(x,y)\in f[1,\alpha]$, hence $(x,y)\in V$. We conclude that η is a uniform embedding.

The mapping property (*siii*) remains to be shown. Let (T, W) be an abelian uniform group, $\omega: [X, U]\to(T, W)$ be uniformly continuous and $\Omega: AG(X)\to T$ be the group homomorphism satisfying $\Omega\circ\eta=\omega$. For each subinvariant pseudometric g in $E(W)$ there exists some $f \in E(U)$ such that ω is a uniformly continuous mapping from the pseudometric space $[X, f]$ to the pseudometric space $[T, g]$. Now select $\alpha \in A$ satisfying (riv) of 2.8. Letting $U = (\varrho \times \varrho)[f^2[n, \alpha]]$ and $W = \{(s, t) : s, t \in T \text{ and } g(s, t) \leqslant 2^{-n}\}$, it will be shown that $(\Omega \times \Omega)[U] \subseteq W$, thus implying the uniform continuity of Ω.

Suppose $(\varrho(a, b), \varrho(c, d)) \in U$ with (a, c) and (b, d) in $f[n, \alpha]$ and suppose that a and c have the length m and that b and d have the length k. Then for some choice of diadic rationals $r_1, \ldots, r_m$ and $q_1, \ldots, q_k$ for which $\sum_1^m r_i = \sum_1^k q_i = 2^{-1-n}$, $f(a_i, c_i) \leqslant \alpha(r_i)$ and $f(b_j, d_j) \leqslant \alpha(q_j)$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$. Observing that

$$\Omega\varrho(a, b) = \omega(a_1) + \ldots + \omega(a_m) - \omega(b_1) - \ldots - \omega(b_k)$$

and

$$\Omega\varrho(c, d) = \omega(c_1) + \ldots + \omega(c_m) - \omega(d_1) - \ldots - \omega(d_k),$$

we conclude that

$$\begin{aligned} g(\Omega\varrho(a, b), \Omega\varrho(c, d)) &\leqslant \sum_1^m g(\omega(a_i), \omega(c_i)) + \sum_1^k g(\omega(b_i), \omega(d_i)) \\ &\leqslant \sum_1^m r_i + \sum_1^k q_i. \\ &= 2^{-n}. \end{aligned}$$

3.3. Theorem. *Let X be a completely regular space and let U be the largest admissible uniformity on X. Then $(AG(X), \tau(U_G))$ is the free abelian topological group on X.*

Proof. Let T be an abelian topological group and let W be its right (= left) uniformity. If ω is a continuous mapping from X to T, it is a uniformly continuous mapping from $[X, U]$ to $[T, W]$ [4, 15G5, p. 234]. Now let $\Omega : (AG(X), U_G) \to (T, W)$ denote the uniform homomorphism satisfying $\Omega \circ \eta = \omega$. Since ω must be continuous as a mapping from $(AG(X), \tau(U_G))$ to T, the proof is complete.

3.4. Remark. Free uniform semigroups can also be used to topologize the free group $G(X)$ on a completely regular space X. Let U be an admissible uniformity on X, $[X', U']$ be a uniformly isomorphic copy of $[X, U]$, and $(S(X \cup X'), W_S)$ be the free uniform semigroup on the disjoint union of $[X, U]$ and $[X', U']$. If $\psi : S(X \cup X') \to G(X)$ is the natural homomorphism, then $V = \{(\psi \times \psi)[W] : W \in W_S\}$ is a uniformity on $G(X)$ having a subinvariant base. Thus $(G(X), \tau(V))$ is a topological group. In general, $\tau(V)$ is too small to be the topology of the free topological group [11], even if U is the largest admissible structure on X.

3.5. Question. For a uniform space $[X, U]$, $(S(X), \tau(U_S))$ is known to be the free topological semigroup on the space $[X, \tau(U)]$ 10, 4.2].Since $(AG(X), U_G)$ is a uniform quotient of $(S(X)\times S(X), U_S^2)$, it is natural to ask if the free abelian topological group is a topological quotient of the direct product of the free topological semigroup with itself. The situation is complicated by the fact that the topology of a quotient uniformity need not be the quotient of the uniform topology [6, 5(a), p. 32].

4. A base for the neighborhood system of e.

The identity element of $AG(X)$ will be denoted by e or $a-a$ for any $a \in S(X)$. In place of the nonzero elements $\varrho(ab, a)$ and $\varrho(a, ab)$ we will write b and $-b$ respectively. If $a=b$ or if (a, b) is a reduced pair, i. e., if a and b have no common letter, we will write $a-b$ in place of $\varrho(a, b)$. The elements b and $-b$ will be called positive and negative, respectively, and for the element $a-b$ we will refer to a as the positive part and b as the negative part.

