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TOPOLOGICAL DIFFERENCE POSETS

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ABSTRACT. Difference posets (D-posets) are partially ordered sets with a partial difference operation. Special cases of D-posets are orthomodular posets or systems of fuzzy sets. In this paper, we define a topological D-poset as a D-poset with a topology guaranteeing the continuity of the difference operation, and a topological lattice D-poset as a lattice D-poset with a topology guaranteeing the continuity of the difference operation and lattice operations. If these topologies are uniform and the operations are uniformly continuous, we speak of uniform D-posets and uniform lattice D-posets. In the paper, several examples of uniform D-posets are exhibited. The main result is the theorem asserting that the topological completion of a uniform Hausdorff lattice D-poset in which all monotone nets are Cauchy is also a uniform Hausdorff lattice D-poset, which is a complete lattice. This is the generalization of a known result for orthomodular lattices ([14]).

1. Introduction

In recent decades, many extensions of Kolmogoroff axiomatics were introduced. After Boolean algebras, there followed quantum logics, orthomodular lattices and fuzzy sets. Several years ago, orthoalgebras were defined (see [1]), and the most recent notion is that of D-posets (see [6], [7]), which include all the previously mentioned structures.

DEFINITION 1.1. A *difference poset* (briefly D-poset) is a quadruple $(D, \leqq, \ominus, 1)$, where D is a nonempty set partially ordered by $\leqq$, 1 is the largest element of D, and $\ominus$ is the *difference operation* which defines for every $a, b \in D$, $a \leqq b$, an element $b \ominus a$ in such a way that the following conditions are true:

i) $b \ominus a \leqq b$,
ii) $b \ominus (b \ominus a) = a$,
iii) $a \leqq b \leqq c$ implies $c \ominus b \leqq c \ominus a$, and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

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It can easily be seen that in any D-poset, $0 = 1 \ominus 1$ is the smallest element of D.

DEFINITION 1.2. An *orthomodular poset* (OMP) is a triple $(P, \leqq, \perp)$, where P is a nonempty set partially ordered by $\leqq$, possessing the largest element 1 and the smallest element 0, and $\perp : P \to P$ is a map with properties:

i) $a \leqq b$ implies $b^{\perp} \leqq a^{\perp}$,
ii) $(a^{\perp})^{\perp} = a$,
iii) $a \vee a^{\perp} = 1$,
iv) $a \leqq b$ implies $b = a \vee (b \wedge a^{\perp})$.

Two elements a, b of P are called *orthogonal* (written $a \perp b$) if $a \leqq b^{\perp}$. For a, b orthogonal, there exists $a \vee b$ in P. An OMP which is a σ-lattice is called *quantum logic.*

It is clear that every OMP becomes a D-poset if we put for every $a, b \in P$, $a \leqq b$, $b \ominus a = b \wedge a^{\perp}$.

In the following, D denotes always a D-poset. Let us write $G = \{(a, b) \in D \times D \ ; \ a \leqq b\}$. A net a_{α} of elements of D is called *increasing* (*decreasing*) if $\alpha \leqq \beta$ implies $a_{\alpha} \leqq a_{\beta}$ ($a_{\alpha} \geqq a_{\beta}$). Increasing and decreasing nets are called monotone.

DEFINITION 1.3. A function $\mu : D \to R$ is called a *signed measure* if for every $a, b \in D$, $a \leqq b$, $\mu(b) = \mu(a) + \mu(b \ominus a)$. If $\mu(a) \geqq 0$ for every $a \in D$, we say that μ is a *measure.*

A set $\mathcal{M}$ of signed measures on D is called *separating* if for every $a, b \in D$, $a \neq b$, there exists $\mu \in \mathcal{M}$ such that $\mu(a) \neq \mu(b)$.

2. D-poset as a topological space

It is well known in classical measure theory that, if μ is a finite measure on the σ-algebra $\mathcal{S}$ of subsets of some set X, then the function $\varrho_{\mu}(A, B) = \mu(A \triangle B)$, $A, B \in \mathcal{S}$, ($A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of A and B) is a pseudometric on $\mathcal{S}$ (see [4]). Quantum logics as topological spaces were investigated, for example, in [8], [9], [10], [11], [12], [13] and [14]. Some considerations of these papers are extendable also to D-posets.

