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ALGEBRAIC STRUCTURES GENERATED BY ALMOST CONTINUOUS FUNCTIONS

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ABSTRACT. There are investigated the group, the lattice and the Baire system generated by the family of almost continuous in the Husain sense functions.

I. Preliminaries

Let us establish some of the terminology to be used. $\mathbf{R}$ denotes the real line. Let $(X,\mathcal{T})$ be a topological space. A function $f\colon X \to \mathbf{R}$ is said to be $\mathcal{T}$ *almost continuous (in the Husain sense) at a point* $x_0 \in X$ iff for every $\varepsilon > 0$, $x_0 \in \operatorname{Int}\Big(\operatorname{Cl}\big(f^{-1}(f(x_0)-\varepsilon, f(x_0)+\varepsilon)\big)\Big)$, where Cl denotes the closure operation (in the topology $\mathcal{T}$) and Int — the interior operation, respectively ([2]). If $\mathfrak{K}$ is a family of functions $f\colon X \to \mathbf{R}$, then

(i) $G(\mathfrak{K})$ denotes the *group generated by* $\mathfrak{K}$, i.m. the least family for which $\mathfrak{K} \subset G(\mathfrak{K})$ and $f+g \in G(\mathfrak{K})$ for any f, $g \in G(\mathfrak{K})$;

(ii) $B(\mathfrak{K})$ denotes the *collection of all pointwise limits of sequences taken from* $\mathfrak{K}$;

(iii) $L(\mathfrak{K})$ denotes the *lattice generated by* $\mathfrak{K}$, i.e. the least family for which $\mathfrak{K} \subset L(\mathfrak{K})$ and $\max(f,g) \in L(\mathfrak{K})$ and $\min(f,g) \in L(\mathfrak{K})$ for any f, $g \in L(\mathfrak{K})$.

Let $(w_n)_{n=0}^{\infty}$ be an enumeration of all rationals.

Denote by $\mathfrak{C}_H$ the *family of all* $\mathcal{T}$ *almost continuous (in the Husain sense) functions* $f\colon X \to \mathbf{R}$ and by $\mathfrak{M}_1$ the family of all $\mathcal{M}$ measurable functions, where $\mathcal{M}$ is a σ-field of subsets of X.

II. General theorems

Theorem 1. *Suppose that the topological space* $(X,\mathcal{T})$ *is such that*

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(1) *there is a sequence* $(A_n)_{n=1}^{\infty}$ *of pairwise disjoint sets from* $\mathcal{M}$ *with* $\operatorname{Cl} A_n = \operatorname{Int}\operatorname{Cl} A_n \supset X'$ *for* $n = 1,2,\ldots$, *where* X' *denotes the set of all accumulation points of* X.
Then for each $\mathcal{M}$ *measurable function* $f\colon X \to \mathbf{R}$ *there are two* $\mathcal{T}$ *almost continuous,* $\mathcal{M}$ *measurable functions* $f_1, f_2\colon X \to \mathbf{R}$ *such that* $f = f_1 + f_2$.

P r o o f. Let us put

$$f_1(x) = \begin{cases} f(x) & \text{for } x \in X - \bigcup_{n=1}^{\infty} A_n \\ w_n & \text{for } x \in A_{2n},\ n = 1,2,\ldots \\ f(x) - w_n & \text{for } x \in A_{2n-1},\ n = 1,2,\ldots \end{cases}$$

and

$$f_2(x) = \begin{cases} 0 & \text{for } x \in X - \bigcup_{n=1}^{\infty} A_n \\ f(x) - w_n & \text{for } x \in A_{2n},\ n = 1,2,\ldots \\ w_n & \text{for } x \in A_{2n-1},\ n = 1,2,\ldots\ . \end{cases}$$