Let $N(e)$ denote the neighborhood system of e relative to the topology $\tau(U_G)$. Since the collection $\{(\varrho\times\varrho)[V] : V \in B^2\}$ is a base for the free uniformity U_G (see the proof of 3.1), it determines a base for $N(e)$. Associated with the entourage $(\varrho\times\varrho)[f^2[n, \alpha]]$ is the following neighborhood of e:

$$\{a-b : \varrho(a, b)=\varrho(c, d) \quad \text{and} \quad (c, u), (d, u) \in f[n, \alpha] \quad \text{for some} \quad c, d, u \in S(X)\}.$$

In order that $a-b$ be an element of this set, the pair (a, b) must satisfy conditions involving words other than a and b. In this section we provide another base for $N(e)$, where the condition for the membership of $a-b$ in a given neighborhood (in the base) involves only a and $P(b)$, the set of permutations of b. It will be helpful to first establish several additional properties of the sets $f[n, \alpha]$ ($f \in E(U)$, $n \in N$, $\alpha \in A$). The first of these follows directly from the definition of $f[n, \alpha]$.

4.1. If $(a_1 \dots a_m, b_1 \dots b_m) \in f[n, \alpha]$, then $(a_{\sigma(1)} \dots a_{\sigma(m)}, b_{\sigma(1)} \dots b_{\sigma(m)}) \in f[n, \alpha]$ for every permutation σ of $\{1, \dots, m\}$.

4.2. If $(ca, (cb)') \in f[n, \alpha]$ where $(cb)' \in P(cb)$, then $(a, b') \in f[n, \alpha]$ for some $b' \in P(b)$.

Proof. It will be sufficient to prove this for c a word of length one. Set $u=ca$ and $v=(cb)'$ and suppose $v_j=c$. If $j=1$, the result follows from 2.7. Assume $j\neq 1$. If m is the length of u and v, there exist $r_1, \dots, r_m \in D$ such that $\sum_1^m r_i = 2^{1-n}$ and $f(u_i, v_i)\leq\alpha(r_i)$, $1\leq i\leq m$. Letting σ denote the permutation $(1, j)$ of $\{1, \dots, m\}$, we observe that

$$\begin{aligned} f(u_j, v_{\sigma(j)}) = f(u_j, v_1) &\leq f(u_j, v_j) + f(u_1, v_1) \\ &\leq \alpha(r_j) + \alpha(r_1) \\ &\leq \alpha(r_j + r_1). \end{aligned}$$

Thus $(u_2 \dots u_m, v_{\sigma(2)} \dots v_{\sigma(m)}) \in f[n, \alpha]$.

4.3. Let (a, b) be a reduced pair in $S(X) \times S(X)$. If $(s, t) \in f[n, \alpha]$ and $\varrho(s, t) = \varrho(a, b)$, then $(a, b') \in f[n, \alpha]$ for some $b' \in P(b)$.

Proof. Since $bs \in P(at)$ and a and b have no common letters, either (*i*) $s \in P(a)$ and $t \in P(b)$ or (*ii*) there exist words c and d in $S(X)$ such that $s \in P(ca)$ and $t \in P(db)$. If (*i*) is true, the result becomes obvious. Suppose that (*ii*) holds. Then $bca \in P(adb)$, hence $c \in P(d)$ and $s \in P(da)$. Using 4.1 and the symmetry of $f[n, \alpha]$, we conclude that $(da, (db)') \in f[n, \alpha]$ for some $(db)' \in P(db)$. The result now follows from 4.2.

For each $f \in E(U)$, $n \in N$ and $\alpha \in A$ set

$$B(f, n, \alpha) = \{a - b : (a, b') \in f[n, \alpha] \quad \text{for some} \quad b' \in P(b)\}$$

and let M be the collection of all such sets.

4.4. Theorem. *The collection M is a base for the neighborhood system of $e \in (AG(X), U_G)$.*

Proof. Since $B(f, n, \alpha) \subseteq \{a - b : (e, a - b) \in (\varrho \times \varrho)[f^2[n, \alpha]]\}$, we have only to show that each member of M is a neighborhood of e. Consider the set $M \in N(e)$, where $M = \{a - b : (e, a - b) \in (\varrho \times \varrho)[f^2[n+1, \alpha]]\}$. If $a - b \in M$, there exist words c, d and u in $S(X)$ such that $\varrho(c, d) = \varrho(a, b)$ and $((c, d), (u, u)) \in f^2[n+1, \alpha]$. But then $(c, d) \in f[n, \alpha]$ (2.6) and, by 4.3, $(a, b') \in f[n, \alpha]$ for some $b' \in P(b)$. Thus $B(f, n, \alpha)$ contains the neighborhood M.