If a D-poset D with a topology $\mathcal{T}$ form a topological space $(D, \mathcal{T})$, and $\mathcal{T} \times \mathcal{T}$ is the usual product topology on $D \times D$, let $\mathcal{T}_0$ be the relative topology on G induced by $\mathcal{T} \times \mathcal{T}$. If $\mathcal{T}$ is an uniform topology induced by an uniformity $\mathcal{U}$, $\mathcal{U} \times \mathcal{U}$ is the product of uniformities, and $\mathcal{U}_0$ is the relative uniformity on G induced by $\mathcal{U} \times \mathcal{U}$, then, of course, $\mathcal{U}_0$ induces $\mathcal{T}_0$.

DEFINITION 2.1. $(D,\mathcal{T})$ is called a *topological D-poset* if $\ominus\colon (G,\mathcal{T}_0) \to (D,\mathcal{T})$ is continuous. If $\mathcal{T}$ is induced by the uniformity $\mathcal{U}$, then $(D,\mathcal{U})$ is called a *uniform D-poset* if $\ominus\colon (G,\mathcal{U}_0) \to (D,\mathcal{U})$ is uniformly continuous.

If $\mathcal{T}$ is uniform, we do not distinguish in the notation between $(D,\mathcal{T})$ and $(D,\mathcal{U})$.

It is obvious that the discrete topology on any D forms a Hausdorff uniform D-poset.

The following lemma is routine.

LEMMA 2.2. *Let $\mathcal{B}$ be a prebase of $\mathcal{U}$. Then $(D,\mathcal{U})$ is a uniform D-poset if and only if for every $U \in \mathcal{B}$ there exists $V \in \mathcal{U}$ such that $(x_1, x_2) \in V$, $(y_1, y_2) \in V$, $x_1 \leqq y_1$, $x_2 \leqq y_2$ implies $(y_1 \ominus x_1, y_2 \ominus x_2) \in U$.*

We exhibit several uniform D-posets. All of them are Hausdorff.

EXAMPLE 1. If $\mathcal{S}$ is the σ-algebra of subsets of X, μ a finite measure on $\mathcal{S}$, let us define en equivalence relation on S via: $A \sim B$ if $\mu(A \triangle B) = 0$. Let us denote by $\overline{\mathcal{S}}$ the system of all equivalence classes $[A]$, $A \in \mathcal{S}$, and by $\leqq$, the partial ordering on $\overline{\mathcal{S}}$, where $[A] \leqq [B]$ if $A_1 \subset B_1$ for some $A_1 \in [A]$, $B_1 \in [B]$. If we define the orthocomplementation $\perp$ on $\overline{\mathcal{S}}$: $[A]^\perp = [X \setminus A]$, then $\overline{\mathcal{S}}$ becomes a Boolean algebra and, hence, a D-poset. If we define the metric ϱ_μ on $\overline{\mathcal{S}}$ by $\varrho_\mu([A],[B]) = \mu(A \triangle B)$, and $\overline{\mathcal{T}}_\mu$ is the topology induced by ϱ_μ, then $(\overline{\mathcal{S}}, \overline{\mathcal{T}}_\mu)$ is a uniform D-poset.

EXAMPLE 2. Let $\mathcal{L}(H)$ be the set of all closed subspaces of the separable Hilbert space H (complex or real) with $\dim H \geqq 3$. If $\mathcal{L}(H)$ is partially ordered by inclusion, and, for $M \in \mathcal{L}(H)$, $M^\perp$ is the usual orthogonal complement of M, then $\mathcal{L}(H)$ is a complete orthomodular lattice. Then the sets $U_{\varphi,\varepsilon} = \{(M,N) \in \mathcal{L}(H) \times \mathcal{L}(H);\ \|P^M\varphi - P^N\varphi\| < \varepsilon\}$, $\varphi \in H$, $\varepsilon > 0$, (P^M denotes the orthogonal projector corresponding to M) form a prebase of a uniformity $\mathcal{U}$. Let us denote by τ_{strong} the topology induced by $\mathcal{U}$. We can also obtain τ_{strong} as the relative topology induced by the strong topology on the space of all bounded linear operators operating from H to H (identifying closed subspaces with orthogonal projectors projecting on them). $(\mathcal{L}(H), \tau_{\text{strong}})$ is a uniform D-poset. Every increasing (decreasing) net M_α converges in this space to $M = \bigvee M_\alpha$ ($\bigwedge M_\alpha$).