It is clear that $f = f_1 + f_2$ and f_1, f_2 are $\mathcal{M}$ measurable. Fix $x_0 \in X$ and $\varepsilon > 0$. If $x_0 \notin X'$, then the functions f_1, f_2 are $\mathcal{T}$ continuous at x_0, hence also $\mathcal{T}$ almost continuous. If $x_0 \in X' - \bigcup_{n=1}^{\infty} A_n$, then there is w_{n_0} such that $|f(x_0) - w_{n_0}| < \varepsilon$. Since $x_0 \in \operatorname{Int}\operatorname{Cl} A_{2n_0}$, the function f_1 is $\mathcal{T}$ almost continuous at x_0. There is also n_1 such that $|w_{n_1}| < \varepsilon$. Since $x_0 \in \operatorname{Int}\operatorname{Cl} A_{2n_1-1}$, f_2 is $\mathcal{T}$ almost continuous at x_0. If $x_0 \in X' \cap \bigcup_{n=1}^{\infty} A_n$, then there is n_2 such that $x_0 \in A_{n_2}$. The function $f_1|A_{n_2}$ ($f_2|A_{n_2}$) is constant for even (odd) n_2 and $x_0 \in \operatorname{Int}\operatorname{Cl} A_{n_2}$, so in this case f_1 (f_2) is $\mathcal{T}$ almost continuous at x_0. If $f_1(x_0) = f(x_0) - w_{n_3}$ ($f_2(x_0) = f(x_0) - w_{n_3}$), then there is n_4 such that $|f(x_0) - w_{n_3} - w_{n_4}| < \varepsilon$. Because $x_0 \in \operatorname{Int}\operatorname{Cl} A_{2n_4}$ ($x_0 \in \operatorname{Int}\operatorname{Cl} A_{2n_4-1}$) and $|f(x_0) - w_{n_3} - w_{n_4}| = |f_i(x_0) - f_i(x)| < \varepsilon$ ($i = 1,2$) for $x \in A_{2n_4}$ ($x \in A_{2n_4-1}$), so f_1 (f_2) is $\mathcal{T}$ almost continuous at x_0. □

Theorem 2. *Assume the hypothesis* (1) *from Theorem 1. For each* $\mathcal{M}$ *measurable function* $f\colon X \to \mathbf{R}$ *there are four* $\mathcal{T}$ *almost continuous and* $\mathcal{M}$ *measurable functions* $f_1, f_2, f_3, f_4\colon X \to \mathbf{R}$ *such that*

(2) $f = \min(\max(f_1, f_2), \max(f_3, f_4))$.

P r o o f. For $i = 1,2,3,4$, let us put

$$f_i(x) = \begin{cases} w_n & \text{for } x \in A_{4n+i},\ n = 0,1,\ldots \\ f(x) & \text{for } x \notin \bigcup_{n=0}^{\infty} A_{4n+i}. \end{cases}$$

Likewise as in the proof of Theorem 1 we prove that the function f_i ($i = 1,2,3,4$) are $\mathcal{T}$ almost continuous and $\mathcal{M}$ measurable.

Now we will prove that (2) holds. Fix $x \in X$. If $x \notin \bigcup_{n=1}^{\infty} A_n = \bigcup_{i=1}^{4} \bigcup_{N=0}^{\infty} A_{4N+i}$, then $f_1(x) = f_2(x) = f_3(x) = f_4(x) = f(x)$ and (2) holds. If $x \in \bigcup_{n=1}^{\infty} A_n$, then there are $i_0 \leq 4$ and n_0 such that $x \in A_{4n_0+i_0}$ and $x \notin A_n$ for $n \neq 4n_0+i_0$. So we have $f_{i_0}(x) = w_{n_0}$ and $f_i(x) = f(x)$ for $i \neq i_0$ ($i = 1,2,3,4$). Consequently, $\max(f_1(x), f_2(x)) \geq f(x)$, $\max(f_3(x), f_4(x)) \geq f(x)$ and $\max(f_1(x), f_2(x)) = f(x)$ or $\max(f_3(x), f_4(x)) = f(x)$. Thus (2) holds. □

Corollary 1. *If the space $(X, \mathcal{T})$ and the σ-field $\mathcal{M}$ fulfil the condition* (1) *from Theorem 1, then*

$$G(\mathfrak{C}_H \cap \mathfrak{M}_1) = L(\mathfrak{C}_H \cap \mathfrak{M}_1) = \mathfrak{M}_1 .$$

Theorem 3. *If $(X, \mathcal{T})$ and $\mathcal{M}$ fulfil the condition* (1) *from Theorem 1, then for each $\mathcal{M}$ measurable function $f\colon X \to \mathbf{R}$ there is a sequence of $\mathcal{T}$ almost continuous and $\mathcal{M}$ measurable functions $f_k\colon X \to \mathbf{R}$ such that $f = \lim_{k\to\infty} f_k$.*

