Mathematica Slovaca

Michał Karoński; Zbigniew Palka

On the size of a maximal induced tree in a random graph

Mathematica Slovaca, Vol. 30 (1980), No. 2, 151--155

Persistent URL: http://dml.cz/dmlcz/130499

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1980

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Math. Slovaca 30, 1980, No. 2, 151—155

ON THE SIZE OF A MAXIMAL INDUCED TREE IN A RANDOM GRAPH

MICHAL KAROŃSKI—ZBIGNIEW PALKA

1. Introduction

Let $G_{n,p}$ be a random graph with n labelled vertices, where each of $\binom{n}{2}$ possible edges occurs with the same probability p independently of all other edges.

In this paper, bounds for the size of a maximal induced tree in $G_{n,p}$, i.e. such a tree which is not properly contained in any other tree, are established.

A similar problem of the size of a clique in $G_{n,p}$ has been considered by Matula [2], [3] as well as by Bollobás and Erdős [1].

It should be noted that we do not consider an isolated vertex in $G_{n,p}$ as a maximal induced tree. Henceforth, in this paper the term "tree" means "induced tree".

2. Maximal trees in a random graph

Let Y_k and Z_k denote the number of trees and maximal trees of the size (the number of vertices) k, $2 \leqslant k \leqslant n$, in $G_{n,p}$, respectively. For the sake of simplicity let us assume that: $p_1(k) = (8\sqrt{\pi}(1+r(k))+1)^{-1}$, where $0<r(k)<1/12k$, $q=1-p$, $d=1/q$, $\lambda=\lambda(p)=d\sqrt{pe^3}/2$.

First, we shall investigate the behavior of the expected values $E(Z_k)$ and $E(Y_k)$ of the random variables Z_k and Y_k, respectively. Let n_1 be the least integer such that, for a given p and all $n \geqslant n_1$, the inequality $2 \leqslant l(n,p) \leqslant n$ holds, where

$$(1) \qquad l(n,p) = \log_d(np) - \log_d \log_e(n\lambda) + 1.$$

Theorem 1. *If $p_1(k) \leqslant p < 1$ and $n \geqslant n_1$, then $E(Z_k) \leqslant 1$ for any integer k, $2 \leqslant k \leqslant l(n,p)$.*

Proof. The random variable Z_k, i.e. the number of maximal trees of the size k in a random graph $G_{n,p}$, has the expectation

$$(2) \qquad E(Z_k) = \binom{n}{k} t_k (1-kpq^{k-1})^{n-k},$$

where

$$t_k = k^{k-2} p^{k-1} q^{(k-1)(k-2)/2}, \tag{3}$$

is the probability that the random graph restricted to the k-membered subset of vertices is a tree. Formula (2) follows from the fact that a subgraph of $G_{n,p}$ of the size k is a maximal tree iff it is a tree and no other vertex is incident with exactly one of the vertices of this subgraph. By the Stirling formula

$$\binom{n}{k} \leq \frac{n^k}{k!} = \left(\frac{ne}{k}\right)^k (\sqrt{2\pi k}\, e^{r(k)})^{-1},$$

where $r(k)$ satisfies the condition $0 < r(k) < 1/12k$, and so we have

$$\binom{n}{k} t_k \leq (n\, epq^{(k-3)/2})^k (dpk^2\sqrt{2\pi k}\, e^{r(k)})^{-1}. \tag{4}$$

It is easy to check that for $k \geq 2$ and $p \geq p_1(k)$

$$dpk^2\sqrt{2\pi k}\, e^{r(k)} \geq 1.$$

Hence we arrive at

$$\binom{n}{k} t_k \leq (n\, epq^{(k-3)/2})^k. \tag{5}$$

By the use of the following inequalities

$$1 - x \leq e^{-x}, \quad \text{for} \quad x \geq 0,$$
$$npq^{k-1} \leq n/4,$$

and

$$kpq^{k-1} \leq 1/2,$$

both for $k \geq 2$, one can get that

$$E(Z_k) \leq (npq^{(k-3)/2} \exp(1 - (n-k)pq^{k-1}))^k$$
$$\leq \left(\frac{1}{2} nd\sqrt{p} \exp\left(\frac{3}{2} - npq^{k-1}\right)\right)^k.$$

Finally we obtain the inequality

$$E(Z_k) \leq (n\lambda \exp(-npq^{k-1}))^k. \tag{6}$$

Now by elementary calculations one can check that $E(Z_k) \leq 1$ for all

$$k \leq \log_d(np) - \log_d \log_e(n\lambda) + 1;$$

thus we prove the theorem.