EXAMPLE 3. If we define a metric d on $\mathcal{L}(H)$ by $d(M,N) = \|P^M - P^N\|$, $M, N \in \mathcal{L}(H)$, where $\|\cdot\|$ denotes the usual operator norm, and τ_{unif} is the topology induced by d, then $(\mathcal{L}(H), \tau_{\text{unif}})$ is uniform D-poset.

D-posets in Examples 4 – 9 are systems of fuzzy sets, i.e., systems of functions defined on some set A with values in the interval $\langle 0,1\rangle$. In all these D-posets,

$\leqq$ is defined via: $f \leqq g$ if $f(t) \leqq g(t)$ for every $t \in A$. Then $(g \ominus f)(t) = g(t) - f(t)$, $t \in A$. The largest element is the constant function equal to 1. On these D-posets, we can define two topologies in a natural way. The first one is the uniform topology of pointwise convergence, where $f_\alpha \to f$ if and only if $f_\alpha(t) \to f(t)$ for every $t \in A$. Let us denote it by τ_{pc}. The second is the topology induced by the metric $d(f,g) = \sup\{|f(t) - g(t)|;\ t \in A\}$, denoted by $\tau_{\sup}$. Then in Examples 4–9, we have further Hausdorff uniform D-posets.

EXAMPLE 4. (D, τ_{pc}), where D is the set of all functions defined on an arbitrary set A with values in $\langle 0,1\rangle$.

EXAMPLE 5. $(D, \tau_{\sup})$, where D is the same as in E4.

EXAMPLE 6. (D, τ_{pc}), where D is the set of all continuous functions defined on $\langle 0,1\rangle$ with values in $\langle 0,1\rangle$.

EXAMPLE 7. $(D, \tau_{\sup})$, where D is the same as in E6.

EXAMPLE 8. (D, τ_{pc}), where D is the set of all convergent sequences with values in $\langle 0,1\rangle$.

EXAMPLE 9. $(D, \tau_{\sup})$, where D is the same as in E8.

EXAMPLE 10. Let D be the subset of the normed space $\mathcal{L}^p(\langle 0,1\rangle)$, $p \geqq 1$, such that $[f] \in D$ if $0 \leqq f_1(t) \leqq 1$, $t \in \langle 0,1\rangle$, for some $f_1 \in [f]$. For $[f],[g] \in D$, $[f] \leqq [g]$ if $f_1(t) \leqq g_1(t)$, $t \in \langle 0,1\rangle$, for some $f_1 \in [f]$, $g_1 \in [g]$. If for $[f] \leqq [g]$, $[g] \ominus [f] = [g - f]$, then D is a D-poset. If $\mathcal{T}$ is the topology induced by the metric $d([f],[g]) = \left(\int\limits_0^1 |f(t) - g(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}$, then $(D, \mathcal{T})$ is a uniform D-poset.

On any D-poset with a separating set of signed measures, it is possible to define some nontrivial Hausdorff uniform topologies.

If $\mathcal{M}$ is a separating set of signed measures, let $\tau(\mathcal{M})$ be the uniform topology with a prebase containing sets $U_{m,\varepsilon} = \{(a,b) \in D \times D;\ |m(a) - m(b)| < \varepsilon\}$, $\varepsilon > 0$, $m \in \mathcal{M}$. Let $\mathcal{T}(\mathcal{M})$ be the topology induced by the metric $\varrho_{\mathcal{M}}(a,b) = \sup\{|m(a) - m(b)|;\ m \in \mathcal{M}\}$.

THEOREM 2.3. *If $\mathcal{M}$ is a separating set of signed measures on D, then $(D, \tau(\mathcal{M}))$ and $(D, \mathcal{T}(\mathcal{M}))$ are Hausdorff uniform D-posets.*

All uniform D-posets in Examples 1–10 are equal to $(D, \tau(\mathcal{M}))$ or $(D, \mathcal{T}(\mathcal{M}))$ for some $\mathcal{M}$.

Proof. If $(a_1,a_2), (b_1,b_2) \in U_{m,\frac{\varepsilon}{2}}$, $a_1 \leqq b_1$, $a_2 \leqq b_2$, then $(b_1 \ominus a_1, b_2 \ominus a_2) \in U_{m,\varepsilon}$. Hence, by Lemma 2.2, $(D, \tau(\mathcal{M}))$ is a uniform D-poset. Similarly, if $\varrho_{\mathcal{M}}(a_1,a_2) < \frac{\varepsilon}{3}$, $\varrho_{\mathcal{M}}(b_1,b_2) < \frac{\varepsilon}{3}$, $a_1 \leqq b_1$, $a_2 \leqq b_2$, then $\varrho_{\mathcal{M}}(b_1 \ominus a_1, b_2 \ominus a_2) < \varepsilon$.