Proof. Let us define the functions f_k ($k = 1,2,\dots$) in the following way:

$$f_k(x) = \begin{cases} w_n & \text{for } x \in A_n,\ n \geq k \\ f(x) & \text{for } x \in X - \bigcup_{n\geq k} A_n . \end{cases}$$

Similarly as in the proof of Theorem 1 we can prove that the functions f_k ($k = 1,2,\dots$) are $\mathcal{T}$ almost continuous and $\mathcal{M}$ measurable. For every $x \in X$ there is k_0 such that $x \notin A_n$ for $n \geq k_0$. So for $k \geq k_0$ we have $f_k(x) = f(x)$ and consequently $f(x) = \lim_{k\to\infty} f_k(x)$. □

Corollary 2. *If $(X, \mathcal{T})$ and $\mathcal{M}$ fulfil* (1), *then $B(\mathfrak{M}_1 \cap \mathfrak{C}_H) = \mathfrak{M}_1$.*

Example 1. Let $X = \{a,b,c\}$ and $\mathcal{T} = \{\emptyset, X, \{a\}, \{a,b\}, \{a,c\}\}$. Then $f \in \mathfrak{C}_H$ iff f is constant. Consequently, if $\mathcal{M} = 2^X$, then

$$\mathfrak{C}_H = G(\mathfrak{C}_H) = L(\mathfrak{C}_H) = B(\mathfrak{C}_H) \neq \mathfrak{M}_1 = \mathbf{R}^X .$$

III. The case of the Euclidean topology

If $X = \mathbf{R}$, $\mathcal{T}$ is the *Euclidean topology* in $\mathbf{R}$ and $\mathcal{M}$ is a *σ-field containing all denumerable sets*, then evidently Theorems 1, 2, 3 hold. In the considered case we can prove some more special versions of these theorems.

A function $f\colon \mathbf{R} \to \mathbf{R}$ is said to be *almost continuous in the Stallings sense (A_K almost continuous)* iff for every open set $V \subset \mathbf{R}^2$ containing the graph $\mathbf{G}(f)$ of the function f there exists a continuous function $g\colon \mathbf{R} \to \mathbf{R}$ such that $\mathbf{G}(g) \subset V$ ([3]). A set $W \subset \mathbf{R}^2$ is said to be a *blocking set for a function* f iff W is closed, $\mathbf{G}(f) \cap W = \emptyset$ and $W \cap \mathbf{G}(g) \neq \emptyset$ for every continuous function $g\colon \mathbf{R} \to \mathbf{R}$. A blocking set W is a *minimal blocking set for* f iff for every blocking set V for f we have $W \subset V$. A minimal blocking set W for a function $f\colon \mathbf{R} \to \mathbf{R}$ is closed and its projection $\operatorname{Pr} W$ on the axis OX is a closed nondegenerate interval. A function $f\colon \mathbf{R} \to \mathbf{R}$ is A_K *almost continuous* iff there is not any blocking set for f ([3]).

Denote by $\mathfrak{A}_K$ the family of all A_K almost continuous functions $f\colon \mathbf{R} \to \mathbf{R}$. Every function $f \in \mathfrak{A}_K$ has the Darboux property, but there are Darboux functions $f\colon \mathbf{R} \to \mathbf{R}$ which are not in $\mathfrak{A}_K$ ([3]).

In the following theorems 4, 5,6 we suppose that

(3) $X = \mathbf{R}$, $\mathcal{T}$ is the Euclidean topology in $\mathbf{R}$ and $\mathcal{M}$ is a σ-field of subsets of $\mathbf{R}$ such that all denumerable sets are in $\mathcal{M}$ and there exists a set $B \subset \mathbf{R}$ with $\operatorname{Cl}(\mathbf{R} - B) = \mathbf{R}$, $2^B \subset \mathcal{M}$ and $B \cap I$ is of the continuum power for every open interval $I \subset \mathbf{R}$.

Theorem 4. *If the condition* (3) *holds, then every* $\mathcal{M}$ *measurable function* $f\colon \mathbf{R} \to \mathbf{R}$ *is the sum of two* $\mathcal{M}$ *measurable functions* $f_1, f_2 \in \mathfrak{C}_H \cap \mathfrak{A}_K$.

