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# ADOMATIC AND IDOMATIC NUMBERS OF GRAPHS

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E. J. Cockayne and S. T. Hedetniemi [1] have defined the domatic number of a graph and also some related concepts, among others the adomatic number of a graph and the idomatic one. Here we shall present some results concerning adomatic and idomatic numbers. We consider finite undirected graphs without loops and multiple edges. First we shall give definitions.

A dominating set in a graph $G$ is a subset $D$ of the vertex set $V(G)$ of $G$ with the property that to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ adjacent to $x$. A dominating set $D$ of $G$ is called indivisible if it is not a union of two disjoint dominating sets of $G$. A partition of $V(G)$, all of whose classes are dominating sets in $G$, is called a domatic partition of $G$. The maximum number of classes of a domatic partition of $G$ is called the domatic number of $G$ and denoted by $d(G)$. The minimum number of classes of a partition of $V(G)$, all of whose classes are indivisible dominating sets in $G$, is called the adomatic number of $G$ and denoted by $ad(G)$. If there exists at least one domatic partition of $G$, all of whose classes are independent sets, then the maximum number of classes of such a partition is called the idomatic number of $G$ and denoted by $id(G)$. If no such partition exists, we put $id(G) = 0$. A graph $G$ for which $id(G) \neq 0$ is called idomatic.

First we prove some assertions concerning the adomatic number.

**Proposition 1.** *A connected graph $G$ has the adomatic number equal to* 1 *if and only if it consists of one vertex.*

Proof. If $G$ consists of one vertex, the unique partition of its vertex set consists of one class and this class is a dominating set in $G$. On the other hand, if $ad(G) = 1$, then $V(G)$ must be an indivisible dominating set in $G$. If $G$ is a connected graph with more than one vertex, then according to [1] its domatic number is at least 2 and there exists a partition $\{D_1, \ldots, D_d\}$ of $V(G)$, where $d$ is the domatic number of $G$ and all classes of the partition are dominating sets in $G$. Then $V(G)$ is the union of two disjoint sets $D_1$, $\bigcup_{i=2}^{d} D_i$, which are both dominating sets in $G$, thus it is not an indivisible dominating set in $G$ and $ad(G) \geqq 2$.

**Theorem 1.** *Let $G$ be a disconnected graph without isolated vertices. Then* $\mathrm{ad}(G)=2$.

Proof. Let $H_1, \ldots, H_k$ be connected components of $G$. As each of these components has at least two vertices, its domatic number is at least 2. For each $i=1, \ldots, k$ choose a domatic partition $\mathscr{D}_i$ of $H_i$ with the maximum number of classes. In each $\mathscr{D}_i$ choose one class $E_i$ and by $F_i$ denote the set of all vertices of $H_i$ not belonging to $E_i$. Evidently $E_i$ is an indivisible dominating set and $F_i$ is a dominating set in $H_i$. Now put $D_1=E_1\cup\bigcup_{i=2}^{k} F_i$, $D_2=F_1\cup\bigcup_{i=2}^{k} E_i$. Evidently $D_1$ and $D_2$ are dominating sets in $G$ and $D_1\cap D_2=\emptyset$. Suppose that $D_1$ is the union of two disjoint sets $A_1$, $A_2$ which are both dominating sets in $G$. Each of the sets $A_1$, $A_2$ must have a non-empty intersection with the vertex sets of all connected components of $G$; thus let $B_1=A_1\cap V(H_1)$, $B_2=A_2\cap V(H_1)$. Then $B_1$, $B_2$ are disjoint dominating sets in $H_1$ and $B_1\cup B_2=E_1$, which is a contradiction with the indivisibility of $E_1$. We have proved that $D_1$ is an indivisible dominating set in $G$. Analogously (using $H_2$ instead of $H_1$) we prove that so is $D_2$. Hence $ad(G)\leqq 2$. According to Proposition 1 it cannot be 1, therefore $ad(G)=2$.

Before proving a further theorem we shall prove a lemma.

**Lemma.** *Let $u$, $v$ be two vertices of a connected graph $G$, let their distance be at least* 3. *Then there exists a spanning tree $T$ of $G$ which contains all edges incident with $u$ and all edges incident with $v$.*

Proof. Choose a shortest path $P$ connecting $u$ and $v$ in $G$; it contains exactly one vertex adjacent to $u$ and exactly one vertex adjacent to $v$. Let $T_0$ be the subgraph of $G$ whose edge set consists of all edges of $P$, all edges incident with $u$ and all edges incident with $v$ and whose vertex set consists of all end vertices of these edges. As the distance between $u$ and $v$ is at least 3, the graph $T_0$ is a tree. Each circuit in $G$ contains at least one edge not belonging to $T_0$, therefore it is possible to destroy all circuits of $G$ by successive deleting edges not belonging to $T_0$ and then a spanning tree $T$ is obtained which contains $T_0$ as a subtree.