Hence, $\big(D, \mathcal{T}(\mathcal{M})\big)$ is a uniform D-poset. Since $\mathcal{M}$ is separating, $\tau(\mathcal{M})$ and $\mathcal{T}(\mathcal{M})$ are Hausdorff.

Let us prove that all topologies in Examples 1 – 10 are special cases of $\tau(\mathcal{M})$ or $\mathcal{T}(\mathcal{M})$.

E1. $\overline{\mathcal{T}}_\mu$ is equal to $\tau(\mathcal{M})$, where $\mathcal{M}$ is the family of all measures m_A on $\overline{\mathcal{S}}$ of the form $m_A([E]) = \mu(A \cap E)$, $[E] \in \overline{\mathcal{S}}$, $A \in \mathcal{S}$.

E2. A measure μ on $\mathcal{L}(H)$ is called *Gleason measure* if μ is of the form $\mu(M) = \operatorname{tr} TP^M$, $m \in \mathcal{L}(H)$ ($\operatorname{tr} TP^M$ is the trace of TP^M), where T is a nonnegative hermitean trace class operator. Let $\mathcal{M}$ be the set of all Gleason measures μ such that $\mu(H) = 1$. It was proved in [8] that $\tau_{\text{strong}} = \tau(\mathcal{M})$.

E3. Let $\mathcal{M}$ be the same as in E2. It was proved in [2] that $\|P^M - P^N\| = \sup\big\{|m(M) - m(N)|;\ m \in \mathcal{M}\big\}$. Hence, $\tau_{\text{unif}} = \mathcal{T}(\mathcal{M})$.

E4, E6, E8. For every t from the domain of functions in D, let us define the measure m_t, $m_t(f) = f(t)$, $f \in D$. If $\mathcal{M}$ is the family of all measures m_t, then $\tau_{\text{pc}} = \tau(\mathcal{M})$.

E5, E7, E9, E10. Every topological D-poset $(D, \mathcal{T})$ in these examples is a topological subspace of some linear norm space X. In E5, X is the space of all bounded real functions defined on A, in E7, X is the space of all real continuous functions defined on $\langle 0,1 \rangle$, in E9, X is the space of all real convergent sequences. The norm of X in these examples is the usual supremum norm. In E10, X is the space $\mathcal{L}^p(\langle 0,1 \rangle)$ with the usual norm. Let X' be the dual space to X, i.e., the space of all bounded functionals defined on X, and $X'' = (X')'$ be the second dual space. If $J\colon X \to X''$ is the canonical mapping, i.e., for $x \in X$, $Jx = x''$, where $x''(x') = x'(x)$, $x' \in X'$, then $\|x\| = \|Jx\|$ (see [15]). Hence, we have

$$\begin{aligned} \|x\| = \|Jx\| &= \sup\big\{|x''(x')|;\ x' \in X',\ \|x'\| \leqq 1\big\} \\ &= \sup\big\{|x'(x)|;\ x' \in X',\ \|x'\| \leqq 1\big\}. \end{aligned}$$

Then for every net $a_\alpha \in D$, $a \in D$,

$$\begin{aligned} \|a_\alpha - a\| &= \sup\big\{|x'(a_\alpha - a)|;\ x' \in X',\ \|x'\| \leqq 1\big\} \\ &= \sup\big\{|x'(a_\alpha) - x'(a)|;\ x' \in X',\ \|x'\| \leqq 1\big\}, \end{aligned}$$

and, hence, $a_\alpha \to a$ in $(D, \mathcal{T})$ if and only if $a_\alpha \to a$ in $\big(D, \mathcal{T}(\mathcal{M})\big)$, where $\mathcal{M}$ contains restrictions of all bounded linear functionals x', $\|x'\| \leqq 1$, from X to D. Hence, $\mathcal{T} = \mathcal{T}(\mathcal{M})$. □