Proof. Let $W_1, \ldots, W_\alpha, \ldots$, ($\alpha < \omega_1$ and ω_1 denotes the first ordinal number of the continuum power) be a transfinite sequence of all minimal blocking sets in $\mathbf{R}^2$.

Let us fix two distinct points $x_{1,1}, x_{1,2} \in B \cap \operatorname{Pr} W_1$. If $1 < \alpha < \omega_1$ then we choose two distinct points $x_{\alpha,1}, x_{\alpha,2} \in B \cap \operatorname{Pr} W_\alpha$ such that

$$x_{\alpha,1}, x_{\alpha,2} \neq x_{\beta,1}, x_{\beta,2} \quad \text{for} \quad \beta < \alpha .$$

For each point $x_{\alpha,i}$, $\alpha < \omega_1$, $i = 1,2$, we choose some $y_{\alpha,i}$ such that $(x_{\alpha,i}, y_{\alpha,i}) \in W_\alpha$. Let $(A_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint denumerable dense sets contained in $\mathbf{R} - B$. Define

$$f_1(x) = \begin{cases} w_n & \text{for} \quad x \in A_{2n},\ n = 1,2,\ldots \\ f(x) - w_n & \text{for} \quad x \in A_{2n-1},\ n = 1,2,\ldots \\ y_{\alpha,1} & \text{for} \quad x = x_{\alpha,1},\ \alpha < \omega_1 \\ f(x) - y_{\alpha,2} & \text{for} \quad x = x_{\alpha,2},\ \alpha < \omega_1 \\ f(x) & \text{in the remaining cases,} \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x) - w_n & \text{for} \quad x \in A_{2n},\ n = 1,2,\dots \\ w_n & \text{for} \quad x \in A_{2n-1},\ n = 1,2,\dots \\ f(x) - y_{\alpha,1} & \text{for} \quad x = x_{\alpha,1},\ \alpha < \omega_1 \\ y_{\alpha,2} & \text{for} \quad x = x_{\alpha,2},\ \alpha < \omega_1 \\ 0 & \text{in the remaining cases}. \end{cases}$$

Evidently $f = f_1 + f_2$. We can also prove likewise as in the proof of Theorem 1 that $f_1, f_2 \in \mathfrak{C}_H$ and are $\mathcal{M}$ measurable. Since the graphs $\mathbf{G}(f_1), \mathbf{G}(f_2)$ intersect every blocking set W_α ($\alpha < \omega_1$), so $f_1, f_2 \in \mathfrak{A}_K$. □

Corollary 3. *If* (3) *holds, then* $G(\mathfrak{M}_1 \cap \mathfrak{C}_H \cap \mathfrak{A}_K) = \mathfrak{M}_1$.

Theorem 5. *Suppose that* (3) *holds. Then for every* $\mathcal{M}$ *measurable function* $f\colon \mathbf{R} \to \mathbf{R}$ *there exist* $\mathcal{M}$ *measurable functions* $f_1, f_2, f_3, f_4 \in \mathfrak{C}_H \cap \mathfrak{A}_K$ *such that* $f = \min(\max(f_1, f_2), \max(f_3, f_4))$.

Proof. As in the proof of Theorem 4 we choose points $(x_{\alpha,i}, y_{\alpha,i}) \in W_\alpha \cap (B \times \mathbf{R})$ ($\alpha < \omega_1$; $i = 1,2,3,4$) such that $x_{\alpha_1,i_1} \neq x_{\alpha_2,i_2}$ if $(\alpha_1, i_1) \neq (\alpha_2, i_2)$ ($\alpha_1, \alpha_2 < \omega_1$ and $i_1, i_2 = 1,2,3,4$). Let $(A_n)_{n=1}^{\infty}$ be the same as in the proof of Theorem 4. Define, for $i = 1,2,3,4$,

$$f_i(x) = \begin{cases} w_n & \text{for} \quad x \in A_{4n+i},\ n = 0,1,\dots \\ y_{\alpha,i} & \text{for} \quad x = x_{\alpha,i},\ \alpha < \omega_1 \\ f(x) & \text{in the remaining case}. \end{cases}$$

As in the proof of Theorem 2 we verify that $f_1, f_2, f_3, f_4 \in \mathfrak{M}_1 \cap \mathfrak{C}_H$ and $f = \min(\max(f_1, f_2), \max(f_3, f_4))$. Since the graphs $\mathbf{G}(f_i)$ ($i = 1,2,3,4$) intersect all the blocking sets W_α ($\alpha < \omega_1$), so $f_1, f_2, f_3, f_4 \in \mathfrak{A}_K$. □

Corollary 4. *If* (3) *holds, then* $L(\mathfrak{M}_1 \cap \mathfrak{C}_H \cap \mathfrak{A}_K) = \mathfrak{M}_1$.