Let now

$$u(n, p) = 2 \log_d(npe) + 3. \tag{7}$$

We have

Theorem 2. *If $p_1(k) \leqslant p < 1$ and $n \geqslant 6$, then $E(Y_k) \leqslant 1$ for any integer k, $u(n, p) \leqslant k \leqslant n$.*

Proof. To prove this it should be noticed only that

$$E(Y_k) = \binom{n}{k} t_k,$$

where t_k is given by the formula (3). Using (5) one can check that $E(Y_k) \leqslant 1$ for all $k \geqslant u(n, p)$.

Now we shall show that $l(n, p)$ and $u(n, p)$ given by (1) and (7), are threshold functions for the occurrence of maximal trees in $G_{n,p}$. It means that, with some restrictions on n and p, a random graph $G_{n,p}$ most likely does not contain a maximal tree of the size k for any $2 \leqslant k \leqslant [l(n, p)]$ and $k \geqslant \{u(n, p)\}$. As usual $[x]$ and $\{x\}$ denote the greatest integer not greater than x and the least integer not less than x, respectively.

Let $\alpha_{n,p}$ denote the size of the smallest maximal tree in a random graph $G_{n,p}$. Let n_2 be the least integer such that for a given p and all $n \geqslant n_2$ the following inequality

$$2 < l(n, p) \leqslant n \left(1 - \frac{1}{\log_e(n\lambda)}\right) \tag{8}$$

holds.

Theorem 3. *Let $1 - e^{-1/2} \leqslant p < 1$, $n \geqslant n_2$ and $l = l(n, p)$ be the threshold functions given by the formula (1). Then for any integer k, $2 \leqslant k < l(n, p)$*

$$\text{Prob}\,(\alpha_{n,p} \leqslant k) \leqslant (k-1)(n\lambda)^{-kf}, \tag{9}$$

where $f = f(\delta) = d^\delta - 1$ and $\delta = \delta(n, k, p) = l(n, p) - k$.

Proof. Let Z_k denote, as before, the number of maximal trees of the size $k \geqslant 2$. Then

$$\text{Prob}\,(\alpha_{n,p} \leqslant k) = \text{Prob}\left(\bigcup_{j=2}^{k} (Z_j > 0)\right) \leqslant \sum_{j=2}^{k} \text{Prob}\,(Z_j > 0),$$

by Bool's inequality. Moreover it is obvious that

$$\text{Prob}\,(Z_j > 0) \leqslant E(Z_j) = g_j.$$

Now we shall show that for $p \geqslant 1 - e^{-1/2}$, $g_2 \leqslant g_3 \leqslant \ldots \leqslant g_k$. Here

$$\frac{g_{j+1}}{g_j} = \left(1 + \frac{1}{j}\right)^{j-2} (n-j)pq^{j-1} \frac{h(j+1)}{h(j)},$$

where

$$h(j)=(1-jpq^{j-1})^{n-j}.$$

Consider $h(j)$ to be the function of the continuous argument j on the interval $\langle 2, l(n,p)\rangle$. One can check that $h(j)$ is increasing for all $j \geq (\log_e d)^{-1}$. Thus the function $h(j)$ is increasing on the interval $\langle 2, l(n,p)\rangle$ if $(\log_e d)^{-1} \leq 2$, which is true for all $p \geq 1-e^{-1/2}$. From this fact it follows that g_{j+1}/g_j is greater than or equal to one if

$$(n-j)pq^{j-1} \geq 1.$$

This inequality is satisfied for all $2 \leq j \leq l(n,p)$ and $p \geq 1-e^{-1/2}$ if

$$\frac{l(n,p)}{n}+\frac{1}{\log_e(n\lambda)} \leq 1,$$

which is true for all $n \geq n_2$.