3. Uniform lattice D-posets

If $(D, \mathcal{T})$ is a topological D-poset and D is a lattice, then the continuity of the lattice operations $\vee$ and $\wedge$ is not guaranteed, in general. $\big(\mathcal{L}(H), \tau_{\text{strong}}\big)$

and $(\mathcal{L}(H), \tau_{\text{unif}})$ are uniform D-posets, but $\vee$ and $\wedge$ are not continuous. For a nonzero vector $\varphi \in H$, let $[\varphi]$ be the one dimensional subspace generated by φ. If $\varphi_n \to \varphi$, $\psi_n \to \psi$, $\|\varphi_n\| = \|\psi_n\| = \|\varphi\| = 1$ and $(\varphi_n, \psi_n) = 0$, $n = 1, 2 \dots$, then $[\varphi_n] \to [\varphi]$, $[\psi_n] \to [\psi]$ in $(\mathcal{L}(H), \tau_{\text{strong}})$ and in $(\mathcal{L}(H), \tau_{\text{unif}})$ as well, but $[\varphi_n] \vee [\psi_n]$ does not converge to $[\varphi] \vee [\varphi] = [\varphi]$ in any of these topologies. However, on orthomodular posets, a topology giving a D-poset guarantees at least partial continuity of $\vee$ (and hence, also of $\wedge$). The following lemma is true:

LEMMA 3.1. *If D is an OMP, then $(D, \mathcal{T})$ is topological D-poset if and only if the following conditions are true:*

i) *if $a_\alpha \to a$ in $(D, \mathcal{T})$, then $a_\alpha^\perp \to a^\perp$ in $(D, \mathcal{T})$;*

ii) *if $a_\alpha \to a$, $b_\alpha \to b$ in $(D, \mathcal{T})$, $a_\alpha \perp b_\alpha$, $a \perp b$, then $a_\alpha \vee b_\alpha \to a \vee b$.*

Proof. For $a \leqq b$, $b \ominus a = b \wedge a^\perp = (b^\perp \vee a)^\perp$, where $a \perp b^\perp$. So, if i) and ii) are true, then $\ominus$ is continuous. Conversely, $a^\perp = 1 \ominus a$, and for $a, b \in D$, $a \perp b$, $a \vee b = 1 \ominus ((1 \ominus a) \ominus b))$. Hence, the continuity of $\ominus$ implies i), ii). □

Orthomodular posets with topologies with properties i) and ii) were studied in [8] before D-posets were introduced.

In the following, we introduce topologies on lattice D-posets, which also guarantee the continuity of lattice operations. Such topologies were studied in the last decade on orthomodular lattices (see [10], [11], [12], [14]). In the following, D is assumed to be a lattice.

DEFINITION 3.2. We say that $(D, \mathcal{T})$ is a *topological lattice D-poset* if

i) the mapping $\ominus\colon (G, \mathcal{T}_0) \to (D, \mathcal{T})$ is continuous,

ii) the mappings $\vee, \wedge\colon (D \times D, \mathcal{T} \times \mathcal{T}) \to (D, \mathcal{T})$ are continuous.

If $\mathcal{T}$ is uniform, we say that $(D, \mathcal{T})$ is an *uniform lattice D-poset* if the mappings $\ominus$, $\vee$ and $\wedge$ are uniformly continuous.

The following lemma is routine.

LEMMA 3.3. *If $\mathcal{T}$ is induced by a uniformity $\mathcal{U}$, then $(D, \mathcal{T})$ is a uniform lattice D-poset if and only if for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $(x_1, x_2), (y_1, y_2) \in V$ implies $(x_1 \vee y_1, x_2 \vee y_2) \in U$, $(x_1 \wedge y_1, x_2 \wedge y_2) \in U$, and, if moreover $x_1 \leqq y_1$ and $x_2 \leqq y_2$, then also $(y_1 \ominus x_1, y_2 \ominus x_2) \in U$.*

It can be easily seen that D-posets in E1 and E4–E9 are uniform lattice D-posets.

It was proved in [14] that the topological completion of a Hausdorff uniform orthomodular lattice in which all monotone nets are Cauchy is also an orthomodular lattice, which is complete. As we shall see, this result is true also for D-posets.

Let $(D,\mathcal{U})$ be a uniform lattice D-poset. Two nets a_α, b_β of elements of D are called *equivalent* ($a_\alpha \sim b_\beta$) if for every $U \in \mathcal{U}$ there exist indices α_0, β_0 such that $(a_\alpha, b_\beta) \in U$ for $\alpha \geqq \alpha_0$, $\beta \geqq \beta_0$.