Theorem 6. *If* (3) *holds, then for every* $\mathcal{M}$ *measurable function* $f\colon \mathbf{R} \to \mathbf{R}$ *there exists a sequence of functions* $f_k \in \mathfrak{M}_1 \cap \mathfrak{C}_H \cap \mathfrak{A}_K$ *such that* $f = \lim_{k\to\infty} f_k$.

Proof. As in the proof of Theorem 4 we choose points $(x_{\alpha,i}, y_{\alpha,i}) \in W_\alpha \cap (B \times \mathbf{R})$ ($\alpha < \omega_1$; $i = 1,2,\dots$) such that $x_{\alpha_1,i_1} \neq x_{\alpha_2,i_2}$ for $(\alpha_1, i_1) \neq (\alpha_2, i_2)$ ($\alpha_1, \alpha_2 < \omega_1$ and $i_1, i_2 = 1,2,\dots$). Let $(A_n)_{n=1}^{\infty}$ be the same as in the proof of the Theorem 4. Define, for $k = 1,2,\dots$,

$$f_n(x) = \begin{cases} w_k & \text{for} \quad x \in A_k,\ n \le k \\ y_{\alpha,k} & \text{for} \quad x = x_{\alpha,k},\ n \le k \quad \text{and} \quad \alpha < \omega_1 \\ f(x) & \text{in the remaining case}. \end{cases}$$

Since

$$\bigcap_{n=1}^{\infty}\Big(\bigcup_{k\geq n}\Big(A_k\cup\bigcup_{\alpha<\omega_1}\{x_{\alpha,k}\}\Big)\Big)=\emptyset\,,$$

there is $f=\lim_{k\to\infty}f_k(x)$. Likewise as in the proofs of the Theorems 4, 5 we show that all $f_k\in\mathfrak{M}_1\cap\mathfrak{C}_H\cap\mathfrak{A}_K$. □

IV. The case of the density topology

Let $X=\mathbf{R}$. Recall that a point x is an *outer density point of a set* $A\subset\mathbf{R}$ iff

$$\lim_{h\to 0+}m^*\big(A\cap(x-h,x+h)\big)/2h=1$$

(m^* denotes the *outer Lebesgue measure in* $\mathbf{R}$). If A is measurable (in the Lebesgue sense) then x is called a *density point of* A. The family of all measurable (L) sets $A\subset\mathbf{R}$ for which every $x\in A$ is a density point of A forms a topology. This topology is said to be a *density topology* in $\mathbf{R}$ ([1]). We denote it by $\mathcal{T}_{\mathrm{d}}$.

In the paper [4] Sierpiński introduced a property (P). A function $f\colon\mathbf{R}\to\mathbf{R}$ has the property (P) at a point $x\in\mathbf{R}$ iff there exists a set $E\subset\mathbf{R}$ such that $x\in E$, x is an outer density point of E and the function $f|E$ is continuous at x. He proved also that every function $f\colon\mathbf{R}\to\mathbf{R}$ has the property (P) at almost all points $x\in\mathbf{R}$.

Remark 1. *A function $f\colon\mathbf{R}\to\mathbf{R}$ has the property* (P) *at a point $x\in\mathbf{R}$ iff f is almost continuous in the Husain sense at x with respect to the topology $\mathcal{T}_{\mathrm{d}}$.*

Proof. If f has the property (P) at x, then there is a set $E\ni x$ having the outer density 1 at x such that $f|E$ is continuous at x. Fix $\varepsilon>0$. Since $f^{-1}\Big(\big(f(x)-\varepsilon,f(x)+\varepsilon\big)\Big)\supset E\cap I$ for some open interval $I\ni x$, so $x\in\mathrm{Int}_{\mathcal{T}_{\mathrm{d}}}\Big(\mathrm{Cl}_{\mathcal{T}_{\mathrm{d}}}\Big(f^{-1}\big(f(x)-\varepsilon,f(x)+\varepsilon\big)\Big)\Big)$. The proof that the $(\mathcal{T}_{\mathrm{d}})_H$ almost continuity (i.m. the Husain almost continuity with respect to $\mathcal{T}_{\mathrm{d}}$) of f at x implies the property (P) of f at x is the same as the proof of the Theorem 5.6 in [1]. □

