Miroslav Fiedler; Thomas L. Markham

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on the occasion of his 80th birthday

ON A THEOREM OF EVERITT, THOMPSON, AND de PILLIS

MIROSLAV FIEDLER* — THOMAS L. MARKHAM**

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ABSTRACT. It is proved that if $H = (H_{ik})$ is a partitioned positive semidefinite matrix with square blocks, then the matrix $\big(E_r(H_{ik})\big)$, where $E_r(X)$ denotes the rth elementary symmetric function of the eigenvalues of X, is again a positive semidefinite matrix.

1. Introduction

In 1958, W. N. Everitt [2] proved that if $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$, where each H_{ij} is a $k \times k$ matrix, is positive definite hermitian, then

$$\det H \le \det(H_{11}) \det(H_{22}) - |\det(H_{12})|^2 .$$

In 1961, R. C. Thompson [5] extended this result to the case where $H = (H_{ij})$, $1 \le i,j \le n$, is an $nk \times nk$ matrix. His main result was the following:

Theorem ([5]): If H is positive definite hermitian with $H = (H_{ij})$, $1 \le i,j \le n$, with each block H_{ij} of order k, then let $\widehat{H} = (\det H_{ij})$. Then $\widehat{H}$ is positive definite hermitian and $\det(H) \le \det(\widehat{H})$. Equality holds if and only if $H_{ij} = 0$ whenever $i \ne j$.

Thompson's proof used an identity for the inner product of Grassmann products as his main weapon.

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John de Pillis [1] showed that the first part of Thompson's result holds in general for the elementary symmetric functions $E_i(A)$, $i = 1, \dots, k$, of the eigenvalues of the matrix A. For a general $k \times k$ matrix A, we have $E_1(A) = \mathrm{tr}(A)$ and $E_k(A) = \det(A)$.

Before we state this result, we introduce a few preliminary notions and cite two useful theorems.

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of size $m \times n$, the Hadamard product of A and B, denoted $A \circ B$, is the $m \times n$ matrix $(a_{ij}b_{ij})$.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then tensor product of A and B, denoted $A \otimes B$, is the $mp \times nq$ matrix $(a_{ij}B)$, in partitioned form. It is well known that $A \circ B$ is a principal submatrix of $A \otimes B$, whenever A and B are square of the same order.

I. Schur proved [4] that if each of A and B is positive semidefinite hermitian of the same order n, then $A \circ B$ is a positive semidefinite hermitian matrix. He also stated the result that if A and B are positive semidefinite hermitian, then $\det(A \circ B) \geq \max\left\{ \left(\prod_{i=1}^{n} a_{ii} \right) \det B, \left(\prod_{i=1}^{n} b_{ii} \right) \det A \right\}$.

Later, Sir Alexander Oppenheim [3] proved this result, and strengthened it.

As we mentioned above, Thompson's result was generalized by John de Pillis, as in the following Theorem 1, of which we will give a new proof. Finally, we give a determinantal inequality, and then state an observation concerning Thompson's result.

2. Main result

THEOREM 1. *Let $H = (H_{ij})$, $1 \leq i, j \leq n$, be a positive semidefinite hermitian matrix with each block H_{ij} of order k. Let E_i denote the ith elementary symmetric function, $1 \leq i \leq k$. Denote $\widehat{H}_r = \big(E_r(H_{ij})\big)$, $1 \leq r \leq k$, and let $E_0(H_{ij}) = 1$ for each pair (i, j). Then $\widehat{H}_r$ is positive semidefinite for $r = 0, 1, \dots, k$.*

Proof. For $r = 0$, we have $\widehat{H}_0 = J_n \otimes I_k$, where J_n is the matrix of all 1's, so $\widehat{H}_0$ is clearly positive semidefinite since J_n and I_k are positive semidefinite. Assume $1 \leq r \leq k$.