Now we can write that

$$\text{Prob}\,(\alpha_{n,p} \leq k) \leq (k-1)E(Z_k). \tag{10}$$

From formula (1) we have

$$q^{k-1}=q^{l-1}d^{\delta}=\frac{\log_e(n\lambda)}{np}\,d^{\delta},$$

hence using (6) we obtain

$$E(Z_k) \leq (n\lambda \exp(-d^{\delta}\log_e(n\lambda)))^k=(n\lambda)^{-kf}.$$

Putting the above estimate into formula (10) we obtain the thesis.

It should be mentioned that the restriction imposed on n in theorem 3, i.e. on the size of a random graph $G_{n,p}$, is not too significant if p is small. For example, in the best case, when $p=1-e^{-1/2}$, n_2 is equal to 15.

Now we would like to state a similar result for the upper bound of the size of a maximal tree in $G_{n,p}$. Let random variables $\beta_{n,p}$ and $\tau_{n,p}$ denote the size of the largest maximal tree and the largest tree in a random graph $G_{n,p}$, respectively.

Theorem 4. *Let $p_1(k) \leq p < 1$, $n \geq 6$ and $u=u(n,p)$ be the threshold function given by the formula (7). Then for any integer k, $u(n,p) < k \leq n$,*

$$\text{Prob}\,(\beta_{n,p} \geq k) \leq \left(\frac{c}{n}\right)^{\varepsilon},$$

where

$$c=c(\varepsilon,p)=q^{(\varepsilon+3)/2}(pe)^{-1}$$

and

$$\varepsilon=\varepsilon(n,k,p)=k-u(n,p).$$

Proof. Let Y_k denote the number of trees of the size k in $G_{n,p}$. Then

$$\text{(11)} \qquad \begin{aligned} \text{Prob}\,(\beta_{n,p} \geqslant k) &= \text{Prob}\,(\tau_{n,p} \geqslant k) \\ &= \text{Prob}\,(Y_k > 0) \leqslant E(Y_k). \end{aligned}$$

By formula (5)

$$\text{(12)} \qquad E(Y_k) \leqslant (np \ \mathrm{e} q^{(k-3)/2})^k,$$

but

$$k(k-3)/2 = (k+\varepsilon)(u-3)/2 + \varepsilon(\varepsilon+3)/2,$$

and from formula (7)

$$q^{(u-3)/2} = \frac{1}{np \ \mathrm{e}}.$$

Putting both into (12) and (11) we obtain the thesis.

3. Final remarks

We were not succesful in determining what will happen with the lower bound for the size of a maximal tree when $p_1(k) \leqslant p < 1 - \mathrm{e}^{-1/2}$. It seems possible that formula (9) holds for such probabilities also.

A problem of the size of a maximal tree remains open in the case when p is very small, i.e. $0 < p < p_1(k)$.

REFERENCES

[1] BOLLOBÁS, B.—ERDÖS, P.: Cliques in random graphs. Math. Proc. Cambridge Philos. Soc., 80, 1976, 419—427.

[2] MATULA, D. W.: On the complete subgraphs of a random graph. Comb. Math. and Its Appl. Chapel Hill, N. C., 1970.

[3] MATULA, D. W.: The largest clique size in a random graph. Technical Report CS 7608, 1976.

Received February 6, 1978

Institute of Mathematics
A. Mickiewicz University
60-769 Poznan
Poland

О ПОРЯДКЕ МАКСИМАЛЬНОГО ИНДУЦИРОВАННОГО ДЕРЕВА
В СЛУЧАЙНОМ ГРАФЕ

Михал Каронски–Збигниев Палка

Резюме

В работе даны ограничения для числа вершин максимального дерева в случаном графе.