Since $\mathbf{R}=\bigcup_{n=1}^{\infty}A_n$, where A_n are disjoint pairwise and of plenty outer Lebesgue measure ([5]), then Theorems 1, 2, 3 hold for the topology $\mathcal{T}_{\mathrm{d}}$ in the case where the σ-field $\mathcal{M}$ contains nonmeasurable (L) sets. From Remark 2 it results that if $f\colon\mathbf{R}\to\mathbf{R}$ is Lebesgue measurable and $\mathcal{T}_{\mathrm{d}}$ almost continuous in the Husain sense, then it is approximately continuous, i.e. $\mathcal{T}_d$ continuous ([1]), so Theorems 1, 2, 3 do not hold for the $\mathcal{T}_d$ topology and the σ-field $\mathcal{M}$ of Lebesgue measurable sets. We have in this case $G(\mathfrak{C}_H \cap \mathfrak{M}_1) = L(\mathfrak{C}_H \cap \mathfrak{M}_1) = \mathfrak{C}_H \cap \mathfrak{M}_1$ ([1]) and $B(\mathfrak{C}_H \cap \mathfrak{M}_1) = \mathfrak{B}_2$, where $\mathfrak{B}_2$ denotes the *family of all Baire 2 functions* ([6]).

V. The maximal additive, multiplicative and lattice families for the class $\mathfrak{C}_H$

Define:

$A(\mathfrak{C}_H) = \{f\colon X \to \mathbb{R};\ \text{ for every } g \in \mathfrak{C}_H \text{ the sum } f+g \in \mathfrak{C}_H\}$,
$P(\mathfrak{C}_H) = \{f\colon X \to \mathbb{R};\ \text{ for every } g \in \mathfrak{C}_H \text{ the product } fg \in \mathfrak{C}_H\}$,
$S_{\max}(\mathfrak{C}_H) = \{f\colon X \to \mathbb{R};\ \text{ for every } g \in \mathfrak{C}_H\ \max(f,g) \in \mathfrak{C}_H\}$,
$S_{\min}(\mathfrak{C}_H) = \{f\colon X \to \mathbb{R};\ \text{ for every } g \in \mathfrak{C}_H\ \min(f,g) \in \mathfrak{C}_H\}$.

Remark 2. $\mathfrak{C}_H \supset A(\mathfrak{C}_H) \cup P(\mathfrak{C}_H) \cup S_{\max}(\mathfrak{C}_H) \cup S_{\min}(\mathfrak{C}_H)$.

Proof. As $g = 0 \in \mathfrak{C}_H$, so $A(\mathfrak{C}_H) \subset \mathfrak{C}_H$. Similarly $g = 1 \in \mathfrak{C}_H$ implies $P(\mathfrak{C}_H) \subset \mathfrak{C}_H$. If $f \notin \mathfrak{C}_H$, then there exists a point $x \in X$ and a positive number ε such that $x \notin \operatorname{Int}\operatorname{Cl}\big(f^{-1}(f(x)-\varepsilon, f(x)+\varepsilon)\big)$. If $g = f(x)-\varepsilon$, then $g \in \mathfrak{C}_H$ and $x \notin \operatorname{Int}\operatorname{Cl}(\{t \in X : |\max(f,g)(t) - \max(f,g)(x)| < \varepsilon\}) \subset \subset \operatorname{Int}\operatorname{Cl}(\{t \in X : |f(t)-f(x)| < \varepsilon\})$ and $\max(f,g)$ is not in $\mathfrak{C}_H$. So $S_{\max}(\mathfrak{C}_H) \subset \mathfrak{C}_H$. Analogously we can prove that $S_{\min}(\mathfrak{C}_H) \subset \mathfrak{C}_H$. □

Remark 3. *Let $\mathfrak{C}$ denote the family of all $\mathcal{T}$ continuous functions $f\colon X \to \mathbb{R}$. We have $\mathfrak{C} \subset A(\mathfrak{C}_H) \cap P(\mathfrak{C}_H) \cap S_{\max}(\mathfrak{C}_H) \cap S_{\min}(\mathfrak{C}_H)$.*