4.5. Remark. Recall that a and b' must be of equal length in order that (a, b') be in $f[n, \alpha]$. Thus a condition necessary (but not sufficient) for a nonzero word $a - b$ to be an element of $B(f, n, \alpha)$ is that its positive and negative parts have an equal length.

4.6. *If $[X, U]$ is a Hausdorff uniform space, then $(AG(X), U_G)$ is Hausdorff.*

Proof. Suppose $a - b$ is a nonzero word in $AG(X)$ and that a and b have the equal length m. Since a and b have no common letter, $f(a_1, b_i) \neq 0$, $1 \leq i \leq m$, for some $f \in E(U)$. If $\alpha : D \to [0,1]$ represents the inclusion mapping and $n \in N$ satisfies $2^{1-n} < \{\min f(a_1, b_i) : i = 1, \dots, m\}$, then $(a, b) \notin B(f, n, \alpha)$.

According to 3.1 the free uniformity U_G is the image filter of U_S^2 under the mapping $\varrho \times \varrho$. Consequently each base for U_S^2 is carried by $\varrho \times \varrho$ onto a base for U_G. In addition to B^2 we will consider one other base for U_S^2.

Let $A_\circ$ be the collection of all monotone increasing functions $\beta : D \to [0,1]$ such that $\beta(r) = 0$ if and only if $r = 0$. For $f \in E(U)$ and $\beta \in A_\circ$ define $f[1, \beta]$ as in §2 and set $f^2[1, \beta] = \{((a, b), (c, d)) : (a, c), (b, d) \in f[1, \beta]\}$. Let $B^2 = \{f^2[1, \beta] : f \in E(U) \text{ and } \beta \in A\}$.

4.7. *The collection B^2 is a base for U_G^2.*

Proof. Given $\beta \in A$, $\alpha : r \to r\beta(r)$ is an element of A and $\alpha < \beta$. Therefore $f^2[1, \alpha] \subseteq f^2[1, \beta]$. It follows that $B^2 \subseteq U_G^2$. Conversely, if $\alpha \in A$ and $n \in N$ are given, define β by $\beta(r) = \alpha(2^{-n}r)$ for each $r \in D$. Then $f^2[1, \beta] \subset f^2[n, \alpha]$, whence B^2 generates a uniformity finer than U_G^2.

4.8. The collection M of all sets $\{a - b : (a, b') \in f[1, \beta]$ for some $b' \in P(b)\}$ $(f \in E(U),\ \beta \in A)$ is a base for the neighborhood system of $e \in (AG(X), U_G)$

5. Properties equivalent to discreteness in $(AG(X), U_G)$. For the remainder of this paper $[X, U]$ will denote a Hausdorff uniform space.

In this section it is shown that no sequence of words in $(AG(X), U_G)$ whose lengths increase without bound can have a limit (Abels [1] has shown this to be a property of the free topological group). Using this theorem we obtain some familiar results concerning the countability axioms and local compactness on the free (abelian) topological group.

The length of a word w in $AG(X)$ will be denoted by $|w|$.

5.1. *Let $\{s(n)\}$ be a sequence in $(AG(X), U_G)$ such that $n \leq |s(n)| < |s(n+1)|$ for all $n \in N$. Then there exists a neighborhood M of e for which $s(n) \notin M$ for all n.*

Proof. According to 4.5 we may assume that for each n, $|a(n)| = |b(n)|$, where $a(n)$ and $b(n)$ denote, respectively, the positive and negative parts of $s(n)$. For $n \in N$ let m_n indicate the length of $|a(n)|$, and let Y be the set of all letters appearing in the words $s(n)$, i.e., let $Y = \{x : x = s(n)_i$ for some $n \in N$ and $1 \leq i \leq |s(n)|\}$. Since $[X, U]$ is Hausdorff and Y is countable, there exists an $f \in E(U)$ for which $f(x, y) \neq 0$ for all distinct points x and y in Y. Setting $\varepsilon_{m_n} = \min\{f(a(n)_i, b(n)_j) : i, j = 1, \ldots, m_n\}$, we observe that $\varepsilon_{m_n} > 0$. Now select $\beta \in A$ such that $\beta(2^{-k}) < \varepsilon_{m_n}$ for the finitely many m_n satisfying

$$2^{-k-1} < m_n^{-1} \leq 2^{-k}.$$

Such a β is easily shown to exist. Finally, set $M = \{a - b : (a, b') \in f[1, \beta]$ for some $b' \in P(b)\}$ (4.8). If $n \in N$ and $r_1, \ldots, r_{m_n} \in D$ with $\sum_1^{m_n} r_i = 1$, then $r_j \leq m_n^{-1}$ for some $j \leq m_n$. Thus if

$$2^{-k-1} < m_n^{-1} \leq 2^{-k},$$

$\beta(r_j) \leq \beta(2^{-k}) < \varepsilon_{m_n}$, whence

$$f(a(n)_i, b'(n)_i) \not< \beta(r_i)$$

for every permutation $b'(n)$ of $b(n)$. Consequently $a(n)-b(n) \notin M$.