At the first stage, we construct the matrix $K = \big(C^r(H_{ij})\big)$, where $C^r(\cdot)$ denotes the rth compound matrix. Thus each block matrix in K has order $\binom{k}{r}$. The matrix K is a principal submatrix of $C^r(H)$, and hence is positive semidefinite hermitian. For any square matrix A, the eigenvalues of $C^r(A)$ are all possible products $\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_r}$ of the eigenvalues γ_j of A. It is thus clear that $\operatorname{tr}(C^r(A)) = E_r(A)$, so that

$$\operatorname{tr}\big[C^r(H_{ij})\big] = E_r(H_{ij})\,. \tag{1}$$

Let J_n denote as before the $n \times n$ matrix of all 1's. Consider the product $K \circ \left(J_n \otimes I_{\binom{k}{r}}\right)$. This matrix is again positive semidefinite by Schur's result that the Hadamard product of two positive semidefinite matrices is again positive semidefinite. At this stage,

$$K \circ \left(J_n \otimes I_{\binom{k}{r}}\right) = \begin{bmatrix} h_1^{(11)} & & & & h_1^{(1n)} & & \\ & \ddots & & \cdots & & \ddots & \\ & & h_{\binom{k}{r}}^{(11)} & & & & h_{\binom{k}{r}}^{(1n)} \\ & \vdots & & \ddots & & \vdots & \\ h_1^{(n1)} & & & & h_1^{(nn)} & & \\ & \ddots & & \cdots & & \ddots & \\ & & h_{\binom{k}{r}}^{(n1)} & & & & h_{\binom{k}{r}}^{(nn)} \end{bmatrix}.$$

This matrix is permutationally similar to a matrix of the form

$$\begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_{\binom{k}{r}} \end{bmatrix},$$

where

$$D_i = \begin{bmatrix} h_i^{(11)} & h_i^{(12)} & \ldots & h_i^{(1n)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ h_i^{(n1)} & h_i^{(n2)} & \ldots & h_i^{(nn)} \end{bmatrix}, \qquad i = 1, \ldots, \binom{k}{r}.$$

This matrix is again a positive semidefinite matrix, so each block D_i is positive semidefinite. Finally, the sum $\sum D_i$ is positive semidefinite hermitian, and, by (1), $\widehat{H}_r = \sum D_i$, so the theorem is proved. □

COROLLARY 1. *Under the assumption of Theorem* 1, *if we take* $E_1 = \operatorname{trace}(\cdot)$, *then*

$$\frac{1}{k^k}\big[\det(\widehat{H}_1)\big]^k \geq \det H\,. \tag{2}$$

Proof. We have

$$\det(\widehat{H}_1) = \det\left(\sum_{j=1}^{k} D_j\right) \geq \sum_{j=1}^{k} \det D_j \geq k \sqrt[k]{\det\left(\prod_{i=1}^{k} D_i\right)}.$$

Thus

$$\left\{\frac{1}{k} \det \widehat{H}_1\right\}^k \geq \det(D_1 \dots D_k) = \det\big(H \circ (J \otimes I_k)\big) \geq \det(H)$$

by Oppenheim's inequality. □

The multiplicative constant $\frac{1}{k^k}$ in (2) is the best possible since equality can be attained. For example, if $H = I_k$ with $n = 1$, then $\widehat{H} = (k)$, and clearly we get equality.

Finally, we observe the following. For H as given in Theorem 1, let $\widehat{H}(t) = \big(\det(H_{ij} + tI_k)\big)$. It is easy to see that

$$\det\big(\widehat{H}(t)\big) = \det\big[\widehat{H}_k + t\widehat{H}_{k-1} + \dots + t^k \widehat{H}_0\big].$$

By Thompson's theorem, we get that $\det\big(H + t(J_n \otimes I_k)\big) \leq \det\big(\widehat{H}(t)\big)$ for $t \geq 0$.

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* *Mathematics Institute*
Academy of Sciences of Czech Republic
CZ-115 67 Prague
Czech Republic

** *Department of Mathematics*
University of South Carolina
Columbia, SC 29208
U.S.A.