Proof. Fix $f \in \mathfrak{C}$, $g \in \mathfrak{C}_H$, $x \in X$ and $\varepsilon > 0$. Since $x \in \operatorname{Int}(\{t \in X\colon |f(t)-f(x)| < \varepsilon/2\}) \cap \operatorname{Int}\operatorname{Cl}(\{t\colon |g(t)-g(x)| < \varepsilon/2\}) \subset \operatorname{Int}\operatorname{Cl}(\{t\colon |f(t)+g(t)-f(x)-g(x)| < \varepsilon\})$, $f+g$ is $\mathcal{T}$ almost continuous (in the Husain sense) at x. This proves that $\mathfrak{C} \subset A(\mathfrak{C}_H)$.

Similarly, $x \in \operatorname{Int}(\{t \in X\colon |f(t)-f(x)| < \varepsilon/2\max(|g(x)|,1)\}) \cap \operatorname{Int}\operatorname{Cl}(\{t\colon |g(t)-g(x)| < \varepsilon/2\max(1, |f(x)|+\varepsilon/2)\}) \subset \operatorname{Int}\operatorname{Cl}(\{t\colon |fg(x)-fg(t)| < \varepsilon\})$, so $P(\mathfrak{C}_H) \supset \mathfrak{C}$.

Now if $f(x) < g(x)$, then for $0 < \varepsilon < (g(x)-f(x))/2$ there exists an open neighbourhood U of x such that
$U \cap \{t \in X\colon |\max(f,g)(x) - \max(f,g)(t)| < \varepsilon\} \cap \{t\colon |f(x)-f(t)| < \varepsilon\} = U \cap \cap \{t\colon |g(t)-g(x)| < \varepsilon\}$. Thus $x \in \operatorname{Int}\operatorname{Cl}(\{t\colon |\max(f,g)(t) - \max(f,g)(x)| < \varepsilon\})$ and $\max(f,g)$ is $\mathcal{T}$ almost continuous in the Husain sense at x. If $f(x) \geq \geq g(x)$, then also $x \in \operatorname{Int}\operatorname{Cl}\Big(\{t\colon \big|\max(f(t),g(t)) - \max(f(x),g(x))\big| < \varepsilon\}\Big)$.

So $\mathfrak{C} \subset S_{\max}(\mathfrak{C}_H)$. Similarly we can show that $\min(f,g)$ is in $\mathfrak{C}_H$. So $\mathfrak{C} \subset \subset S_{\min}(\mathfrak{C}_H)$. $\square$

Theorem 7. $S_{\min}(\mathfrak{C}_H) = S_{\max}(\mathfrak{C}_H) = P(\mathfrak{C}_H) = A(\mathfrak{C}_H) = \mathfrak{C}$.

Proof. It suffices to prove that $S_{\min}(\mathfrak{C}_H), S_{\max}(\mathfrak{C}_H), P(\mathfrak{C}_H), A(\mathfrak{C}_H) \subset \mathfrak{C}$. If $f \in \mathfrak{C}_H$ ($f\colon X \to \mathbf{R}$) is not continuous at $x \in X$, then there is $\varepsilon > 0$ such that for every open neighbourhood U of x there is some point $t \in U$ with $|f(t) - f(t)| > \varepsilon$ ($\varepsilon < |f(x)|/2$ whenever $f(x) \neq 0$). Let $V \subset X$ be an open set such that $x \in V$ and $V \subset \operatorname{Int}\operatorname{Cl}\Big(f^{-1}\big(f(x)-\varepsilon, f(x)+\varepsilon\big)\Big)$. There is some point $u \in V$ such that $|f(u) - f(x)| > \varepsilon$. As $f \in \mathfrak{C}_H$, there is an open neighbourhood $W \subset V$ of u such that $W \subset \operatorname{Int}\operatorname{Cl}(\{t \in V\colon |f(t) - f(u)| < \eta\})$, where 2η is a positive number $\leq |f(u) - f(x)| - \varepsilon$ ($\eta < |f(u)|/2$ whenever $f(u) \neq 0$). Remark that
$W \subset \operatorname{Int}\operatorname{Cl}(\{t \in W\colon |f(t) - f(x)| < \varepsilon\}) \cap \operatorname{Int}\operatorname{Cl}(\{t\colon |f(t) - f(u)| < \eta\})$ and $\{t \in W\colon |f(t) - f(u)| < \eta\} \cap \{t\colon |f(t) - f(x)| < \varepsilon\} = \emptyset$. Define

$$g(t) = \begin{cases} 2\eta & \text{if } t \in W,\ t \neq u \text{ and } |f(t) - f(u)| < \eta \\ 0 & \text{in the remaining case.} \end{cases}$$