**Theorem 2.** *Let $G$ be a connected graph whose diameter is at least* 3. *Then* $\mathrm{ad}(G)=2$.

Proof. Let $u$, $v$ be two vertices of $G$ whose distance is at least 3. Let $T$ be a spanning tree of $G$ described in Lemma. We shall colour the vertices of $T$ by the colours 1 and 2. The vertex $u$ will be coloured by the colour 1 and all vertices adjacent to it by the colour 2. The vertex $v$ will be coloured by 2 and all vertices adjacent to it by 1. Now let $P$ be the path described in the proof of Lemma. Let the vertices of $P$ be $u=x_0, x_1, \ldots, x_k=v$ and let its edges be $x_ix_{i+1}$ for $i=0, 1, \ldots, k-1$. If $k$ is odd, then $x_i$ will be coloured by 1 for $i$ even and by 2 for $i$ odd. If $k$ is even, then $x_i$ will be coloured by 1 for $i$ even, $i\leqq k-2$ and by 2 for $i$ odd, $i\leqq k-3$; further $x_{k-1}$ will be coloured by 1 and $x_k$ by 2. Thus all vertices of $T_0$ are coloured.

To each vertex $y$ of $T$ not belonging to $T_0$ there exists exactly one vertex $z$ of $T_0$ whose distance from $y$ in $T$ is minimal. If this distance is even, we colour $y$ by the same colour as $z$, if it is odd, we colour it by the colour other than that of $z$. Let $D_1$ (or $D_2$) be the set of all vertices coloured by 1 (or by 2 respectively). Then $\{D_1, D_2\}$ is a domatic partition of $T$ and also of $G$. Suppose that $D_1$ is not indivisible. Then $D_1$ is the union of two disjoint dominating sets $A_1$, $A_2$ of $G$. Exactly one of the sets $A_1$, $A_2$ contains $u$; without loss of generality let it be $A_1$. Then $A_2$ does not contain $u$ and no vertex of $A_2$ is adjacent to $u$ (all vertices adjacent to $u$ belong to $D_2$). This is a contradiction with the assumption that $A_2$ is a dominating set in $G$. We have proved that $D_1$ is an indivisible dominating set in $G$. Analogously we prove that so is $D_2$. Hence $ad(G)=2$.

**Theorem 3.** *Let $a$, $n$ be integers such that $2 \leqq a \leqq n-2$ or $2 \leqq a = n$. Then there exists a connected graph $G$ with $n$ vertices such that $ad(G)=a$.*

Proof. If $2 \leqq a = n$, the required graph is the complete graph with $n$ vertices. Thus suppose $2 \leqq a \leqq n-2$. Let $V_1$, $V_2$ be two disjoint sets, let $|V_1| = a$, $|V_2| = n-a$. Let $G(a, n)$ be the graph with the vertex set $V = V_1 \cup V_2$ in which two vertices are adjacent if and only if at least one of them belongs to $V_1$. Let $x_1$, $x_2$ be two distinct vertices of $V_1$, let $y$ be a vertex of $V_2$. Consider the sets $D_1, \ldots, D_a$ such that $D_1 = \{x_1, y\}$, $D_2 = \{x_2\} \cup (V_2 - \{y\})$ and the sets $D_3, \ldots, D_a$ (if $a \geqq 3$) as one-element subsets of $V_1 - \{x_1, x_2\}$. The sets $D_1, \ldots, D_a$ form a domatic partition of $G(a, n)$. Moreover, each of these sets is an indivisible dominating set in $G(a, n)$; this follows from the fact that neither $\{y\}$, nor $V_2 - \{y\}$ is a dominating set. Hence $ad(G(a, n)) \leqq a$. Suppose that there exists a partition of the vertex set of $G(a, n)$ into less than $a$ indivisible dominating sets. Then according to the Pigeon Hole Principle at least one of these sets contains two distinct vertices of $V_1$. If we denote it by $E$ and the mentioned vertices by $u$, $v$, then $E$ is the union of disjoint sets $\{u\}$, $E - \{u\}$. The set $\{u\}$ is evidently dominating in $G(a, n)$ and so is $E - \{u\}$, because it contains a dominating set $\{v\}$ as a subset. This is a contradiction with the indivisibility of $E$. We have proved that $ad(G(a, n)) = a$.