5.2. Theorem. *Let $\{w(n)\}$ be a sequence in $(AG(X), U_G)$ for which $\{|w(n)| : n \in N\}$ is unbounded in N. Then $\{w(n)\}$ fails to converge.*

Proof. Let u be any element of $AG(X)$ and define $v: N \to AG(X)$ by $v(n) = w(n) - u$ for all n. Then $\{|v(n)| : n \in N\}$ is unbounded in N and thus $\{v(n)\}$ has a subsequence $\{s(n)\}$ satisfying the hypothesis of 5.1. It follows that $\{v(n)\}$ does not converge to e, hence $\{w(n)\}$ does not converge to u.

5.3. *If $[X, U]$ is not discrete, then every neighborhood of e in $(AG(X), U_G)$ contains a sequence $\{w(n)\}$ for which $\{|w(n)| : n \in N\}$ is strictly increasing.*

Proof. Consider the neighborhood $B(f, m, \alpha) \in M$ (4.4) and let x be a nonisolated point in X. If for each $n \in N$ we select $x_n \in X$ satisfying

$$0 < f(x_n, x) < \alpha(2^{1\ \ m-n}),$$

then $(2^n x_n,\ 2^n x) \in f[m, \alpha]$. Setting $w(n) = 2^n x_n - 2^n x$, we observe that $w(n) \in B(f, m, \alpha)$ for all n.

The following theorem is an immediate consequence of 5.3 and 5.2.

5.4. Theorem. *If the uniform space* [X, U] *is Hausdorff, the following properties are equivalent in $(AG(X), U_G)$:*

(*i*) *discreteness*

(*ii*) *1st countability*

(*iii*) *metri: ability*

(*iv*) *local compactness.*

In particular, if $(AG(X), U_G)$ is 2nd countable, it must be discrete.

5.5. *Let X be a completely regular T_1-space. In the free (abelian) topological group on X, the properties (i)—(iv) of 5.4 are equivalent.*

Proof. This result follows from 3.3 and the fact that each of these properties is preserved under open-continuous homomorphisms.

The equivalence of (*i*), (*ii*) and (*iii*) in the free (abelian) topological group was established by Graev [5]. The equivalence of (*i*) and (*iv*) follows from a stronger result, which is due to Dudley [3, p. 589].

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Marquette University
Milwaukee, Wisconsin 53233

BOOK REVIEWS

J. Farkas and M. Farkas: INTRODUCTION TO LINEAR ALGEBRA. Akadémiai Kiado, Budapest 1975. 205 strán.

V poslednom čase sa veľa pozornosti venuje metodike vyučovania matematiky. Je iste správne, ak je úsilie naučiť študenta myslieť prostredníctvom moderných matematických pojmov a metód. Veľakrat však študenti ťažko prijímajú modernú matematiku, lebo im chýba príprava, ktorá by im umožnila chápať moderný matematický jazyk prirodzene a nielen ako systém axióm a pod. Autori tejto knihy sa pokúsili napísať knihu, ktorá by umožnila študentom prvých semestrov vysokých škôl pripraviť sa na štúdium modernej algebry a myslím, že sa im to aj podarilo.

Kniha je rozdelená do šiestich kapitol:

V kapitole I sú vysvetlené základy vektorovej algebry a na konci tejto kapitoly sú uvedené príklady aplikácie vektorov v analytickej geometrii a v mechanike.

Kapitola II je venovaná komplexným číslam.

V kapitole III sú vysvetlené základy maticovej algebry a teórie determinantov.

Kapitola IV je venovaná systémom lineárnych algebraických rovníc. Ako príklad aplikácie teórie systémov algebraických rovníc je uvedený základný problém lineárneho programovania.

V kapitole V sú uvedené definície grupy, okruhu, telesa, vektorového priestoru nad telesom, bázy a transformácie bázy.

V kapitole VI sú vysvetlené základy teórie lineárnych operátorov a kvadratických foriem.

Na konci každej kapitoly sú cvičenia. Odpovede a návody na riešenie týchto cvičení sú uvedené na konci knihy.

Kniha bude cennou pomôckou pre poslucháčov matematiky na vysokých školách univerzitneho aj technického smeru.

Milan Medveď, Bratislava