Since $\operatorname{Int}\big(g^{-1}(2\eta)\big) = \emptyset$, g is $\mathcal{T}$ almost continuous in the Husain sense at each point $t \in X$ with $g(t) = 0$. The almost continuity in the Husain sense of g at every point $t \in X$ with $g(t) = 2\eta$ results from the inclusion

$$W \subset \operatorname{Int}\operatorname{Cl}(\{t \in W\colon |f(t) - f(u)| < \eta\}).$$

So $g \in \mathfrak{C}_H$. But $f + g$ is not $\mathcal{T}$ almost continuous in the Husain sense at u, because $(f+g)(u) = f(u)$, $f(t) + g(t) = f(t)$ for $t \in W$ with $|f(t) - f(u)| \geq \eta$ and $|f(t) + g(t) - f(u)| \geq |g(t)| - |f(t) - f(u)| \geq 2\eta - \eta = \eta$ for $t \in W$ with $t \neq u$ and $|f(u) - f(t)| < \eta$. Thus $f \notin A(\mathfrak{C}_H)$ and $A(\mathfrak{C}_H) \subset \mathfrak{C}$. For the proof of the inclusion $P(\mathfrak{C}_H) \subset \mathfrak{C}$ we fix a point $w \in W$ such that $|f(w) - f(u)| < \eta$ whenever $f(u) \neq 0$ or $|f(w) - f(x)| < \varepsilon$ in the other case. If $f(u) \neq 0$, we define

$$h(t) = \begin{cases} c & \text{if } t \in W,\ t \neq w \text{ and } |f(t) - f(u)| < \eta \\ 1 & \text{in the remaining case,} \end{cases}$$

where c is such that $|cy| > |f(w)| + 1$ for each $y \in \big(f(u) - \eta, f(u) + \eta\big)$. Analogously to the case of the function g we prove that $h \in \mathfrak{C}_H$. Since $f(w)h(w) = f(w) \neq 0$, $f(t)h(t) = cf(t)$ for $t \in W$ with $|f(t) - f(u)| < \eta$ and $f(t)h(t) = f(t)$ for $t \in W$ with $|f(t) - f(u)| \geq \eta$, so the function fh is not $\mathcal{T}$ almost continuous in the Husain sense at w.

If $f(u) = 0$, then we define

$$h(t) = \begin{cases} c & \text{if} \quad t \in W,\ t \neq x \quad \text{and} \quad |f(t) - f(x)| < \varepsilon \\ 1 & \text{in the remaining case,} \end{cases}$$

where c is such that $|cy| > |f(w)| + 1$ for each $y \in \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$ and analogously as above we prove that $h \in \mathfrak{C}_H$ and that the function fh is not $\mathcal{T}$ almost continuous in the Husain sense at w. Thus $f \notin P(\mathfrak{C}_H)$ and $P(\mathfrak{C}_H) \subset \mathfrak{C}$.

For the proof of the inclusion $S_{\max}(\mathfrak{C}_H) \subset \mathfrak{C}$ we define

$$k(x) = \begin{cases} f(u) + \eta & \text{if} \quad t \in W,\ t \neq u \quad \text{and} \quad |f(t) - f(u)| < \eta \\ f(u) - \eta & \text{in the remaining case} \end{cases}$$

and analogously to the case of the function g we prove that $k \in \mathfrak{C}_H$. Since $\max\big(f(u), k(u)\big) = f(u)$ and $\max\big(f(t), g(t)\big) \notin \big(f(u) - \eta, f(u) + \eta\big)$ for $t \in W$ with $t \neq u$, we have $f \notin S_{\max}(\mathfrak{C}_H)$ and $S_{\max}(\mathfrak{C}_H) \subset \mathfrak{C}$. Analogously we can prove that $S_{\min}(\mathfrak{C}_H) \subset \mathfrak{C}$. □

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