REMARK 3.4. If $a_\alpha \sim a'_\alpha$, $b_\alpha \sim b'_\alpha$, then $a_\alpha \vee b_\alpha \sim a'_\alpha \vee b'_\alpha$ and $a_\alpha \wedge b_\alpha \sim a'_\alpha \wedge b'_\alpha$. Specially, if $a_\alpha \leqq b_\alpha$, $b'_\alpha \leqq a'_\alpha$, we obtain $b_\alpha = b_\alpha \vee a_\alpha \sim b'_\alpha \vee a'_\alpha = a'_\alpha \sim a_\alpha$.

LEMMA 3.5. *Let $(D,\mathcal{U})$ be a uniform lattice D-poset. If a_α, b_α, b'_α, c_α are nets in D and $a_\alpha \leqq b_\alpha$, $b'_\alpha \leqq c_\alpha$, $b_\alpha \sim b'_\alpha$, then there exist nets $b''_\alpha \sim b_\alpha$ and $c'_\alpha \sim c_\alpha$ such that $a_\alpha \leqq b''_\alpha \leqq c'_\alpha$.*

Proof. Put $b''_\alpha = b_\alpha \vee b'_\alpha \sim b'_\alpha \vee b'_\alpha \sim b_\alpha$. Then we put $c'_\alpha = c_\alpha \vee b''_\alpha \sim c_\alpha \vee b'_\alpha = c_\alpha$. Clearly, $a_\alpha \leqq b''_\alpha \leqq c'_\alpha$. □

We can embed every uniform space $(X,\mathcal{U})$ into a complete uniform space $(\hat{X},\hat{\mathcal{U}})$ in a standard way (see [5]).

THEOREM 3.6. *If $(D,\mathcal{U})$ is a Hausdorff uniform lattice D-poset in which all monotone nets are Cauchy, then there exist extensions of $\leqq$ and $\ominus$ on $\hat{D}$ such that $(\hat{D},\hat{\mathcal{U}})$ is also a Hausdorff uniform lattice D-poset, and $\hat{D}$ is a complete lattice.*

Proof.

I. In this first part, we define extensions of $\leqq$ and $\ominus$. (For the extension of the partial ordering we shall use the same symbol $\leqq$.) For $a, b \in \hat{D}$ we put $a \leqq b$ if there exist nets $\{x_{\hat{U}}\}_{\hat{U}\in\hat{\mathcal{U}}}$, $\{y_{\hat{U}}\}_{\hat{U}\in\hat{\mathcal{U}}}$ of elements of D such that $x_{\hat{U}} \leqq y_{\hat{U}}$, $x_{\hat{U}} \to a$, and $y_{\hat{U}} \to b$. The reflexivity of $\leqq$ is clear, the antisymmetricity follows from Remark 3.4, and Lemma 3.5 implies the transitivity. Obviously, this partial ordering is the extension of that on D.

Let us define the difference operation $\overline{\ominus}$ on $\hat{D}$. For $a, b \in \hat{D}$, $a \leqq b$, there exist nets $x_{\hat{U}}$, $y_{\hat{U}} \in D$, $x_{\hat{U}} \leqq y_{\hat{U}}$, $x_{\hat{U}} \to a$ and $y_{\hat{U}} \to b$. The net $y_{\hat{U}} \ominus x_{\hat{U}}$ is Cauchy, let us denote its limit by $b \,\overline{\ominus}\, a$. Clearly, $\overline{\ominus}$ is the extension of $\ominus$, and the uniform continuity of $\ominus$ implies that the definition of $\overline{\ominus}$ is correct.