**Theorem 4.** *If $G$ is a connected graph with $n$ vertices, $n \geqq 4$, then $ad(G) \neq n-1$.*

Proof. Suppose that $ad(G) = n-1$ and let $\mathscr{D}$ be a domatic partition of $G$ with $n-1$ classes. Then exactly one class of $\mathscr{D}$ consists of two vertices and all others are one-element sets. As all of these sets are dominating in $G$, the graph $G$ is either complete, or obtained from a complete graph by deleting one edge. In the first case $ad(G) = n$. In the second case $G$ is isomorphic to the graph $G(a, n)$ from the proof of Theorem 3 for $a = n-2$ and thus $ad(G) = n-2$.

For $n = 3$ the assertion does not hold. A path of the length 2 has 3 vertices and its adomatic number is 2.

Now we turn to the idomatic number of a graph. E. J. Cockayne and S. T. Hedetniemi [1] have suggested the problem to characterize idomatic graphs. We shall give a simple characterization of them.

**Proposition 2.** *A graph $G$ is idomatic if and only if its vertex set $V(G)$ is the union of pairwise disjoint maximal independent sets.*

Proof. Let $M$ be a maximal independent set in a graph $G$. Then $M$ is a dominating set in $G$; otherwise there would exist a vertex $x \in V(G) - M$ adjacent to no vertex of $M$ and $M \cup \{x\}$ would be an independent set, which would be a contradiction with the maximality of $M$. This implies the sufficiency of the condition. On the other hand, if $N$ is an independent dominating set in $G$, then it is evidently a maximal independent set in $G$ and this implies the necessity.

**Proposition 3.** *Let $G$ be an idomatic graph. Then*

$$ad(G) \leqq id(G) \leqq d(G).$$

Proof. The inequality $id(G) \leqq d(G)$ is evident. Now let $M$ be an independent dominating set in $G$. As we have shown in the proof of Proposition 2, no proper subset of $M$ is dominating in $M$, because it is an independent set which is not maximal. Hence $M$ is an indivisible dominating set and this implies $ad(G) \leqq id(G)$.

**Proposition 4.** *Let $G$ be an idomatic graph. Then $\chi(G) \leqq id(G)$ and consequently $\chi(G) \leqq d(G)$, where $\chi(G)$ is the chromatic number of $G$.*

The proof is left to the reader.

**Theorem 5.** *Let $c$, $d$ be integers, $2 \leqq c \leqq d$. Then there exists a graph $G$ such that $id(G) = c$, $d(G) = d$.*

Proof. Let the vertex set of $G$ be the union of disjoint sets $X = \{x_1, \ldots, x_{d-c+2}\}$, $Y = \{y_1, \ldots, y_{d-c+2}\}$, $z = \{z_1, \ldots, z_{c-2}\}$. Two vertices of $G$ are adjacent if and only if they belong neither both to $X$, nor both to $Y$. Evidently $\{X, Y, \{z_1\}, \ldots, \{z_{c-2}\}\}$ is a partition of the vertex set of $G$, all of whose classes are independent dominating sets. It has $c$ classes and evidently there exists no such partition with more than $c$ classes, because any other independent set in $G$ is a proper subset of $X$ or of $Y$ and is not dominating in $G$. Hence $id(G) = c$. Now $\{\{x_1, y_1\}, \ldots, \{x_{d-c+2}, y_{d-c+2}\}, \{z_1\}, \ldots, \{z_{c-2}\}\}$ is a domatic partition of $G$ with $d$ classes. Any partition of the vertex set of $G$ with more than $d$ classes would contain a class being a proper subset of $X$ or of $y$ and such a class would not be a dominating set in $G$. Therefore $d(G) = d$.

## REFERENCE


[1] COCKAYNE, E. J.—HEDETNIEMI, S. T.: Towards a theory of domination in graphs. Networks 7, 1977, 247—261.



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## АДОМАТИЧЕСКИЕ И ИДОМАТИЧЕСКИЕ ЧИСЛА ГРАФОВ

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### Резюме

Доминантное множество в графе $G$ называется неразложимым, если оно не является объединением двух непересекающихся доминантных множеств в $G$. Минимальное число классов разбиения множества вершин $V(G)$ графа $G$, все классы которого являются неразложимыми доминантными множествами в $G$, называется адоматическим числом графа $G$ и обозначается через $ad(G)$. Максимальное число классов разбиения множества $V(G)$, все классы которого являются независимыми доминантными множествами в $G$, называется идоматическим числом графа $G$ и обозначается через $id(G)$. Изучаются свойства этих чисел.