We shall prove that $\overline{\ominus}$ is a difference operation. Obviously, $y_{\hat{U}} \ominus x_{\hat{U}} \leqq y_{\hat{U}}$ implies $b \,\overline{\ominus}\, a \leqq b$. Since $y_{\hat{U}} \ominus (y_{\hat{U}} \ominus x_{\hat{U}}) = x_{\hat{U}}$ and $y_{\hat{U}} \ominus (y_{\hat{U}} \ominus x_{\hat{U}}) \to b \,\overline{\ominus}\, (b \,\overline{\ominus}\, a)$, we have $b \,\overline{\ominus}\, (b \,\overline{\ominus}\, a) = a$. Let $a \leqq b \leqq c$. By Lemma 3.5, there exist nets $x_{\hat{U}} \leqq y_{\hat{U}} \leqq z_{\hat{U}}$ converging to a, b, c, respectively. Then $z_{\hat{U}} \ominus y_{\hat{U}} \leqq z_{\hat{U}} \ominus x_{\hat{U}}$ implies $c \,\overline{\ominus}\, b \leqq c \,\overline{\ominus}\, a$. Moreover, $(z_{\hat{U}} \ominus x_{\hat{U}}) \ominus (z_{\hat{U}} \ominus y_{\hat{U}}) = y_{\hat{U}} \ominus x_{\hat{U}}$, and this implies $(c \,\overline{\ominus}\, a) \,\overline{\ominus}\, (c \,\overline{\ominus}\, b) = b \,\overline{\ominus}\, a$. Hence, $\overline{\ominus}$ is a difference operation, and $(\hat{D}, \leqq, \overline{\ominus}, 1)$ is a D-poset.

Let us prove that $\hat{D}$ is a lattice. If $a, b \in \hat{D}$ are given, then there exist nets $x_{\hat{U}}$, $y_{\hat{U}}$ of elements of D such that $x_{\hat{U}} \to a$, $y_{\hat{U}} \to b$. Then $x_{\hat{U}} \vee y_{\hat{U}}$ is Cauchy, and it converges to some $c \in \hat{D}$, $a \leqq c$, $b \leqq c$. If $d \in \hat{D}$ is given, $a \leqq d$, $b \leqq d$, then there exist nets $a_{\hat{U}} \to a$, $d_{\hat{U}} \to d$, $b_{\hat{U}} \to b$, $d'_{\hat{U}} \to d$, $a_{\hat{U}} \leqq d_{\hat{U}}$, $b_{\hat{U}} \leqq d'_{\hat{U}}$. Then $d''_{\hat{U}} = d_{\hat{U}} \vee d'_{\hat{U}} \to d$, $a_{\hat{U}} \vee b_{\hat{U}} \leqq d''_{\hat{U}}$. Then $c \leqq d$ and, hence, $c = a \vee b$. We can prove the existence of $a \wedge b$ similarly. $\hat{D}$ is a lattice.

II. In this step, we shall prove that $(\hat{D}, \hat{\mathcal{U}})$ is uniform lattice D-poset. We shall use the fact that closures of all $U \in \mathcal{U}$ in the product space $\hat{D} \times \hat{D}$ form a base of $\hat{\mathcal{U}}$. Let $\hat{U} \in \hat{\mathcal{U}}$ be given, $\hat{U} = \overline{U}$ for some $U \in \mathcal{U}$. Since $(D, \mathcal{U})$ was a uniform lattice D-poset, by Lemma 3.3, there exists $V \in \mathcal{U}$ such that (x_1, x_2), $(y_1, y_2) \in V$ implies $(x_1 \vee y_1, x_2 \vee y_2)$ and $(x_1 \wedge y_1, x_2 \wedge y_2) \in U$, and, if moreover $x_1 \leqq y_1$, $x_2 \leqq y_2$, then also $(y_1 \ominus x_1, y_2 \ominus x_2) \in U$.

Let $V_1 \in \mathcal{U}$, $V_1 \circ V_1 \circ V_1 \subset V$. If $(x_1, x_2) \in \overline{V}_1$, $(y_1, y_2) \in \overline{V}_1$, $x_1 \leqq y_1$, $x_2 \leqq y_2$, then there exist nets $(x^1_{\hat{U}}, x^2_{\hat{U}}) \in V_1$, $(y^1_{\hat{U}}, y^2_{\hat{U}}) \in V_1$ converging to (x_1, x_2) and (y_1, y_2) in $\hat{D} \times \hat{D}$. By the definition of partial ordering in $\hat{D}$, there exist nets $\overline{x}^1_{\hat{U}}, \overline{x}^2_{\hat{U}} \in D$ converging to x_1, x_2, and nets $\overline{y}^1_{\hat{U}}, \overline{y}^2_{\hat{U}} \in D$ converging to y_1, y_2 such that $\overline{x}^1_{\hat{U}} \leqq \overline{y}^1_{\hat{U}}$ and $\overline{x}^2_{\hat{U}} \leqq \overline{y}^2_{\hat{U}}$. Then $y^1_{\hat{U}} \sim y^1_{\hat{U}} \vee \overline{y}^1_{\hat{U}} = z^1_{\hat{U}}$, and, starting from some index, we have $(y^1_{\hat{U}}, z^1_{\hat{U}}) \in V_1$. Similarly, if $z^2_{\hat{U}} = y^2_{\hat{U}} \vee \overline{y}^2_{\hat{U}}$, we have $(y^2_{\hat{U}}, z^2_{\hat{U}}) \in V_1$, starting from a certain index. Then $(z^1_{\hat{U}}, z^2_{\hat{U}}) \in V$, starting from a certain index. At the same time, we have $(x^1_{\hat{U}}, x^2_{\hat{U}}) \in V$ and $x^1_{\hat{U}} \leqq z^1_{\hat{U}}$, $x^2_{\hat{U}} \leqq z^2_{\hat{U}}$. This implies $(z^1_{\hat{U}} \ominus x^1_{\hat{U}}, z^2_{\hat{U}} \ominus x^2_{\hat{U}}) \in U$. Then $(y_1 \overline{\ominus} x_1, y_2 \overline{\ominus} x_2) \in \overline{U}$.

Similarly, for any $(x_1, x_2) \in \overline{V}$, $(y_1, y_2) \in \overline{V}$, we have $(x_1 \vee y_1, x_2 \vee y_2)$ and $(x_1 \wedge y_1, x_2 \wedge y_2) \in \overline{U}$. Hence, $(\hat{D}, \leqq, \overline{\ominus}, 1)$ is a Hausdorff uniform D-poset.

III. The proof of the completeness of $\hat{D}$ does not differ from the case where D is an OML ([14]). First, we show that $\bigvee a_n$ exists in $\hat{D}$ for every increasing sequence $a_n \in \hat{D}$. If a_n is given, let us prove that it is Cauchy. For $\hat{W} \in \hat{\mathcal{U}}$ there exists a sequence $\hat{U}_n \in \hat{\mathcal{U}}$ with the properties:

1) $\hat{U}_1 \circ \hat{U}_1 \circ \hat{U}_1 \subset \hat{W}$,
2) $(x_1, x_2), (y_1, y_2) \in \hat{U}_{n+1}$ implies $(x_1 \vee y_1, x_2 \vee y_2) \in \hat{U}_n$.

For every n natural, there exists a net $a^n_\alpha \in D$ converging to a_n. For n there exists α_n such that $(a^n_{\alpha_n}, a_n) \in \hat{U}_{n+1}$. If we put $b_k = \bigvee\limits_{n=1}^{k} a^n_{\alpha_n}$, then $(b_k, a_k) \in \hat{U}_1$, $k = 1, 2 \dots$. Since b_k is Cauchy, we have $(a_n, a_m) \in \hat{U}_1 \circ \hat{U}_1 \circ \hat{U}_1 \subset \hat{W}$, starting from some index. Hence, a_n is Cauchy in $\hat{D}$. Then there exists $a \in \hat{D}$, $a_n \to a$ in $(\hat{D}, \hat{\mathcal{U}})$, and this implies $a = \bigvee a_n$. Hence, for every sequence a_n (not only increasing) there exists $\bigvee a_n$ in $\hat{D}$.

Let us assume that $\hat{D}$ is not complete. Let α_0 be the smallest ordinal number such that there exists $M \subset \hat{D}$ such that $M = \{a_\alpha\}_{\alpha < \alpha_0}$, and $\bigvee M$ does not exist. For every $\alpha < \alpha_0$ put $b_\alpha = \bigvee\limits_{\beta \leqq \alpha} a_\beta$ (by assumption, b_α exists). The net b_α is increasing. If it were not Cauchy, a non-Cauchy increasing subsequence of b_α would exist, which is not possible. Hence, b_α is Cauchy, and there exists $b \in \hat{D}$, $b_\alpha \to b$, and this implies $b = \bigvee b_\alpha = \bigvee M$, a contradiction. Hence, $\hat{D}$ is a complete lattice.

Theorem is proved. □

EXAMPLE 3.7. In the Hausdorff uniform lattice D-poset (D, τ_{pc}) from Example 8 all monotone sequences of elements of D are Cauchy. Then its completion $(\hat{D}, \hat{\mathcal{U}})$ is the uniform lattice D-poset of Example 4, where A is the set of all natural numbers.

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